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Differential subordination and Bazilevič functions

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Abstract. Let $M(z)=z^n+\cdots, N(z)=z^n+\cdots$ be analytic in the unit disc Δ and let $\lambda(z)=N(z)/zN'(z)$. The classical result of Sakaguchi–Libera shows that $\mathrm{Re}(M'(z)/N'(z))>0$ implies $\mathrm{Re}(M(z)/N(z))>0$ in Δ whenever $\mathrm{Re}(\lambda(z))>0$ in Δ. This can be expressed in terms of differential subordination as follows: for any p analytic in Δ, with $p(0)=1$,

$$p(z)+\lambda(z)zp'(z)\prec\frac{1+z}{1-z}\quad \text{implies } p(z)\prec\frac{1+z}{1-z},\quad \text{for } \mathrm{Re}\,\lambda(z)>0,\quad z\in\Delta.$$

In this paper we determine different type of general conditions on $\lambda(z)$, $h(z)$ and $\phi(z)$ for which one has

$$p(z)+\lambda(z)zp'(z)\prec h(z)\quad \text{implies } p(z)\prec\phi(z)\prec h(z),\quad z\in\Delta.$$

Then we apply the above implication to obtain new theorems for some classes of normalized analytic funotions. In particular we give a sufficient condition for an analytic function to be starlike in Δ.

Keywords. Differential subordination; univalent; starlike and convex functions.

1. Introduction

Let f and g be analytic in the unit disc Δ. The function f is *subordinate* to g, written $f\prec g$, or $f(z)\prec g(z)$, if g is univalent, $f(0)=g(0)$ and $f(\Delta)\subset g(\Delta)$. Define $\mathscr{A}=\{f: f(0)=f'(0)-1=0\}$, $\mathscr{A}_k=\{f: f(z)=z+a_{k+1}z^{k+1}+\cdots\}$, and $\mathscr{A}'=\{f: f(0)=1\}$. Let $\lambda(z)$ be a function defined on Δ with $\mathrm{Re}\lambda(z)>\eta>0$, $z\in\Delta$ and let $p\in\mathscr{A}'$. Then a recent paper [8, Theorem 1] establishes the following:

$$\mathrm{Re}[p(z)+\lambda(z)zp'(z)]>\beta \text{ implies } \mathrm{Re}\,p(z)>\frac{2\beta+\eta}{2+\eta},\quad \text{for } z\in\Delta. \tag{1}$$

Let μ and λ satisfy $|\mathrm{Im}\,\mu(z)|\leqslant \mathrm{Re}\,\lambda(z)$, $z\in\Delta$ and let $p\in\mathscr{A}'$. Then a result of Miller and Mocanu [5, Theorem 8] shows that

$$\mathrm{Re}[\mu(z)p(z)+\lambda(z)zp'(z)]>0 \text{ implies } \mathrm{Re}\,p(z)>0,\quad \text{for } z\in\Delta. \tag{2}$$

(2) is equivalent to (1) if we take $\mu(z)=1$ in (2) and $\beta=\eta=0$ in (1).

Let M and N be analytic in Δ, with $M'(0)/N'(0)=1$ and let β be real. If N maps Δ onto a multisheeted starlike domain with respect to the origin, then from [4, Theorem 10] we get

$$\operatorname{Re}\frac{M'(z)}{N'(z)}<\beta \text{ (or } >\beta \text{ resp.) implies } \operatorname{Re}\frac{M(z)}{N(z)}<\beta \text{ (or } >\beta \text{ resp.)}, \quad \text{for } z\in\Delta. \tag{3}$$

A well-known condition for a function $p\in\mathscr{A}$ subordinate to q is that [6]

$$p(z)+\frac{zp'(z)}{p(z)}\prec q(z)+\frac{zq'(z)}{q(z)},$$

under some conditions on $q(z)$. Suppose we let $p(z)=zf'(z)/f(z)$ and $q(z)=2(1+z)/(2-z)$, then we get

$$\operatorname{Re}\left[\frac{(zf'(z))'}{f'(z)}\right]<\frac{3}{2} \text{ implies } \left|\frac{zf'(z)}{f(z)}-\frac{2}{3}\right|<\frac{2}{3}, \quad \text{for } z\in\Delta.$$

Similarly it follows from a result of Mocanu *et al* [7] that for $p\in\mathscr{A}'$,

$$\operatorname{Re}[p(z)+zp'(z)]>0 \text{ implies } |\arg p(z)|<0<\pi/3, \quad \text{for } z\in\Delta,$$

where θ lies between 0.911621904 and 0.911621907. This improves the relation (2) whenever $\mu(z)=\lambda(z)=1$ for $z\in\Delta$.

However the example $M(z)=zf'(z)$, $N(z)=f(z)$ and $\beta=3/2$ [or $M(z)=zf'(z)$, $N(z)=z$ and $\beta=0$ resp.] in (3) suggests that there may exist some conditions on M and N so that

$$\operatorname{Re}\left[(1-\alpha)\frac{M(z)}{N(z)}+\alpha\frac{M'(z)}{N'(z)}\right]\begin{cases}<\beta_1\\ >\beta_2\end{cases} \text{implies } \frac{M(z)}{N(z)}\prec\begin{cases}h_1(z)\\ h_2(z)\end{cases}, \quad \text{for } z\in\Delta \tag{4}$$

for some h_i $(i=1,2)$ to be specified.

Thus it is interesting to ask whether there exist such conditions for our implication. By writing (4) in terms of differential subordination, in this article we determine some new sufficient conditions on $\lambda(z)$, β_i and $h_i(z)$, $(i=1,2)$ for $\operatorname{Re}[p(z)+\lambda(z)zp'(z)]>\beta_i$ to imply $p(z)$ is subordinate to $h_i(z)$. Some interesting applications of this are given. In particular they improve the previous works of different authors [1, 8, 9, 12].

All of the inequalities in this article involving functions of z, such as (2), hold uniformly in the unit disc Δ. So the condition 'for $z\in\Delta$' will be omitted in the remaining part of the paper.

2. Preliminaries

Let $f\in\mathscr{A}$ and $S^*=\{f\in\mathscr{A}: f(\Delta) \text{ is starlike}\}$. Then for $\gamma>0$ and $\beta<1$, we say $f\in B(\gamma,\beta)$ if, and only if, there exists $g\in S^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)^{1-\gamma}g(z)^{\gamma}}\right)>\beta,$$

where all powers are chosen as principal ones.

Denote by $B_1(\gamma,\beta)$, the subclass consisting of those functions in $B(\gamma,\beta)$ for which $g\in S^*$ can be taken as the identity map on Δ. As usual we let $B_1(1,\beta)=R(\beta)$ and $B_1(0,\beta)=S^*(\beta)$. From (1), for $0\leqslant\beta<1$, $\gamma>0$ and for $f\in B_1(\gamma,\beta)$, we easily have

$$\mathrm{Re}\left(\frac{f(z)}{z}\right)^{\gamma}>\frac{2\beta\gamma+1}{2\gamma+1}. \tag{5}$$

In Lemma 1 of section 3 below, we obtain a more general result which improves the above inequality. Lemma 1 has been used in [9] to obtain new sufficient conditions for starlikeness.

We use the following two lemmas in our proofs.

Lemma A. [5] *Let F be analytic in Δ and let G be analytic and univalent on $\bar{\Delta}$, with $F(0)=G(0)$. If F is not subordinate to G, then there exist points $z_0\in\Delta$ and $\zeta_0\in\partial\Delta$, and $m\geqslant 1$ for which $F(|z|<|z_0|)\subset G(|z|<|z_0|)$, $F(z_0)=G(\zeta_0)$, and $z_0F'(z_0)=m\zeta_0G'(\zeta_0)$.*

Lemma B. [5,6] *Let $\Omega\subset\mathbb{C}$ and let q be analytic and univalent on $\bar{\Delta}$ except for those $\zeta\in\partial\Delta$ for which $\mathrm{Lt}_{z\to\zeta}q(z)=\infty$. Suppose that $\psi\colon\mathbb{C}^2\times\Delta\to\mathbb{C}$ satisfies the condition*

$$\psi(q(\zeta),m\zeta q'(\zeta);z)\notin\Omega \tag{6}$$

when $q(z)$ is finite, $m\geqslant k\geqslant 1$ and $|\zeta|=1$. If p and q are analytic in Δ, $p(z)=p(0)+p_kz^k+\cdots$, $p(0)=q(0)$, and further if

$$\psi(p(z),zp'(z);z)\in\Omega$$

then $p(z)\prec q(z)$ in Δ.

Suppose that $p\in\mathscr{A}'$ with $p(z)=1+p_kz^k+\cdots$, and $q(z)=(1+z)/(1-z)$. Then the condition (6) reduces to

$$\psi(ix,y;z)\notin\Omega \tag{7}$$

when x is real and $y\leqslant -k(1+x^2)/2$. Except for Theorems 5 and 6, we, in our results, consider the situations where $k=1$.

3. Main results

We now state and prove our main results.

Lemma 1. *Let $p\in\mathscr{A}'$, $\alpha\in\mathbb{C}$ with $\mathrm{Re}\,\alpha\geqslant 0$ $(\alpha\neq 0)$, $\beta<1$ be such that*

$$\mathrm{Re}\{p(z)+\alpha zp'(z)\}>\beta, \tag{8}$$

then

$$\mathrm{Re}\,p(z)>\beta+(1-\beta)[2\delta-1], \tag{9}$$

where

$$\delta=\delta(\mathrm{Re}\,\alpha)=\int_0^1\frac{\mathrm{d}t}{1+t^{\mathrm{Re}\,\alpha}} \tag{10}$$

and $\delta(\mathrm{Re}\,\alpha)$ is an increasing function of $\mathrm{Re}\,\alpha$ with $(1+\mathrm{Re}\,\alpha)/(1+2\mathrm{Re}\,\alpha)\leqslant\delta<1$. The estimate cannot be improved in general.

Proof. We use the well-known result of Hallenbeck an Ruscheweyh [2], namely,

$$p(z)+\alpha zp'(z)\prec h(z) \text{ implies } p(z)\prec \frac{1}{\alpha}z^{-1/\alpha}\int_0^z h(t)t^{1/\alpha-1}\,\mathrm{d}t \tag{11}$$

for $p\in\mathscr{A}'$ and h a convex(univalent) function with $h(0)=1$. If we let

$$h(z)=2\beta-1+\frac{2(1-\beta)}{1-z}$$

then h is convex and univalent on Δ, $h(0)=1$ and $\operatorname{Re} h(z)>\beta$. For this choice of h, the condition that (8) implies – in fact is equivalent to –

$$p(z)+\alpha zp'(z)\prec h(z).$$

Therefore from a straightforward calculation, Inequality (8) implies

$$p(z)\prec 2\beta-1+2(1-\beta)\phi(z) \tag{12}$$

where ϕ defined by

$$\phi(z)=1+\sum_{n=1}^{\infty}\frac{z^n}{n\alpha+1}=\int_0^1\frac{\mathrm{d}t}{1-zt^\alpha}$$

is convex in Δ.

Let

$$W=\frac{1}{1-zt^\alpha},\quad z\in\Delta$$

so that

$$1-\frac{1}{W}=zt^\alpha.$$

Then, for $|z|=r$ and $0\leqslant t\leqslant 1$, we have

$$\left|1-\frac{1}{W}\right|\leqslant rt^{\operatorname{Re}\alpha}.$$

This implies that

$$\left|W-\frac{1}{1-r^2t^{2\operatorname{Re}\alpha}}\right|\leqslant\frac{rt^{\operatorname{Re}\alpha}}{1-r^2t^{2\operatorname{Re}\alpha}}$$

and so

$$\frac{1}{1+rt^{\operatorname{Re}\alpha}}\leqslant\operatorname{Re}W\leqslant\frac{1}{1-rt^{\operatorname{Re}\alpha}}.$$

(Note that if $\operatorname{Re}\alpha<0$, $rt^{\operatorname{Re}\alpha}$ need not be less than one and the above will not work.)

Therefore, we have

$$\operatorname{Re}\phi(z)\geqslant K(r)=\int_0^1\frac{\mathrm{d}t}{1+rt^{\operatorname{Re}\alpha}},\quad \text{for } |z|=r,\quad 0<r<1.$$

Observe that the series $K(r)$ is absolutely convergent for $0<r<1$. Suitably rearranging the pairs of terms in $K(r)$ it can be shown that $1/2 \leqslant K(r) < 1$.

In particular for $r \to 1^-$ the above inequality reduces to

$$\operatorname{Re}\phi(z) \geqslant K(r) > K(1) = \delta(\operatorname{Re}\alpha),$$

where δ is as in (10).

Next we show that δ satisfies the inequality $(1+\operatorname{Re}\alpha)/(1+2\operatorname{Re}\alpha) \leqslant \delta < 1$. Since $2\beta - 1 + 2(1-\beta)\phi(z)$ is the best dominant for (8), we obtain taking $\lambda(z) = \alpha$, with $\operatorname{Re}\alpha > \eta$ in (1),

$$\operatorname{Re}\{2\beta - 1 + 2(1-\beta)\phi(z)\} > 2\beta - 1 + 2(1-\beta)K(1) \geqslant \frac{2\beta+\eta}{2+\eta},$$

$$\text{i.e., } \operatorname{Re}\phi(z) > K(1) \geqslant \frac{1+\eta}{2+\eta}.$$

Thus making $\eta \to \operatorname{Re}\alpha^+$, we get

$$\operatorname{Re}\phi(z) > K(1) = \delta(\operatorname{Re}\alpha) \geqslant \frac{1+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}.$$

This from (12) proves (9).

To complete the proof we need only to show that the bound in (9) cannot be improved in general. For this we let

$$q(z) = 2\beta - 1 + 2(1-\beta)\int_0^1 \frac{\mathrm{d}t}{1-zt^{\alpha}}.$$

Then q is the best dominant for (8), because it satisfies the differential equation

$$q(z) + \alpha z q'(z) - 2\beta - 1 + \frac{2(1-\beta)}{1-z}h(z).$$

Therefore the function $q(z)$ shows that the bound in (9) cannot be improved. □□

Remark. In fact the second assertion, namely,

$$\frac{1+\operatorname{Re}\alpha}{1+2\operatorname{Re}\alpha} \leq \delta,$$

can be seen directly. If $\operatorname{Re}\alpha \geq 1$, then

$$\begin{aligned}\delta(\operatorname{Re}\alpha) &= \int_0^1 \left[1 - t^{\operatorname{Re}\alpha} + \frac{t^{2\operatorname{Re}\alpha}}{1+t^{\operatorname{Re}\alpha}}\right]\mathrm{d}t \\ &> \frac{\operatorname{Re}\alpha}{1+\operatorname{Re}\alpha} + \frac{1}{2(1+2\operatorname{Re}\alpha)} \\ &\geq \frac{1+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}.\end{aligned}$$

Similarly if $0 \leqslant \operatorname{Re}\alpha < 1$, then

$$\begin{aligned}\delta(\operatorname{Re}\alpha) &= \frac{1}{2} + \frac{1}{2}\int_0^1 \left(\frac{1-t^{\operatorname{Re}\alpha}}{1+t^{\operatorname{Re}\alpha}}\right) \mathrm{d}t \\ &= \frac{1}{2} + \frac{1}{2}\int_0^1 (1-t^{\operatorname{Re}\alpha})\left[\frac{1}{2} + \frac{1-t^{\operatorname{Re}\alpha}}{2(1+t^{\operatorname{Re}\alpha})}\right] \mathrm{d}t \\ &\geqslant \frac{1}{2} + \frac{\operatorname{Re}\alpha}{4(1+\operatorname{Re}\alpha)} + \frac{(\operatorname{Re}\alpha)^2}{4(1+\operatorname{Re}\alpha)(1+2\operatorname{Re}\alpha)} \\ &\geqslant \frac{1}{2} + \frac{\operatorname{Re}\alpha}{2(2+\operatorname{Re}\alpha)} = \frac{1+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}.\end{aligned}$$

Using Lemma 1 in particular for $\operatorname{Re}\alpha \to 0$, $\alpha \neq 0$, $\beta = 0$, one has

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} > 0 \text{ implies } \operatorname{Re} p(z) > 0.$$

In the next result we improve this relation by showing that the same conclusion may be obtained under a weaker hypothesis on p.

Theorem 1. *Let α be a purely imaginary number, i.e., $\alpha = i\alpha_2$, α_2 real. Let Q be the unique-function that maps Δ onto the complement of the ray $\{it: t \leqslant 2^{-1}(\alpha_2^{-1} - \alpha_2)\}$ whenever $\alpha_2 > 0$ ($\{it: t \geqslant 2^{-1}(\alpha_2^{-1} - \alpha_2)\}$ whenever $\alpha_2 < 0$). If $p \in \mathscr{A}'$ satisfies*

$$p(z) + \alpha z p'(z) \prec Q(z)$$

then $\operatorname{Re} p(z) > 0$.

Proof. If we let $\psi(r,s) = r + \alpha s$, then $\psi(p(z), zp'(z))$ is analytic in Δ and the above subordination becomes

$$\psi(p(z), zp'(z)) \prec Q(z).$$

The conclusion of the theorem will follow from Lemma B and (7) if we can show that $\psi(ix, y) \notin Q(\Delta)$ when $y \leqslant -(1+x^2)/2$ and x-real. Suppose that $\alpha = i\alpha_2$, with $\alpha_2 > 0$, then $\psi(ix, y) = i(x + \alpha_2 y)$ and

$$x + \alpha_2 y \leqslant x - \alpha_2(1+x^2)/2 \leqslant 2^{-1}[\alpha_2^{-1} - \alpha_2]$$

for all x-real.

This shows that for $\alpha_2 > 0$, $\psi(ix, y) \notin Q(\Delta)$. A similar conclusion holds for the case $\alpha_2 < 0$. Hence the theorem. □

However the special case of the following lemma improves the conclusion of Lemma 1 further at least for $\alpha \in \mathbb{C}$ such that $|\operatorname{Im}\alpha| \leqslant \sqrt{3}(\operatorname{Re}\alpha - \eta)$ for a suitable fixed $\eta > 0$.

Lemma 2. *Let λ be a function defined on Δ satisfying*

$$|\arg(\lambda(z) - \eta)| < \pi/3 \tag{13}$$

such that

$$\eta = \inf_{z\in\Delta}\left(\operatorname{Re}\lambda(z) - \frac{|\operatorname{Im}\lambda(z)|}{\sqrt{3}}\right) \quad (>0) \tag{14}$$

and let

$$\beta'(\eta) = \left(\frac{6+5\eta^2+2\sqrt{9+15\eta^2}}{25\eta^2}\right)^{1/3}\left[\frac{9-2\sqrt{9+15\eta^2}}{10}\right] \tag{15}$$

be such that $2\beta'(\eta)+\eta \geqslant 0$. *If* $p\in\mathscr{A}'$ *satisfies*

$$\operatorname{Re}[p(z)+\lambda(z)zp'(z)] > \beta'(\eta) \tag{16}$$

then $|\arg p(z)| < \pi/3$.

Proof. Note that $\beta'(\eta) \leqslant 0$ if, and only if, $\eta \geqslant \sqrt{3}/2$. Now using (1), (16) implies

$$\operatorname{Re} p(z) > \frac{2\beta'(\eta)+\eta}{2+\eta}.$$

Since $2\beta'(\eta)+\eta \geqslant 0$, this Inequality further implies $\operatorname{Re} p(z) > 0$ in Δ.

If we let $\Omega = \{\omega\in\mathbb{C}:\operatorname{Re}\omega > \beta'(\eta)\}$ and $q(z) = [(1+z)/(1-z)]^{2/3}$, then $q(\Delta)$ equals $\{\omega\in\mathbb{C}: |\arg\omega| < \pi/3\}$. Then for $\psi(r,s;z) = r+\lambda(z)s$, (16) can be rewritten as

$$\{\psi(p(z), zp'(z); z): |z|<1\} \subset \Omega.$$

So to prove the lemma we need only to show that $p \prec q$.

If p is not subordinate to q, then by Lemma A there exist points $z_0\in\Delta$ and $\zeta_0\in\partial\Delta$, and $m \geqslant 1$ such that

$$p(|z|<|z_0|) \subset q(\Delta), \quad p(z_0) = q(\zeta_0) \text{ and } z_0p'(z_0) = m\zeta_0 q'(\zeta_0).$$

We first discuss the case $p(z_0) \neq 0$ which corresponds to a point on one of the rays on the sector $q(\Delta)$. Since $p(z_0)\neq 0$, $\zeta_0 \neq \pm 1$. Next by letting X and Y be the real and imaginary parts of $\lambda(z_0)$, respectively, from (13) and (14), we find that

$$\left.\begin{matrix} X+Y/\sqrt{3} \\ X-Y/\sqrt{3} \end{matrix}\right\} \geqslant X - |Y|/\sqrt{3} \geqslant \eta > 0. \tag{17}$$

Further if we set $ix = (1+\zeta_0)/(1-\zeta_0)$ and use the above observations, we obtain

$$\psi(p(z_0), z_0p'(z_0); z_0) = (ix)^{2/3}\left[1 + im(X+iY)\frac{1+x^2}{3x}\right].$$

For $x \neq 0$,

$$\operatorname{Re}\psi(p(z_0), z_0p'(z_0); z_0) = \operatorname{Re}\begin{cases} |x|^{2/3}\left(\dfrac{1+i\sqrt{3}}{2}\right)\left(1 - \dfrac{m(Y-iX)(1+x^2)}{3|x|}\right), & \text{if } x>0 \\ |x|^{2/3}\left(\dfrac{1-i\sqrt{3}}{2}\right)\left(1 + \dfrac{m(Y-iX)(1+x^2)}{3|x|}\right), & \text{if } x<0 \end{cases}$$

$$
=\begin{cases} |x|^{2/3}\left[1-\dfrac{m(1+x^2)}{\sqrt{3}|x|}\left(X+\dfrac{Y}{\sqrt{3}}\right)\right]\Big/2, & \text{if } x>0 \\ |x|^{2/3}\left[1-\dfrac{m(1+x^2)}{\sqrt{3}|x|}\left(X-\dfrac{Y}{\sqrt{3}}\right)\right]\Big/2, & \text{if } x<0. \end{cases}
$$

Therefore, for $x\neq 0$, since $\lambda(z_0)$ satisfies (17) and $m\geqslant 1$, we obtain

$$
\operatorname{Re}\psi(p(z_0), z_0p'(z_0); z_0)\leqslant |x|^{2/3}\left[1-\frac{\eta}{\sqrt{3}}\left(|x|+\frac{1}{|x|}\right)\right]\frac{1}{2}=f(|x|),
$$

where

$$
f(t)=t^{2/3}\left[1-\frac{\eta(t^2+1)}{\sqrt{3}t}\right]\frac{1}{2}, \quad \text{with } t=|x|.
$$

Since

$$
t_0=\frac{\sqrt{3}+\sqrt{3+5\eta^2}}{5\eta}
$$

is the maximum for $f(t)$, we have

$$
\operatorname{Re}\psi(p(z_0), z_0p'(z_0); z_0)\leqslant f(|x|)\leqslant f(t_0)\equiv\beta'(\eta).
$$

This implies that $\psi(p(z_0), z_0p'(z_0); z_0)$ lies outside Ω, contradicting (16). Hence we must have $p\prec q$ when $p(z_0)\neq 0$.

Now consider the case $p(z_0)=0$ which corresponds to the corner of the sector $q(\Delta)$. Observe that the sector angle of $q(\Delta)$ is $2\pi/3$ and so $p(|z|=|z_0|)$ cannot pass through such a corner without itself having a corner and hence the case $p(z_0)=0$ cannot occur for the present form of our lemma. This completes the proof. □

Lemmas 1 and 2 yield improvements on most of the results of [8]. As an equivalent form of Lemma 2 we state

Theorem 2. *Let $\beta'(\eta)$ be as defined by (15) so that $2\beta'(\eta)+\eta\geqslant 0$. Let $M(z)=z^n+\cdots$ and $N(z)=z^n+\cdots$ be analytic in Δ and such that for some $\alpha\in\mathbb{C}$, N satisfies*

$$
\left|\operatorname{Im}\frac{\alpha N(z)}{zN'(z)}\right|\leqslant\sqrt{3}\left(\operatorname{Re}\frac{\alpha N(z)}{zN'(z)}-\eta\right), \quad (0<\eta\leqslant\operatorname{Re}\alpha-|\operatorname{Im}\alpha|/\sqrt{3}).
$$

Then

$$
\operatorname{Re}\left[(1-\alpha)\frac{M(z)}{N(z)}+\alpha\frac{M'(z)}{N'(z)}\right]>\beta'(\eta) \text{ implies } \left|\arg\frac{M(z)}{N(z)}\right|<\frac{\pi}{3}.
$$

Proof. Consider the function $p(z)=M(z)/N(z)$ and let $\lambda(z)=\alpha N(z)/zN'(z)$. Then by hypothesis, $p\in\mathscr{A}'$ and all the conditions of Lemma 2 are satisfied. Now it is elementary to show that

$$
(1-\alpha)\frac{M(z)}{N(z)}+\alpha\frac{M'(z)}{N'(z)}=p(z)+\lambda(z)zp'(z)
$$

and hence Theorem 2 follows from Lemma 2.

COROLLARY 1.

If $p \in \mathscr{A}'$ and if λ is a function defined on Δ such that

$$|\operatorname{Im} \lambda(z)| \leqslant \sqrt{3}(\operatorname{Re} \lambda(z) - \sqrt{3}/2)$$

then

$$\operatorname{Re}\{p(z) + \lambda(z) z p'(z)\} > 0 \text{ implies } |\arg p(z)| < \pi/3.$$

Proof. If we let $\eta = \sqrt{3}/2$ then in this case $\beta'(\eta) = 0$ in (15) and the corollary now follows from Lemma 2. □

COROLLARY 2.

Let $f \in B_1(\gamma, 0)$. Then we have

(i) $\operatorname{Re}\left(\dfrac{f(z)}{z}\right)^{\gamma} > 2\delta(1/\gamma) - 1, \quad$ *for* $\gamma > 0$.

(ii) $\left|\arg\left(\dfrac{f(z)}{z}\right)^{\gamma}\right| < \pi/3, \quad$ *for* $0 < \gamma \leqslant 2/\sqrt{3}$.

For the function F defined by $F^{\gamma}(z) = \dfrac{\gamma + c}{z^c}\displaystyle\int_0^z t^{c-1} f^{\gamma}(t)\,\mathrm{d}t$, *we have*

(iii) $\operatorname{Re} F'(z)\left(\dfrac{F(z)}{z}\right)^{\gamma - 1} > 2\delta(1/(\gamma + c)) - 1, \quad$ *for γ and c real with* $0 < \gamma + c$.

(iv) $\left|\arg F'(z)\left(\dfrac{F(z)}{z}\right)^{\gamma - 1}\right| < \pi/3, \quad$ *for γ and c such that* $0 < \gamma + c \leqslant 2/\sqrt{3}$.

Proof. Proofs of the above inequalities follow from Lemma 1 and Lemma 2 using the techniques of [8]. □

Theorem 3. *Let $\eta > 0$ be such that*

$$\beta'(\eta) + \frac{2\delta(\eta) - 1}{2(1 - \delta(\eta))} \geqslant 0. \tag{18}$$

where $\beta'(\eta)$ is as defined in (15). *Let $p \in \mathscr{A}'$ and $\alpha \geqslant \eta$. Suppose*

$$\operatorname{Re}[p(z) + \alpha z p'(z)] > \frac{\beta'(\eta) - (1 - (\eta/\alpha))(2\delta(\alpha) - 1)}{1 - (1 - (\eta/\alpha))(2\delta(\alpha) - 1)}. \tag{19}$$

Then we have

$$\operatorname{Re}[p(z) + \eta z p'(z)] > \beta'(\eta), \; |\arg p(z)| < \pi/3$$

and

$$\operatorname{Re} p(z) > 2(1 - \delta(\eta))\beta'(\eta) + 2\delta(\eta) - 1.$$

Proof. Observe that

$$p(z) + \eta z p'(z) = \left(1 - \frac{\eta}{\alpha}\right) p(z) + \frac{\eta}{\alpha}[p(z) + \alpha z p'(z)], \quad (\alpha \geqslant \eta).$$

Now Lemmas 1 and (19) yield

$$\mathrm{Re}[p(z)+\eta zp'(z)]>\beta'(\eta). \tag{20}$$

Taking $\lambda(z)=\eta$ in Lemma 2 and $\alpha=\eta$ in Lemma 1, respectively, the theorem follows.

□

We note that, using Lemma 1 and Theorem 2, we can construct several new examples. The result even for the special case $\alpha\geqslant\sqrt{3}/2$ where $\beta'(\sqrt{3}/2)=0$ could not be found in the literature.

For $f\in\mathscr{A}$ and $\alpha\in\mathbb{C}$ with $|\mathrm{Im}\,\alpha|\leqslant\sqrt{3}(\mathrm{Re}\,\alpha-\sqrt{3}/2)$, we have

$$\mathrm{Re}\{f'(z)+\alpha zf''(z)\}>0$$

implies

$$|\arg f'(z)|<\pi/3 \text{ and } \mathrm{Re}\,f'(z)>2\delta(\mathrm{Re}\,\alpha)-1.$$

We can use Corollary 1 to improve the result obtained by Yoshikawa and Yoshikai in [12, Theorem 4] concerning the transformation

$$F(z)=f(z)\exp\left\{-z^{-c}\int_0^z t^{c}\left(\frac{f'(t)}{f(t)}-\frac{1}{t}\right)\mathrm{d}t\right\},\quad \text{for } \mathrm{Re}\,c\geqslant 0, c\neq 0, \tag{21}$$

of the well-known γ-spiral-like functions. His result proves that for $|\gamma|<\pi/2$,

$$\mathrm{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right)>0 \text{ implies } \mathrm{Re}\left(e^{i\gamma}\frac{zF'(z)}{F(z)}\right)>\frac{\mathrm{Re}(1/c)}{2+\mathrm{Re}(1/c)}\cos\gamma.$$

From Corollary 1, with $\lambda(z)=1/c$, we see that we can improve the above implication to

$$\frac{zf'(z)}{f(z)}\prec\frac{1-z}{1+z}$$

implies

$$\left|\arg\frac{zF'(z)}{F(z)}\right|<\pi/3 \text{ whenever } \left|\arg\left(\frac{1}{c}-\frac{\sqrt{3}}{2}\right)\right|<\pi/3;$$

or, equivalently, if $\left|\arg\left(\frac{1}{c}-\frac{\sqrt{3}}{2}\right)\right|<\pi/3$, then

$$e^{i\gamma}\frac{zf'(z)}{f(z)}\prec\frac{e^{i\gamma}-e^{-i\gamma}z}{1+z} \text{ implies } e^{i\gamma}\left[\frac{zF'(z)}{F(z)}\right]^{3/2}\prec\frac{e^{i\gamma}-e^{-i\gamma}z}{1+z}.$$

We next prove the following lemma and then apply this to derive Theorem 4.

Lemma 3. *Let* $\alpha^*\approx 0.407\cdots$ *be the root of the equation*

$$\alpha^*=\tan[(2\pi-3\pi\alpha^*)/6] \tag{22}$$

and $\theta = \alpha^*\pi/2$. *Suppose that* β *is the smallest positive root of the cubic equation*

$$12\beta^3 - [(6-4\sqrt{3})\cos^2\theta + 18 - 4\sqrt{3}]\beta^2 - [(10+4\sqrt{3})\cos^2\theta + 8\sqrt{3} - 13]\beta + (16-4\sqrt{3})\cos^2\theta - (2-\sqrt{3})^2 = 0.$$

Further let $F(z)$ *be a complex function that satisfies*

$$|\arg F(z)| < \alpha^*\pi/2. \tag{23}$$

If $p \in \mathscr{A}'$ *satisfies*

$$\operatorname{Re} F(z)\{\beta + (1-\beta)p(z) + (\sqrt{3}/2)[(\beta + (1-\beta)p(z)^2) + (1-\beta)zp'(z) - (\beta + (1-\beta)p(z))]\} > 0 \tag{24}$$

then $\operatorname{Re} p(z) > 0$ *in* Δ.

Proof. First, we write

$$\psi(r,s;z) = F(z)\{\beta + (1-\beta)r + (\sqrt{3}/2)[(\beta + (1-\beta)r)^2 + (1-\beta)s - (\beta + (1-\beta)r)]\}$$

and $F(z) = X + iY \equiv \operatorname{Re} F(z) + i\operatorname{Im} F(z)$. Let us now apply Lemma B. Then for all x, y reals and $z \in \Delta$, we have

$$\operatorname{Re}\psi(ix, y; z) = X[\beta + (\sqrt{3}/2)(\beta^2 - \beta - (1-\beta^2)x^2 + (1-\beta)y)] - Y(1-\beta)[1 + (\sqrt{3}/2)(2\beta - 1)]x.$$

From this it is easily verified that

$$\operatorname{Re}\psi(ix, y; z) \leqslant -(Rx^2 + Sx + T)$$

for all x real, $y \leqslant -(1+x^2)/2$ and all $z \in \Delta$, where

$$R = \sqrt{3}X(1-\beta)(3-2\beta),\ S = 2Y(1-\beta)(2-\sqrt{3}+2\sqrt{3}\beta), \text{ and}$$

$$T = [\sqrt{3}(1-\beta)(1+2\beta) - 4\beta] = 2\sqrt{3}\left(\frac{4-\sqrt{3}+\sqrt{43-8\sqrt{3}}}{4\sqrt{3}} - \beta\right)\left(\frac{4-\sqrt{3}-\sqrt{43-8\sqrt{3}}}{4\sqrt{3}} + \beta\right).$$

Therefore $\operatorname{Re}\psi(ix, y; z) \leqslant 0$ if, as usual, $Rx^2 + Sx + T \geqslant 0$ for all real x. The second inequality holds if and only if $S^2 \leqslant 4RT$. By performing further algebraic simplifications, it can be easily seen that this is indeed equivalent to

$$|Y| \leqslant (\tan(\alpha^*\pi/2))X, \quad \text{i.e., } |\arg F(z)| < \alpha^*\pi/2,$$

where the required identity to claim this is

$$\tan^2(\alpha^*\pi/2) = \frac{\sqrt{3}(3-2\beta)(\sqrt{3}(1-\beta)(1+2\beta) - 4\beta)}{(1-\beta)(2-\sqrt{3}+2\sqrt{3}\beta)^2}.$$

Since this automatically follows from the hypothesis, the desired conclusion now follows from Lemma B with $\Omega = \{\omega \in \mathbb{C}\colon \operatorname{Re}\omega > 0\}$ and (7). Therefore the proof is complete. □

Theorem 4. *Let $f \in \mathscr{A}$ and β be as stated in Lemma 3. Suppose that for $\alpha \geqslant \sqrt{3}/2$,*

$$\operatorname{Re}[f'(z) + \alpha z f''(z)] > \frac{-(1-(\sqrt{3}/2\alpha))(2\delta(\alpha)-1)}{1-(1-(\sqrt{3}/2\alpha))(2\delta(\alpha)-1)}. \tag{26}$$

This implies $f \in S^*(\beta)$.

Proof. Suppose that f satisfies (26). Then taking $p(z) = f'(z)$ and $\eta = \sqrt{3}/2$ in Theorem 3, we obtain

$$\operatorname{Re}\{f'(z) + (\sqrt{3}/2) z f''(z)\} > 0 \tag{27}$$

and $|\arg f'(z)| < \pi/3$. Thus from [6, Theorem 5] we get

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\alpha^* \pi}{2}$$

where α^* is as in (22).

Now we need only to show that (27) implies $f \in S^*(\beta)$. For this we let

$$p(z) = \left(\frac{zf'(z)}{f(z)} - \beta\right)(1-\beta)^{-1} \text{ and } F(z) = \frac{f(z)}{z}.$$

Then by performing differentiation and some algebraic simplifications, (27) deduces to

$$\operatorname{Re}\psi(p(z), zp'(z); z) > 0$$

where

$$\psi(r,s;z) = F(z)\{\beta + (1-\beta)r + (\sqrt{3}/2)[(\beta + (1-\beta)r)^2 + (1-\beta)s - (\beta + (1-\beta)r)]\}.$$

The theorem now follows from Lemma 3. □

Taking $\alpha = 1$ in the above theorem we obtain the following.

COROLLARY 3.

Let β be as in Lemma 3. If $g \in \mathscr{A}$ satisfies

$$\operatorname{Re} g'(z) > -\frac{(2-\sqrt{3})(2\ln 2 - 1)}{2-(2-\sqrt{3})(2\ln 2 - 1)},$$

then the Alexander Operator $I(g)$ defined by

$$[I(g)](z) = \int_0^z \frac{g(t)}{t}\,\mathrm{d}t,$$

is in $S^*(\beta)$, where β is as in Lemma 3.

Observe that a little computation shows that β is slightly bigger than the value obtained in [9, Corollary 3]. Further the above corollary favours the existence of a family of analytic functions, containing non-univalent functions, mapping onto $S^*(\beta) \subset S^*$ under the Alexander Operator.

Theorem 5. *Let α be a real number with $\alpha < -2/k\delta$. Let $M(z) = z^n + a_{n+k}z^{n+k} + \cdots$ and $N(z) = z^n + \cdots$ be analytic in Δ $(n \geqslant 1, k \geqslant 1)$ and let N satisfy*

$$\mathrm{Re}(N(z)/zN'(z)) > \delta, \quad (0 < \delta < 1/n). \tag{28}$$

If

$$\mathrm{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] < \beta, \tag{29}$$

then

$$\mathrm{Re}\left[\frac{M(z)}{N(z)}\right] > \frac{2\beta + k\delta\alpha}{2 + k\delta\alpha} \text{ and } \mathrm{Re}\left[\frac{M'(z)}{N'(z)}\right] > \frac{\beta(2+k\delta) - k\delta(1-\alpha)}{2 + k\delta\alpha}. \tag{30}$$

Proof. If we let $\Omega = \{\omega \in \mathbb{C}\colon \mathrm{Re}\,\omega < \beta\}$, $\beta_1 = 2\beta + k\delta\alpha/2 + k\delta\alpha$, $\lambda(z) = N(z)/zN'(z)$ and $p(z) = (1-\beta_1)^{-1}(M(z)/N(z)) - \beta_1)$, then $p(z) = 1 + p_k z^k + \cdots$ is analytic in Δ and the condition (29) implies

$$\psi(p(z), zp'(z); z) \in \Omega$$

where $\psi\colon \mathbb{C}^2 \times \Delta \to \mathbb{C}$ is $\psi(r,s;z) = \beta_1 + (1-\beta_1)[r + \alpha\lambda(z)s]$.

Since N satisfies (28), we have $\mathrm{Re}\,\lambda(z) > \delta$ in Δ. If x is real and $y \leqslant -k(1+x^2)/2$ then for this ψ we have

$$\begin{aligned}\psi(ix, y; z) &= \beta_1 + (1-\beta_1)\alpha[\mathrm{Re}\,\lambda(z)]y \\ &\geqslant \beta_1 - [(1-\beta_1)\alpha\delta k]/2 \equiv \beta,\end{aligned}$$

since $\alpha < 0$, i.e. $\psi(ix, y; z) \notin \Omega$. Hence (7) is satisfied and Lemma B leads to $\mathrm{Re}\,p(z) > 0$. This shows the first part of (30). Since $1-\alpha > 0$, this proves $(1-\alpha)\mathrm{Re}(M(z)/N(z)) > (1-\alpha)\beta_1$. Moreover, from this and (29) we easily have the second inequality of (30). Hence the theorem. □

COROLLARY 4.

Let $|\lambda| < 1$ and $f \in \mathscr{A}_k$. (i) *If*

$$\mathrm{Re}\{(1+\lambda z)[(1+\alpha\lambda z)f'(z) + \alpha(1+\lambda z)zf''(z)]\} < \beta \tag{31}$$

then for $k\alpha(1-|\lambda|) < -2$,

$$\mathrm{Re}(1+\lambda z)f'(z) > \frac{2\beta + (1-|\lambda|)\alpha k}{2 + (1-|\lambda|)\alpha k}.$$

(ii) *If*

$$\mathrm{Re}\left\{e^{-\lambda z}\left[\left(1 - \frac{\lambda\alpha z}{1+\lambda z}\right)f'(z) + \frac{\alpha z}{1+\lambda z}f''(z)\right]\right\} < \beta \tag{32}$$

then for $k\alpha + 2(1+|\lambda|) < 0$,

$$\operatorname{Re} e^{-\lambda z} f'(z) > \frac{2\beta(1+|\lambda|)+\alpha k}{2(1+|\lambda|)+\alpha k}.$$

Proof. For the proof of (i) we choose $M(z) = zf'(z)$ and $N(z) = z/(1+\lambda z)$. Then

$$\frac{M'(z)}{N'(z)} = (1+\lambda z)^2[f'(z) + zf''(z)], \quad \frac{M'(0)}{N'(0)} = 1, \quad \frac{N(z)}{zN'(z)} = 1+\lambda z$$

and

$$(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)} = (1+\lambda z)[(1+\alpha\lambda z)f'(z) + \alpha(1+\lambda z)zf''(z)].$$

Since $f \in \mathscr{A}$ satisfies (31), we have

$$\operatorname{Re}\frac{M(z)}{N(z)} > \frac{2\beta + \alpha\delta k}{2+\alpha\delta k}, \quad \text{whenever } \delta < 1-|\lambda|.$$

But δ can be chosen as close to $1-|\lambda|$ as we please and so we can allow $\delta \to 1-|\lambda|$ from below. Thus making $\delta \to 1-|\lambda|$ we establish our claim. The proof for the case (ii) follows on similar lines taking $M(z) = zf'(z)$ and $N(z) = ze^{\lambda z}$. □

Similar arguments used in Theorem 5 would help us to prove the following more general result.

Theorem 6. *Let* α *be a complex number with* $\operatorname{Re}\alpha < -2n/k\delta$. *Let* $M(z) = z^n + a_{n+k}z^{n+k} + \cdots$ *and* $N(z) = z^n + \cdots$ *be analytic in* Δ $(n \geqslant 1, k \geqslant 1)$ *and let* N *satisfy*

$$\operatorname{Re}(\alpha N(z)/zN'(z)) < \delta, \quad (\operatorname{Re}\alpha/n < \delta < -2/k).$$

Then

$$\operatorname{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] < \beta \quad \textit{implies} \quad \operatorname{Re}\left[\frac{M(z)}{N(z)}\right] > \frac{2\beta + k\delta}{2+k\delta}.$$

COROLLARY 5.

Let $\alpha \in \mathbb{C}$ *be such that* $\operatorname{Re}\alpha < -2m/k$, *where m is a positive integer and let* $\beta > 1$. *If* $f \in \mathscr{A}$ *satisfy*

$$\operatorname{Re}\left\{(1-\alpha)\left(\frac{f(z)}{z}\right)^m + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{m-1}\right\} < \beta$$

then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^m > \frac{2\beta m + k\operatorname{Re}\alpha}{2m + k\operatorname{Re}\alpha}.$$

Proof. The corollary follows from Theorem 6 taking $M(z) = (f(z))^m$ and $N(z) = z^m$. □

In the following theorem we generalize the concept of α-close-to-convexity [1] when α is a complex number.

Theorem 7. *Let $M(z)=z^n+\cdots$ and $N(z)=z^n+\cdots$ be analytic in Δ and suppose that N satisfies*

$$\mathrm{Re}(N(z)/zN'(z))>\delta, \quad (0<\delta<1/n). \tag{33}$$

Further let k be a complex number satisfying

$$|\mathrm{Im}\,k|\leqslant\sqrt{D\delta}, \quad 0<D\leqslant(\delta+2\mathrm{Re}\,k). \tag{34}$$

Then

$$\mathrm{Re}\left[(k-1)\frac{M(z)}{N(z)}+\frac{M'(z)}{N'(z)}\right]>\beta, \quad (\mathrm{Re}\,k>\beta)$$

implies

$$\mathrm{Re}\frac{M(z)}{N(z)}>\frac{\delta+2\beta-D}{\delta+2\mathrm{Re}\,k-D}. \tag{35}$$

Proof. If we let $\beta_1=(\delta+2\beta-D)/(\delta+2\mathrm{Re}\,k-D)$ so that $D=[\delta+2\beta-\beta_1(\delta+2\mathrm{Re}\,k)]/(1-\beta_1)$ and define

$$p(z)=(1-\beta_1)^{-1}\left(\frac{M(z)}{N(z)}-\beta_1\right) \tag{36}$$

then $p\in\mathscr{A}'$. From (36) and (35), we obtain, as before, $\mathrm{Re}\,\psi(p(z),zp'(z);z)>0$, where

$$\psi(r,s;z)=-\beta+k\beta+(1-\beta_1)[kr+s(N(z)/zN'(z))].$$

If we can show that $\mathrm{Re}\,\psi(ix,y;z)\leqslant 0$ when $y\leqslant-(1+x^2)/2$ and x any real, the required conclusion is immediate from Lemma B and (7). But for this ψ we obtain

$$\begin{aligned}\mathrm{Re}\psi(ix,y;z)&=-\beta+\beta_1\mathrm{Re}k+(1-\beta_1)[y\mathrm{Re}(N(z)/zN'(z))-x\,\mathrm{Im}\,k]\\&\leqslant-\beta+\beta_1\mathrm{Re}\,k+(1-\beta_1)\left[-\frac{(1+x^2)}{2}\delta-x\,\mathrm{Im}\,k\right]\\&=-(1-\beta_1)[\delta x^2+2x\,\mathrm{Im}\,k+D]/2.\end{aligned}$$

By (34), we deduce that $\mathrm{Re}\psi(ix,y;z)<0$ and so the proof is complete. □

Examples. Let $M(z)=z^n+\cdots$ and $N(z)=z^n+\cdots$ be analytic in Δ. Then for $k\in\mathbb{C}$ with $|\mathrm{Im}\,k|\leqslant\sqrt{\delta(\delta+2\beta)}$, and $\mathrm{Re}(N(z)/zN'(z))>\delta>0$, Theorem 7 shows

$$\mathrm{Re}\left(\frac{M'(z)}{N'(z)}+(k-1)\frac{M(z)}{N(z)}\right)>\beta \text{ implies } \mathrm{Re}\left(\frac{M(z)}{N(z)}\right)>0.$$

As a special case of Theorem 7, let $f\in\mathscr{A}$ and $k\in\mathbb{C}$ with $|\mathrm{Im}\,k|\leqslant\sqrt{D}<\sqrt{1+2\mathrm{Re}\,k}$. In this case, Theorem 7 leads to

$$\mathrm{Re}(kf'(z)+zf''(z))>\beta \text{ implies } \mathrm{Re}\,f'(z)>\frac{1+2\beta-D}{1+2\mathrm{Re}\,k-D}.$$

In particular, this yields

$$\mathrm{Re}(kf'(z)+zf''(z))>\beta \text{ implies } \mathrm{Re}\, f'(z)>0 \quad (\beta<\mathrm{Re}\, k)$$

provided $|\mathrm{Im}\, k|\leqslant\sqrt{1+2\beta}$. This simple fact for $\beta=0$ has been used in [9, Theorem 3] to obtain an affirmative answer to a problem of Mocanu (for details see [9]).

Problems

Suppose that $p\in\mathscr{A}'$, $\beta<1$, $\rho=\beta+(1-\beta)[2\delta(\mathrm{Re}\,\alpha)-1]$ and H be defined by

$$H(z)=\frac{1-2[(1+\alpha)\rho-\alpha]z-(1-2\rho)z^2}{(1-z)^2}=$$
$$-(1-2\rho)+2(1-\rho)\left[(1-\alpha)\frac{1}{1-z}+\alpha\frac{1}{(1-z)^2}\right].$$

Now by setting $|z|=1$, i.e., $z=e^{i\theta}$ and $H(|z|=1)=U+iV$, we easily obtain

$$U=U(\theta)=\rho-\frac{\mathrm{Re}\,\alpha(1-\rho)}{1-\cos\theta} \text{ and } V=V(\theta)=(1-\rho)\left[\frac{\sin\theta-\mathrm{Im}\,\alpha}{1-\cos\theta}\right].$$

This, upon simplification for the case α real, yields the parabola

$$V^2=\frac{-2(1-\rho)}{\alpha}\left[U-\rho+\frac{\alpha(1-\rho)}{2}\right]=\frac{-4(1-\beta)(1-\delta(\alpha))}{\alpha}$$
$$[U-\beta+(\alpha+1-(\alpha+2)\delta(\alpha))]$$

and so for real α, the function H maps the unit disc $|z|<1$ into the convex domain, say D, bounded by the above parabola. Observe that the domain D contains $\{\omega\in\mathbb{C}$: $\mathrm{Re}\,\omega>\beta+((\alpha+2)\delta(\alpha)-(\alpha+1))\}$ for $\beta<1$.

Also from the sharp subordination relation (11) and a little manipulation we have the following implication

$$p\in\mathscr{A}' \text{ and } p(z)+\alpha zp'(z)\prec H(z) \text{ implies } p(z)\prec\frac{1+(1-2\rho)z}{1-z}$$

provided $\mathrm{Re}\,\alpha\geqslant 0$. From this, it is interesting to note that the same bound in Lemma 1 may be obtained under weaker hypothesis, though the images of Δ under p, respectively under the stated conditions on h and H, are different. Here h is as in the proof of Lemma 1 and H as above.

Problem 1. Find a (convenient) function $G(z)$ such that $G(\Delta)\subset H(\Delta)$ for which

$$f\in\mathscr{A} \text{ and } f'(z)+\alpha zf''(z)\prec G(z) \quad \text{implies} \quad f\in S^*? \qquad \square$$

For $\alpha<-2$, let

$$P(\alpha,\beta)=\{f\in\mathscr{A}\colon \mathrm{Re}(f'(z)+\alpha zf''(z))<\beta\}, \quad (\beta>1).$$

For $f \in \mathscr{A}$, $\alpha < -2$ and $\text{Re}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} < \beta$, by Theorem 5, we have

$$\text{Re}\,\frac{f(z)}{z} > \frac{2\beta + \alpha}{2+\alpha} \text{ and } \text{Re}\,f'(z) > \frac{3\beta + \alpha - 1}{2+\alpha}.$$

However, for $\alpha \in \mathbb{C}$, $\text{Re}\,\alpha < -2$, Theorem 6 yields

$$f \in \mathscr{A} \text{ and } \text{Re}\,(f'(z) + \alpha z f'(z)) < \beta \text{ implies } \text{Re}\,f'(z) > \frac{2\beta + \text{Re}\,\alpha}{2 + \text{Re}\,\alpha}.$$

In particular for $\alpha < -2$ and $\beta \leqslant -\alpha/2$,

$$f \in P(\alpha, \beta) \text{ implies } \text{Re}\,f'(z) > 0$$

and further it is easy to show that

$$P(\alpha, \beta) \subset P\left(\alpha', \frac{\beta(2+\alpha') + (\alpha - \alpha')}{2+\alpha}\right), \quad \text{for } -2 > \alpha > \alpha'.$$

Although a function $f \in \mathscr{A}$ such that $\text{Re}\,f'(z) > 0$ in Δ is univalent, Krzyz [3] showed that such a function need not be starlike in Δ. As pointed out in [10] there are functions, say f in $\mathscr{A}$ satisfying the condition $|f'(z) - 1| < 1$ in Δ, but they are not in general starlike in Δ. However the natural problem is the following:

Problem 2. Find certain subsets Ω of the left half plane, such that $f \in S^*$, whenever $f'(z) + \alpha z f''(z)$ belongs to Ω for all $z \in \Delta$ and $\alpha < -2$. In particular, under what conditions on β and α, $z(F * G)'(z)$ is starlike in Δ whenever F and G belong to $P(\alpha, \beta)$. Here * between two functions denotes Hadamard convolution. □

For $\delta \geqslant 0$, define a δ-neighborhood of $f(z) = z + a_2 z^2 + \cdots \in \mathscr{A}$ by

$$N_\delta(f) = \left\{g\colon g(z) = z + a_2 z^2 + \cdots \in \mathscr{A}, \sum_{k=2}^{\infty} k\,|a_k - b_k| < \delta\right\}.$$

δ-neighborhoods were introduced by Ruscheweyh [11], who used this to generalize the result that $N_1(z) \subset S^*$. Now for $\alpha \geqslant 0$, let

$$R(\alpha) = \{f \in \mathscr{A}\colon R(f'(z) + \alpha z f''(z)) > 0, z \in \Delta\}.$$

It is known [9] that, $R(\alpha) \subset S^*$ at least when $\alpha \geqslant 0.4269\cdots$. Using Lemma 1, it is seen that if $f \in R(1)$ then $\text{Re}\,f'(z) > 2\ln 2 - 1$ and hence proceeding as in [11], it is not difficult to show that $N_{2\ln 2 - 1}(R(1)) \subset R(0)$.

Interestingly Ruscheweyh proved that if f is in $S^*(\beta)$ then there is no value of $\delta > 0$ such that $N_\delta(S^*(\beta)) \subset S^*$ for any $0 \leqslant \beta < 1$.

In spite of this, it seems reasonable to ask the following:

Problem 3. Do there exist some conditions on α and δ such that $N_\delta(R(\alpha)) \subset S^*$? If so, what is the best possible δ for a suitable fixed α? □

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