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Convolution integral equations involving a general class of polynomials and the multivariable H-function

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Abstract. In this paper we first solve a convolution integral equation involving product of the general class of polynomials and the H-function of several variables. Due to general nature of the general class of polynomials and the H-function of several variables which occur as kernels in our main convolution integral equation, we can obtain from it solutions of a large number of convolution integral equations involving products of several useful polynomials and special functions as its special cases. We record here only one such special case which involves the product of general class of polynomials and Appell's function F_3. We also give exact references of two results recently obtained by Srivastava *et al* [10] and Rashmi Jain [3] which follow as special cases of our main result.

Keywords. The convolution integral equation; multivariable H-function; general class of polynomials; Laplace transform.

1. Introduction

On account of the usefulness of convolution integral equations, a large number of authors, notably Srivastava [5], Kalla [4], Buschman *et al* [1], Srivastava and Buschman [8], Srivastava *et al* [10] and Rashmi Jain [3], have done significant work on this topic. In the present paper we develop generalizations of results of the last two papers referred to above. Also, Srivastava and Buschman [7, pp. 34–42 and §4.3] have discussed extensively such family of convolution integral equations as those considered here and in the works cited above.

We start by giving the following definitions and results which will be required later on.

(i) A general class of polynomials [6, p. 1, eq. (1)]

$$S_N^M[x] = \sum_{k=0}^{[M/N]} \frac{(-N)_{Mk} A_{N,k}}{k!} x^k, \quad N = 0, 1, 2, \ldots \tag{1.1}$$

where M is an arbitrary positive integer and the coefficient $A_{N,k}$ $(N, k \geqslant 0)$ are arbitrary constants real or complex. On suitably specializing the coefficient $A_{N,k}$, $S_N^M[x]$ yields a number of known polynomials as its special cases. These include, among others, Laguerre polynomials, Hermite polynomials and several others [12, pp. 158–161].
(ii) A special case of the H-function of r variables [11, p. 271, eq. (4.1)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,0:1,n_1;\ldots;1,n_r}_{p,q:p_1,q_1+1;\ldots;p_r,q_r+1}$$

$$\times\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}(a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}: & (c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\ldots; & (c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j:\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}: & (0,1),(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\ldots; & (0,1),(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\cdots\phi_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)$$

$$\times\,\Gamma(-\xi_1)\cdots\Gamma(-\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r,\quad \omega=\sqrt{-1}. \tag{1.2}$$

Or equivalently [10, p. 64, eq. (1.3)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \sum_{k_1,\ldots,k_r=0}^{\infty}\phi_1(k_1)\cdots\phi_r(k_r)\psi(k_1,\ldots,k_r)\frac{(-z_1)^{k_1}}{k_1!}\cdots\frac{(-z_r)^{k_r}}{k_r!} \tag{1.3}$$

where

$$\phi_i(k_i) = \frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}k_i)}{\prod_{j=1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}k_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}k_i)}\quad (i=1,\ldots,r) \tag{1.4}$$

$$\psi(k_1,\ldots,k_r) = \left\{\prod_{j=1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}k_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}k_i\right)\right\}^{-1}. \tag{1.5}$$

For the convergence, existence conditions and other details of the multivariable H-function refer the book [9, pp. 251–253, eqs. (C.2)–(C.8)].

(iii) The following property of the Laplace transform [2, p. 131]

$$L\{f^{(n)}(x);s\} = s^n\bar{f}(s) \tag{1.6}$$

holds provided that $f^{(i)}(0)=0$, $i=0,1,2,\ldots,n-1$, n being a positive integer, where

$$L\{f(x);s\} = \int_0^\infty e^{-sx}f(x)\,\mathrm{d}x = \bar{f}(s). \tag{1.7}$$

(iv) The well-known convolution theorem for Laplace transform

$$L\left\{\int_0^x f(x-u)g(u)\,\mathrm{d}u;s\right\} = L\{f(x);s\}L\{g(x);s\} \tag{1.8}$$

holds provided that the various Laplace transforms occurring in (1.8) exist.

2. Main result

The convolution integral equation

$$\int_0^x (x-u)^{\rho-1}S_N^M[-z_{r+1}(x-u)]H\begin{bmatrix} z_1(x-u) \\ \vdots \\ z_r(x-u) \end{bmatrix}f(u)\,\mathrm{d}u = g(x) \tag{2.1}$$

has the solution given by

$$f(x)=\int_0^x (x-u)^{l-\rho-\mu-1}\sum_{j=0}^{\infty}\frac{E_j(x-u)^j}{\Gamma(j+l-\rho-\mu)}g^{(l)}(u)\,\mathrm{d}u \tag{2.2}$$

where $\mathrm{Re}(l-\rho-\mu)>0$, $\mathrm{Re}(\rho)>0$

$g^{(i)}(0)=0\ (i=0,1,\ldots,l-1)$, l being a positive integer and E_j is given by the recurrence relation

$$E_0\lambda_\mu=1,\quad \sum_{t=0}^{q}E_t\lambda_{q+\mu-t}=0,\quad q=1,2,3,\ldots \tag{2.3}$$

or by

$$E_j=(-1)^j(\lambda_\mu)^{-j-1}\det\begin{bmatrix}\lambda_{\mu+1} & \lambda_\mu & 0 & 0 & \cdots & 0\\ \lambda_{\mu+2} & \lambda_{\mu+1} & \lambda_\mu & 0 & \cdots & 0\\ \vdots & \vdots & & & & \\ \lambda_{\mu+j} & \lambda_{\mu+j-1} & & & \cdots & \lambda_{\mu+1}\end{bmatrix} \tag{2.4}$$

and μ is least B for which $\lambda_B\neq 0$

$$\lambda_B=(-1)^B\sum_{k_1+\cdots+k_{r+1}=B}\Delta(k_1,\ldots,k_{r+1})\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_{r+1}^{k_{r+1}}}{k_{r+1}!} \tag{2.5}$$

where

$$\Delta(k_1,\ldots,k_{r+1})=\phi_1(k_1)\cdots\phi_{r+1}(k_{r+1})\psi(k_1,\ldots,k_{r+1}) \tag{2.6}$$

$$\psi(k_1,\ldots,k_{r+1})=\Gamma(\rho+k_1+\cdots+k_{r+1})\times\left\{\prod_{j=1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}k_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}k_i\right)\right\}^{-1} \tag{2.7}$$

$$\phi_i(k_i)=\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}k_i)\left\{\prod_{j=n_i+1}^{p_i}\Gamma(c_j-\gamma_j^{(i)}k_i)\prod_{j=1}^{q_i}\Gamma(1-d_j+\delta_j^{(i)}k_i)\right\}^{-1}\quad (i=1,\ldots,r) \tag{2.8}$$

and

$$\phi_{r+1}(k_{r+1})=\begin{cases}(-N)_{Mk_{r+1}}A_{N,k_{r+1}}, & 0\leqslant k_{r+1}\leqslant\left[\dfrac{N}{M}\right]\\[2ex] 0\quad , & k_{r+1}>\left[\dfrac{N}{M}\right]\end{cases} \tag{2.9}$$

Proof. To solve the convolution integral equation (2.1) we first take the Laplace transform of its both sides. We easily obtain by the definition of Laplace transform and its convolution property stated in (1.8), the following result

$$\left[\int_0^\infty e^{-sx}x^{\rho-1}S_N^M[(-z_{r+1})x]H\begin{bmatrix}z_1x\\ \vdots\\ z_rx\end{bmatrix}\mathrm{d}x\ \bar{f}(s)=\bar{g}(s). \tag{2.10}$$

Now expressing the $S_N^M[(-z_{r+1})x]$ and $H\begin{bmatrix} z_1 x \\ \vdots \\ z_r x \end{bmatrix}$ involved in (2.10) in series using (1.1) and (1.3), changing the order of series and integration and evaluating the x-integral, we obtain

$$\left[\sum_{k_1,\ldots,k_{r+1}=0}^{\infty} \Delta(k_1,\ldots,k_{r+1})\frac{(-z_1)^{k_1}}{k_1!}\cdots\frac{(-z_{r+1})^{k_{r+1}}}{k_{r+1}!} \times s^{-\rho-(k_1+\cdots+k_{r+1})}\right]\bar{f}(s) = \bar{g}(s) \quad (2.11)$$

where $\Delta(k_1,\ldots,k_{r+1})$ is defined by (2.6). Now making use of the known formula [10, p. 67, eq. (2.3)], we easily obtain from (2.11)

$$\left[\sum_{B=0}^{\infty} \lambda_B s^{-B}\right] s^{-\rho}\bar{f}(s) = \bar{g}(s) \quad (2.12)$$

where λ_B is defined by (2.5).

Again, (2.12) is equivalent to

$$\bar{f}(s) = s^{\rho}\left[\sum_{B=0}^{\infty} \lambda_B s^{-B}\right]^{-1} \bar{g}(s). \quad (2.13)$$

If μ denotes the least B for which $\lambda_B \neq 0$, the series given by (2.13) can be reciprocated. Writing

$$\left[\sum_{B=0}^{\infty} \lambda_{B+\mu} s^{-B}\right]^{-1} = \sum_{j=0}^{\infty} E_j s^{-j} \quad (2.14)$$

eq. (2.13) takes the following form:

$$\bar{f}(s) = s^{\rho-l+u}\sum_{j=0}^{\infty} E_j s^{-j}[s^l \bar{g}(s)]. \quad (2.15)$$

(2.15) can be written as

$$L\{f(x);s\} = L\left\{\sum_{j=0}^{\infty} E_j \frac{x^{j+l-\mu-\rho-1}}{\Gamma(j+l-\mu-\rho)};s\right\} L\{g^{(l)}(x);s\} \quad (2.16)$$

[on using (1.6)].

Now using the convolution theorem in the RHS of (2.16) we get

$$L\{f(x);s\} = L\left\{\int_0^x \sum_{j=0}^{\infty} \frac{E_j(x-u)^{j+l-\rho-\mu-1}}{\Gamma(j+l-\rho-\mu)} g^{(l)}(u)\,\mathrm{d}u;s\right\}. \quad (2.17)$$

Finally, on taking the inverse of the Laplace transform of both sides of (2.17) we arrive at the desired result (2.2).

3. Special cases

If we put $r=2$ in (2.1) and reduce the H-function of two variables thus obtained to Appell's function F_3 [9, p. 89, eq. (6.4.6)] we find after a little simplification that the convolution equation given by

$$\int_0^x (x-u)^{\rho-1} S_N^M[-z_3(x-u)] F_3[c_1^{(1)}, c_1^{(2)}, c_2^{(1)}, c_2^{(2)}; b; -z_1(x-u), -z_2(x-u)] \times f(u)\,\mathrm{d}u = g(x) \tag{3.1}$$

has the solution

$$f(x) = \frac{\Gamma(c_1^{(1)})\Gamma(c_1^{(2)})\Gamma(c_2^{(1)})\Gamma(c_2^{(2)})}{\Gamma(b)} \int_0^x (x-u)^{l-\rho-\mu-1} \sum_{j=0}^{\infty} \frac{E_j (x-u)^j}{\Gamma(j+l-\rho-\mu)} g^{(l)}(u)\,\mathrm{d}u \tag{3.2}$$

where $\mathrm{Re}(l-\rho-\mu) > 0$, $\mathrm{Re}(\rho) > 0$, $|z_1(x-u)| < 1$, $|z_2(x-u)| < 1$, $g^{(i)}(0) = 0$ $(i = 0, 1, \ldots, l-1)$, l being a positive integer and E_j are given by recurrence relation (2.9) or (2.4) and μ is least B for which $\lambda_B \neq 0$

$$\lambda_B = (-1)^B \sum_{k_1+k_2+k_3=B} \Delta(k_1, k_2, k_3) \frac{z_1^{k_1}}{k_1!} \frac{z_2^{k_2}}{k_2!} \frac{z_3^{k_3}}{k_3!} \tag{3.3}$$

where in (3.3)

$$\Delta(k_1, k_2, k_3) = \frac{\Gamma(c_1^{(1)}+k_1)\Gamma(c_1^{(2)}+k_2)\Gamma(c_2^{(1)}+k_1)\Gamma(c_2^{(2)}+k_2)\Gamma(\rho+k_1+k_2+k_3)}{\Gamma(b+k_1+k_2)} \times \phi_3(k_3) \tag{3.4}$$

and

$$\phi_3(k_3) = \begin{cases} (-N)_{Mk_3} A_{N,k_3}, & 0 \leqslant k_3 \leqslant \left[\dfrac{N}{M}\right] \\ 0, & k_3 > \left[\dfrac{N}{M}\right] \end{cases} \tag{3.5}$$

In the main result if we take $N = 0$ (the polynomial S_0^M will reduce to $A_{0,0}$ which can be taken to be unity without loss of generality), we arrive at a result given by Srivastava *et al* [10, p. 64, eq. (1.1)].

Again, if we put $r = 1$, $p = q = 0$, $z_2 = -1$ in the main result, and further reduce the Fox's H-function thus obtained to $\exp(-z_1)$ [9, p. 18, eq. (2.6.2)] and let $z_1 \to 0$, the Fox's H-function reduces to unity and we arrive at a result which in essence is the same as that given by Rashmi Jain [3, pp. 102–103, eqs (3.5), (3.6)].

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