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Induced representation and Frobenius reciprocity for compact quantum groups

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Abstract. Unitary representations of compact quantum groups have been described as isometric comodules. The notion of an induced representation for compact quantum groups has been introduced and an analogue of the Frobenius reciprocity theorem is established.

Keywords. Induced representation; compact quantum group; Hilbert C^*-module.

Quantum groups, like their classical counterparts, have a very rich representation theory. In the representation theory of classical groups, induced representation plays a very important role. Among other things, for example, one can obtain families of irreducible unitary representations of many locally compact groups as representations induced by one-dimensional representations of appropriate subgroups. Therefore, it is natural to try and see how far this notion can be developed and exploited in the case of quantum groups. As a first step, we do it here for compact quantum groups. First we give an alternative description of a unitary representation as an isometric comodule map. This is trivial in the finite-dimensional case, but requires a little bit of work if the comodule is infinite-dimensional. Using the comodule description, the notion of an induced representation is defined. We then go on to prove that an exact analogue of the Frobenius reciprocity theorem holds for compact quantum groups. As an application of this theorem, an alternative way of decomposing the action of $SU_q(2)$ on the Podleś sphere S^2_{q0} is given.

Notations. $\mathscr{H}, \mathscr{K}$ etc, with or without subscripts, will denote complex separable Hilbert spaces. $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}_0(\mathscr{H})$ denote respectively the space of bounded operators and the space of compact operators on $\mathscr{H}$. $\mathscr{A}, \mathscr{B}, \mathscr{C}$ etc denote C^*-algebras. All the C^*-algebras used in this article have been assumed to act nondegenerately on Hilbert spaces. More specifically, given any C^*-algebra $\mathscr{A}$, it is assumed that there is a Hilbert space $\mathscr{K}$ such that $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ and for $u \in \mathscr{K}$, $a(u) = 0$ for all $a \in \mathscr{A}$ implies $u = 0$. Tensor product of C^*-algebras will always mean their spatial tensor product. The identity operator on Hilbert spaces is denoted by I, and on C^*-algebras by *id*. For two vector spaces X and Y, $X \otimes_{\mathrm{alg}} Y$ denote their algebraic tensor product.

Let $\mathscr{A}$ be a C^*-algebra acting on $\mathscr{K}$. The subalgebras $\{a \in \mathscr{B}(\mathscr{K}): ab \in \mathscr{A}\ \forall b \in \mathscr{A}\}$ and $\{a \in \mathscr{B}(\mathscr{K}): ab, ba \in \mathscr{A}\ \forall b \in \mathscr{A}\}$ of $\mathscr{B}(\mathscr{K})$ are called respectively the *left multiplier algebra* and the *multiplier algebra* of $\mathscr{A}$. We denote them by $LM(\mathscr{A})$ and $M(\mathscr{A})$ respectively. A good reference for multiplier algebras and other topics in C^*-algebra theory is [4]. See [9] for another equivalent description of multiplier algebras that is often very useful.

1. Preliminaries

1.1 Let $\mathscr{A}$ be a unital C^*-algebra. A vector space X having a right $\mathscr{A}$-module structure is called a *Hilbert $\mathscr{A}$-module* if it is equipped with an $\mathscr{A}$-valued inner product that satisfies

(i) $\langle x,y\rangle^* = \langle y,x\rangle$,
(ii) $\langle x,x\rangle \geqslant 0$,
(iii) $\langle x,x\rangle = 0 \Rightarrow x = 0$,
(iv) $\langle x, yb\rangle = \langle x,y\rangle b$ for $x,y\in X$, $b\in\mathscr{A}$,

and if $\|x\| := \|\langle x,x\rangle\|^{1/2}$ makes X a Banach space.

Details on Hilbert C^*-modules can be found in [1], [2] and [3]. We shall need a few specific examples that are listed below.

Examples. (a) Any Hilbert space $\mathscr{H}$ with its usual inner product is a Hilbert $\mathbb{C}$-module.
(b) Any unital C^*-algebra $\mathscr{A}$ with $\langle a,b\rangle = a^*b$ is a Hilbert $\mathscr{A}$-module.
(c) $\mathscr{H}\otimes\mathscr{A}$, the 'external tensor product' of $\mathscr{H}$ and $\mathscr{A}$, is a Hilbert $\mathscr{A}$-module.
(d) $\mathscr{B}(\mathscr{H},\mathscr{K})$, with $\langle S, T\rangle = S^*T$ is a Hilbert $\mathscr{B}(\mathscr{H})$-module.

1.2 We have seen above that $\mathscr{H}\otimes\mathscr{B}(\mathscr{K})$ and $\mathscr{B}(\mathscr{K},\mathscr{H}\otimes\mathscr{K})$ both are Hilbert $\mathscr{B}(\mathscr{K})$-modules. It is easy to see that the map $\vartheta:\Sigma u_i\otimes a_i \mapsto \Sigma u_i\otimes a_i(\cdot)$ from $\mathscr{H}\otimes_{alg}\mathscr{B}(\mathscr{K})$ to $\mathscr{B}(\mathscr{K},\mathscr{H}\otimes\mathscr{K})$ extends to an isometric module map from $\mathscr{H}\otimes\mathscr{B}(\mathscr{K})$ to $\mathscr{B}(\mathscr{K},\mathscr{H}\otimes\mathscr{K})$, i.e. ϑ obeys

$$\langle\vartheta(x),\vartheta(y)\rangle = \langle x,y\rangle,\quad \forall x,y\in\mathscr{H}\otimes\mathscr{B}(\mathscr{K}),$$

$$\vartheta(xb) = \vartheta(x)b,\quad \forall x\in\mathscr{H}\otimes\mathscr{B}(\mathscr{K}),\ b\in\mathscr{B}(\mathscr{K}).$$

Thus ϑ embeds $\mathscr{H}\otimes\mathscr{B}(\mathscr{K})$ in $\mathscr{B}(\mathscr{K},\mathscr{H}\otimes\mathscr{K})$. Observe two things here: first, if $\mathscr{H}=\mathbb{C}$, ϑ is just the identity map. And, ϑ is *onto* if and only if $\mathscr{H}$ is finite-dimensional. The following lemma, the proof of which is fairly straightforward, gives a very useful property of ϑ.

Lemma. Let ϑ_i be the map ϑ constructed above with $\mathscr{H}_i$ replacing $\mathscr{H}$, $i=1,2$. Let $S\in\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)$ and $x\in\mathscr{H}_1\otimes\mathscr{B}(\mathscr{K})$. Then $\vartheta_2((S\otimes id)x) = (S\otimes I)\vartheta_1(x)$.

1.3 For an operator $T\in\mathscr{B}(\mathscr{H}\otimes\mathscr{K})$, and a vector $u\in\mathscr{H}$, let T_u denote the operator $v\mapsto T(u\otimes v)$ from $\mathscr{K}$ to $\mathscr{H}\otimes\mathscr{K}$. It is not too difficult to show that $T_u\in\vartheta(\mathscr{H}\otimes\mathscr{B}(\mathscr{K}))$ if $T\in LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{B}(\mathscr{K}))$. Define a map $\Psi(T)$ from $\mathscr{H}$ to $\mathscr{H}\otimes\mathscr{B}(\mathscr{K})$ by: $\Psi(T)(u) = \vartheta^{-1}(T_u)$. Then Ψ is the unique linear injective contraction from $LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{B}(\mathscr{K}))$ to $\mathscr{B}(\mathscr{H},\mathscr{H}\otimes\mathscr{B}(\mathscr{K}))$ for which $\vartheta(\Psi(T)(u))(v) = T(u\otimes v)$ $\forall u\in\mathscr{H}, v\in\mathscr{K}$, $T\in LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{B}(\mathscr{K}))$. Here are a few interesting properties of this map Ψ.

PROPOSITION

Let $\Psi: LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{B}(\mathscr{K}))\to\mathscr{B}(\mathscr{H},\mathscr{H}\otimes\mathscr{B}(\mathscr{K}))$ be the map described above. Then we have the following:

(i) *Ψ maps isometries in $LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{B}(\mathscr{K}))$ onto the isometries in $\mathscr{B}(\mathscr{H},\mathscr{H}\otimes\mathscr{B}(\mathscr{K}))$.*

(ii) *For any $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K}))$ and $S \in \mathscr{B}_0(\mathscr{H})$,*

$$\Psi(T(S \otimes I)) = \Psi(T) \circ S,\ \Psi((S \otimes I)T) = (S \otimes id) \circ \Psi(T).$$

(iii) *If $\mathscr{A}$ is any C^*-subalgebra of $\mathscr{B}(\mathscr{K})$ containing its identity, then $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A})$ if and only if range $\Psi(T) \subseteq \mathscr{H} \otimes \mathscr{A}$.*

Proof. (i) Suppose $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K}))$ is an isometry. By 1.2, $\langle \Psi(T)u, \Psi(T)v \rangle = \langle \vartheta^{-1}(T_u), \vartheta^{-1}(T_v) \rangle = \langle T_u, T_v \rangle = \langle u, v \rangle I$ for $u, v \in \mathscr{H}$. Thus $\Psi(T)$ is an isometry.

Conversely, take an isometry $\pi : \mathscr{H} \to \mathscr{H} \otimes \mathscr{B}(\mathscr{K})$ and define an operator T on the product vectors in $\mathscr{H} \otimes \mathscr{K}$ by $T(u \otimes v) = \vartheta(\pi(u))(v)$, ϑ being the map constructed in 1.2. It is clear that T is an isometry. It is enough, therefore, to show that $T(|u\rangle\langle v| \otimes S) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K})$ whenever $S \in \mathscr{B}(\mathscr{K})$ and u, v are unit vectors in $\mathscr{H}$ such that $\langle u, v \rangle = 0$ or 1.

Choose an orthonormal basis $\{e_i\}$ for $\mathscr{H}$ such that $e_1 = u, e_r = v$ where

$$r = \begin{cases} 0 & \text{if } \langle u, v \rangle = 0, \\ 1 & \text{if } \langle u, v \rangle = 1. \end{cases}$$

Let $\pi_{ij} = (\langle e_i| \otimes id)\pi(e_j)$. Then $T(|u\rangle\langle v| \otimes S) = \Sigma |e_i\rangle\langle e_r| \otimes \pi_{i1} S$ where the right-hand side converges strongly. Since $\pi(e_1) \in \mathscr{H} \otimes \mathscr{B}(\mathscr{K})$, it follows that $\Sigma_i \pi_{i1}{}^* \pi_{i1}$ converges in norm. Consequently the right-hand side above converges in norm, which means $T(|u\rangle\langle v| \otimes S) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K})$.

(ii) Straightforward.

(iii) Take $T = |u\rangle\langle v| \otimes a, u, v \in \mathscr{H}, a \in \mathscr{A}$. For any $w \in \mathscr{H}, \Psi(T)(w) = \langle v, w \rangle u \otimes a \in \mathscr{H} \otimes \mathscr{A}$. Since Ψ is a contraction, and the norm closure of all linear combinations of such T's is $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$, we have range $\Psi(T) \subseteq \mathscr{H} \otimes \mathscr{A}$ for all $T \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$.

Assume next that $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A})$. Then $T(|u\rangle\langle u| \otimes I) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$ for all $u \in \mathscr{H}$. Hence $\Psi(T(|u\rangle\langle u| \otimes I))(u) \in \mathscr{H} \otimes \mathscr{A}$, which means, by part (ii), that $\Psi(T)(u) \in \mathscr{H} \otimes \mathscr{A}$ for all $u \in \mathscr{H}$. Thus range $\Psi(T) \subseteq \mathscr{H} \otimes \mathscr{A}$.

To prove the converse, it is enough to show that $T(|u\rangle\langle v| \otimes a) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$ whenever $a \in \mathscr{A}$ and $u, v \in \mathscr{H}$ are such that $\langle u, v \rangle = 0$ or 1. Rest of the proof goes along the same lines as the proof of the last part of (i). ■

1.4 Let $\mathscr{K}_1, \mathscr{K}_2$ be two Hilbert spaces, $\mathscr{A}_i$ being a C^*-subalgebra of $\mathscr{B}(\mathscr{K}_i)$ containing its identity. Suppose ϕ is a unital $*$-homomorphism from $\mathscr{A}_1$ to $\mathscr{A}_2$. Then $id \otimes \phi$: $S \otimes a \mapsto S \otimes \phi(a)$ extends to a $*$-homomorphism from $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1$ to $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2$. Moreover $\{((id \otimes \phi)(a))b : a \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1, b \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2\}$ is total in $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2$. Therefore $id \otimes \phi$ extends to an algebra homomorphism by the following prescription: for all $a \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1)$, $b \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1$, $c \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2$,

$$((id \otimes \phi)a)(((id \otimes \phi)b)c) := ((id \otimes \phi)(ab))c.$$

PROPOSITION

Let ϕ be as above, and Ψ_i be the map Ψ constructed earlier with $\mathscr{K}_i$ replacing $\mathscr{K}$. Then for $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1)$,

$$(I \otimes \phi)\Psi_1(T) = \Psi_2((id \otimes \phi)T).$$

Proof. It is enough to prove that

$$(\langle u|\otimes id)((I\otimes\phi)\Psi_1(T)(v))=(\langle u|\otimes id)\Psi_2((id\otimes\phi)T)(v), \forall u,v\in\mathscr{H}.$$

Rest now is a careful application of 1.2. ■

1.5 Consider the homomorphic embeddings $\phi_{12}:\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\to\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\otimes\mathscr{A}_2$ and $\phi_{13}:\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_2\to\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\otimes\mathscr{A}_2$ given on the product elements by

$$\phi_{12}(a\otimes b)=a\otimes b\otimes I,\ \phi_{13}(a\otimes c)=a\otimes I\otimes c,$$

respectively. Each of their ranges contains an approximate identity for $\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\otimes\mathscr{A}_2$, so that their extensions respectively to $LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1)$ and $LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_2)$ are also homomorphic embeddings.

PROPOSITION

Let Ψ_1,Ψ_2 be as in the previous proposition, and let Ψ_0 be the map Ψ with $\mathscr{A}_1\otimes\mathscr{A}_2$ replacing $\mathscr{A}$. Let $S\in LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1)$, $T\in LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_2)$. Then

$$\Psi_0(\phi_{12}(S)\phi_{13}(T))=(\Psi_1(S)\otimes id)\Psi_2(T).$$

Proof. Observe that for $u_1,\ldots,u_n\in\mathscr{H},((\langle\Psi_1(S)(u_i),\Psi_1(S)(u_j)\rangle))\leqslant\|S\|^2((\langle u_i,u_j\rangle I))$. Therefore $\Psi_1(S)\otimes id$ is a well-defined bounded operator from $\mathscr{H}\otimes\mathscr{A}_2$ to $\mathscr{H}\otimes\mathscr{A}_1\otimes\mathscr{A}_2$. Take an orthonormal basis $\{e_i\}$ for $\mathscr{H}$. Define S_{ij}'s and T_{ij}'s as follows:

$$S_{ij}:v\mapsto(\langle e_i|\otimes I)S(e_j\otimes v),\ T_{ij}:v\mapsto(\langle e_i|\otimes I)\,T(e_j\otimes v).$$

Let $P_n:=\Sigma_{i=1}^n|e_i\rangle\langle e_i|$. Then $(\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i)=(\Psi_1(S)\otimes id)(\Sigma_{j\leqslant n}e_j\otimes T_{ij})=\Sigma_{j\leqslant n}(\Sigma_k e_k\otimes S_{kj})\otimes T_{ij}$. Hence for $v\in\mathscr{K}_1,w\in\mathscr{K}_2$,

$$\begin{aligned}&\vartheta((\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i))(v\otimes w)\\&=\sum_{j\leqslant n}\sum_k e_k\otimes S_{kj}(v)\otimes T_{ji}(w)\\&=\left(\sum_{j\leqslant n}\sum_{k,r}|e_k\rangle\langle e_r|\otimes S_{kj}\otimes T_{ji}\right)(e_i\otimes v\otimes w)\\&=\phi_{12}(S)(P_n\otimes I\otimes I)\phi_{13}(T)(e_i\otimes v\otimes w).\end{aligned}$$

This converges to $\phi_{12}(S)\phi_{13}(T)(e_i\otimes v\otimes w)$ as $n\to\infty$. On the other hand,

$$\lim_{n\to\infty}(\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i)=(\Psi_1(S)\otimes id)\Psi_2(T)(e_i),$$

which implies $\lim_{n\to\infty}\vartheta((\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i))=\vartheta((\Psi_1(S)\otimes id)\Psi_2(T)(e_i))$. Therefore $\vartheta((\Psi_1(S)\otimes id)\Psi_2(T)(e_i))(v\otimes w)=\phi_{12}(S)\phi_{13}(T)(e_i\otimes v\otimes w)=\vartheta(\Psi_0(\phi_{12}(S)\phi_{13}(T))(e_i))(v\otimes w)$. Thus $(\Psi_1(S)\otimes id)\Psi_2(T)=\Psi_0(\phi_{12}(S)\phi_{13}(T))$. ■

2. Representations of compact quantum groups

2.1 We start by recalling a few facts from [6] on compact quantum groups.

DEFINITION

Let $\mathscr{A}$ be a separable unital C^*-algebra, and $\mu:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ be a unital $*$-homomorphism. We call $G=(\mathscr{A},\mu)$ a compact quantum group if the following two conditions are satisfied:

(i) $(id\otimes\mu)\mu=(\mu\otimes id)\mu$, and
(ii) $\{(a\otimes I)\mu(b):a,b\in\mathscr{A}\}$ and $\{(I\otimes a)\mu(b):a,b\in\mathscr{A}\}$ both are total in $\mathscr{A}\otimes\mathscr{A}$.

μ is called the *comultiplication map* associated with G. We shall very often denote the underlying C^*-algebra $\mathscr{A}$ by $C(G)$ and the map μ by μ_G.

A representation of a compact quantum group G acting on a Hilbert space $\mathscr{H}$ is an element π of the multiplier algebra $M(\mathscr{B}_0(\mathscr{H})\otimes C(G))$ that obeys $\pi_{12}\pi_{13}=(id\otimes\mu)\pi$, where π_{12} and π_{13} are the images of π in the space $M(\mathscr{B}_0(\mathscr{H})\otimes C(G)\otimes C(G))$ under the homomorphisms ϕ_{12} and ϕ_{13} which are given on the product elements by:

$$\phi_{12}(a\otimes b)=a\otimes b\otimes I,\ \phi_{13}(a\otimes b)=a\otimes I\otimes b.$$

A representation π is called a *unitary representation* if $\pi\pi^*=I=\pi^*\pi$. One also has the notions of irreducibility, direct sum and tensor product of representations. As in the case of classical groups, any unitary representation decomposes into a direct sum of finite-dimensional irreducible unitary representations. Let $A(G)$ be the unital $*$-subalgebra of $C(G)$ generated by the matrix entries of finite-dimensional unitary representations of G. Then one has the following result (see [8]).

Theorem. ([8]) *Suppose G is a compact quantum group. Let $A(G)$ be as above. Then we have the following*:

(a) *$A(G)$ is a dense unital $*$-subalgebra of $C(G)$ and $\mu(A(G))\subseteq A(G)\otimes_{alg}A(G)$.*
(b) *There is a complex homomorphism $\varepsilon:A(G)\to\mathbb{C}$ such that*

$$(\varepsilon\otimes id)\mu=id=(id\otimes\varepsilon)\mu.$$

(c) *There exists a linear antimultiplicative map $\kappa:A(G)\to A(G)$ obeying*

$$m(id\otimes\kappa)\mu(a)=\varepsilon(a)I=m(\kappa\otimes id)\mu(a),\ \text{and}\ \kappa(\kappa(a^*)^*)=a$$

for all $a\in A(G)$, where m is the operator that sends $a\otimes b$ to ab.

The maps ε and κ in the above theorem are called the *counit* and *coinverse* respectively of the quantum group G.

2.2 Let $G=(C(G),\mu_G)$ and $H=(C(H),\mu_H)$ be two compact quantum groups. A C^*-homomorphism ϕ from $C(G)$ to $C(H)$ is called a quantum group homomorphism from G to H if it obeys $(\phi\otimes\phi)\mu_G=\mu_H\phi$.

One can show that if G, H are compact quantum groups, then H is a subgroup of G if and only if there is a homomorphism from G to H that maps $C(G)$ *onto* $C(H)$.

2.3 Let $G=(\mathscr{A},\mu)$ be a compact quantum group. From now onward we shall assume that $\mathscr{A}$ acts nondegenerately on a Hilbert space $\mathscr{H}$, i.e. $\mathscr{A}$ is a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ containing its identity. We call a map π from $\mathscr{H}$ to $\mathscr{H}\otimes\mathscr{A}$ an *isometry* if $\langle\pi(u),\pi(v)\rangle=\langle u,v\rangle I$ for all $u,v\in\mathscr{H}$. If $\pi:\mathscr{H}\to\mathscr{H}\otimes\mathscr{A}$ is an isometry, then $\pi\otimes id$: $u\otimes a\mapsto\pi(u)\otimes a$ extends to a bounded map from $\mathscr{H}\otimes\mathscr{A}$ to $\mathscr{H}\otimes\mathscr{A}\otimes\mathscr{A}$. π is called an *isometric comodule map* if it is an isometry, and satisfies $(\pi\otimes id)\pi=(I\otimes\mu)\pi$. The pair $(\mathscr{H},\pi)$ is called an *isometric comodule.* We shall often just say π is a comodule, omitting the $\mathscr{H}$.

The following theorem says that for a compact quantum group isometric comodules are nothing but the unitary representations.

Theorem. *Let π be an isometric comodule map acting on $\mathscr{H}$. Then $\Psi^{-1}(\pi)$ is a unitary representation acting on $\mathscr{H}$. Conversely, if $\hat{\pi}$ is a unitary representation of G on $\mathscr{H}$, then $(\mathscr{H},\Psi(\hat{\pi}))$ is an isometric comodule.*

We need the following lemma for proving the theorem.

Lemma. Let $(\mathscr{H},\pi)$ be an isometric comodule. Then $\mathscr{H}$ decomposes into a direct sum of finite dimensional subspaces $\mathscr{H}=\oplus\mathscr{H}_\alpha$ such that each $\mathscr{H}_\alpha$ is π-invariant and $\pi|_{\mathscr{H}_\alpha}$ is an irreducible isometric comodule.

Proof. By 1.3, there is an isometry $\hat{\pi}$ in $LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A})$ such that $\Psi(\hat{\pi})=\pi$. Using 1.4 and 1.5, we get $\hat{\pi}_{12}\hat{\pi}_{13}=(id\otimes\mu)\hat{\pi}$ where $\hat{\pi}_{12}=\phi_{12}(\hat{\pi})$, $\hat{\pi}_{13}=\phi_{13}(\hat{\pi})$, ϕ_{12} and ϕ_{13} being as in 1.5 with $\mathscr{A}_1=\mathscr{A}_2=\mathscr{A}$.

Let $\mathscr{I}=\{a\in\mathscr{A}:h(a^*a)=0\}$. From the properties of the haar state, $\mathscr{I}$ is an ideal in $\mathscr{A}$. For any unit vector u in $\mathscr{H}$, let $Q(u)=(id\otimes h)(\hat{\pi}(|u\rangle\langle u|\otimes I)\hat{\pi}^*)$. Then $Q(u)^*=Q(u)\in\mathscr{B}_0(\mathscr{H})$. If $Q(u)=0$, then $|\hat{\pi}(|u\rangle\langle u|\otimes I)\hat{\pi}^*|^{1/2}\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{I}$. Therefore $\hat{\pi}(|u\rangle\langle u|\otimes I)\hat{\pi}^*\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{I}$. It follows then that $|u\rangle\langle u|\otimes I\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{I}$. This forces u to be zero. Thus for a nonzero u, $Q(u)\neq 0$. Choose and fix any nonzero u. Then

$$
\begin{aligned}
&\hat{\pi}(Q(u)\otimes I)\hat{\pi}^*\\
&\quad=(id\otimes id\otimes h)(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u|\otimes I\otimes I)\hat{\pi}_{13}^*\hat{\pi}_{12}^*)\\
&\quad=(id\otimes id\otimes h)(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u|\otimes I\otimes I)(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u|\otimes I\otimes I))^*)\\
&\quad=(id\otimes id\otimes h)((id\otimes\mu)(\hat{\pi})(id\otimes\mu)(|u\rangle\langle u|\otimes I)\\
&\qquad\times((id\otimes\mu)(\hat{\pi})(id\otimes\mu)(|u\rangle\langle u|\otimes I))^*)\\
&\quad=(id\otimes id\otimes h)((id\otimes\mu)(\hat{\pi}(|u\rangle\langle u|\otimes I))((id\otimes\mu)(\hat{\pi}(|u\rangle\langle u|\otimes I)))^*)\\
&\quad=(id\otimes id\otimes h)((id\otimes\mu)(\hat{\pi}(|u\rangle\langle u|\otimes I))(id\otimes\mu)((|u\rangle\langle u|\otimes I)\hat{\pi}^*))\\
&\quad=(id\otimes id\otimes h)(id\otimes\mu)(\hat{\pi}(|u\rangle\langle u|\otimes I)\hat{\pi}^*)\\
&\quad=(id\otimes(id\otimes h)\mu)(\hat{\pi}(|u\rangle\langle u|\otimes I)\hat{\pi}^*)\\
&\quad=Q(u)\otimes I.
\end{aligned}
$$

Thus $\hat{\pi}(Q(u)\otimes I)=(Q(u)\otimes I)\hat{\pi}$. If P is any finite-dimensional spectral projection of $Q(u)$, then $\hat{\pi}(P\otimes I)=(P\otimes I)\hat{\pi}$, which means, by an application of part (ii) of 1.3, that $\pi P=(P\otimes id)\pi$. Standard arguments now tell us that π can be decomposed into a direct sum of finite-dimensional isometric comodules. Finite-dimensional comodules, in turn, can easily be shown to decompose into a direct sum of irreducible isometric comodules. The proof is thus complete. ∎

Proof of the theorem: Let $\hat{\pi}$ be a unitary representation. By 1.3, $\Psi(\hat{\pi})$ is an isometry from $\mathscr{H}$ to $\mathscr{H}\otimes C(G)$. Using 1.4 and 1.5, we conclude that $\Psi(\hat{\pi})$ is an isometric comodule.

For the converse, take an isometric comodule π. If π is finite-dimensional, it is easy to see that $\Psi^{-1}(\pi)$ is a unitary representaticn. So, assume that π is infinite-dimensional. By the lemma above, there is a family $\{P_\alpha\}$ of finite-dimensional projections in $\mathscr{B}(\mathscr{H})$ satisfying

$$P_\alpha P_\beta = \delta_{\alpha\beta}P_\alpha, \sum P_\alpha = I, \pi P_\alpha = (P_\alpha\otimes id)\pi \ \forall\alpha \tag{2.1}$$

such that $\pi|_{P_\alpha\mathscr{H}} = \pi P_\alpha$ is an irreducible isometric comodule. $\pi|_{P_\alpha\mathscr{H}}$ is finite-dimensional, therefore $\Psi^{-1}(\pi|_{P_\alpha\mathscr{H}})$ is a unitary element of $LM(\mathscr{B}_0(P_\alpha\mathscr{H})\otimes\mathscr{A}) = \mathscr{B}(P_\alpha\mathscr{H})\otimes\mathscr{A}$. Let us denote $\Psi^{-1}(\pi)$ by $\hat{\pi}$. Then the above implies that in the bigger space $\mathscr{B}(\mathscr{H}\otimes\mathscr{K})$,

$$(\hat{\pi}(P_\alpha\otimes I))^*(\hat{\pi}(P_\alpha\otimes I)) = P_\alpha\otimes I = (\hat{\pi}(P_\alpha\otimes I))(\hat{\pi}(P_\alpha\otimes I))^*.$$

The second equality implies that $\hat{\pi}(P_\alpha\otimes I)\hat{\pi}^* = P_\alpha\otimes I$ for all α, so that $\hat{\pi}\hat{\pi}^* = I$. We already know by 1.3 that $\hat{\pi}^*\hat{\pi} = I$ and by 1.4 and 1.5 that $\hat{\pi}_{12}\hat{\pi}_{13} = (id\otimes\mu)\hat{\pi}$. Thus it remains only to show that $\hat{\pi}\in M(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A})$. It is enough to show that for any $S\in\mathscr{B}_0(\mathscr{H})$ and $a\in\mathscr{A}$, $(S\otimes a)\hat{\pi}\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}$. Now from (2.1) and 1.3, $\hat{\pi}(P_\alpha\otimes I) = (P_\alpha\otimes I)\hat{\pi}$ for all α. Therefore $(S\otimes a)(P_\alpha\otimes I)\hat{\pi} = (S\otimes a)\hat{\pi}(P_\alpha\otimes I)\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}$. Since $(S\otimes a)\hat{\pi}$ is the norm limit of finite sums of such terms, $(S\otimes a)\hat{\pi}\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}$. Thus $\hat{\pi}$ is a unitary representation acting on $\mathscr{H}$. ∎

2.4 Next we introduce the right regular comodule. Denote by $L_2(G)$ the GNS space associated with the haar state h on G. Then $\mathscr{A}$ is a dense subspace of $L_2(G)$. One can also see that $\mathscr{A}\otimes\mathscr{A}$ can be regarded as a subspace of $L_2(G)\otimes\mathscr{A}$. Consider the map $\mu:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$.

$$\langle\mu(a),\mu(b)\rangle = (h\otimes id)(\mu(a)^*\mu(b)) = (h\otimes id)\mu(a^*b) = h(a^*b)I = \langle a,b\rangle I$$

for all $a,b\in\mathscr{A}$. Therefore μ extends to an isometry from $L_2(G)$ into $L_2(G)\otimes\mathscr{A}$. Denote it by $\mathfrak{R}$. The maps $(I\otimes\mu)\mathfrak{R}$ and $(\mathfrak{R}\otimes id)\mathfrak{R}$ both are isometries from $L_2(G)$ to $L_2(G)\otimes\mathscr{A}\otimes\mathscr{A}$ and they coincide on $\mathscr{A}$. Hence $(I\otimes\mu)\mathfrak{R} = (\mathfrak{R}\otimes id)\mathfrak{R}$. Thus $\mathfrak{R}$ is an isometric comodule map. We call it the *right-regular comodule* of G. By theorem 2.3, $\Psi^{-1}(\mathfrak{R})$ is a unitary representation acting on $L_2(G)$. This is the right-regular representation introduced by Woronowicz ([8]).

Finally let us state here a small lemma which is a direct consequence of the Peter-Weyl theorem for compact quantum groups.

2.5 *Lemma.* $\{u\in L_2(G):\mathfrak{R}(u)\in L_2(G)\otimes_{alg}C(G)\} = A(G)$.

3. Induced representations

In this section we shall introduce the concept of an induced representation and show that Frobenius reciprocity theorem holds for compact quantum groups. Throughout this section $G = (C(G), \mu_G)$ will denote a compact quantum group and $H = (C(H), \mu_H)$, a subgroup of G. We start with a lemma concerning the boundedness of the left convolution operator.

3.1 *Lemma. Let $G = (\mathscr{A}, \mu)$ be a compact quantum group. Then the map $L_\rho: \mathscr{A} \to \mathscr{A}$ given by $L_\rho(a) = (\rho \otimes id)\mu(a)$ extends to a bounded operator from $L_2(G)$ into itself.*

Proof. The proof follows from the following inequality: for any two states ρ_1 and ρ_2 on $\mathscr{A}$, we have

$$\rho_1((\rho_2 * a)^*(\rho_2 * a)) \leqslant \rho_2 * \rho_1(a^* a) \ \forall a \in \mathscr{A},$$

where $\rho_i * a := (\rho_i \otimes id)\mu(a)$. ■

3.2 Let $\hat{\pi}$ be a unitary representation of H acting on the space $\mathscr{H}_0$. $\pi := \Psi(\hat{\pi})$ is then an isometric comodule map from $\mathscr{H}_0$ to $\mathscr{H}_0 \otimes C(H)$. Consider the following map from $\mathscr{H}_0 \otimes L_2(G)$ to $\mathscr{H}_0 \otimes L_2(G) \otimes C(G)$:

$$I \otimes \mathfrak{R}^G : u \otimes v \mapsto u \otimes \mathfrak{R}^G(v)$$

where $\mathfrak{R}^G$ is the right-regular comodule of G. It is easy to see that this is an isometric comodule map acting on $\mathscr{H}_0 \otimes L_2(G)$.

Let p be the homomorphism from G to H (cf. 2.2). Let $\mathscr{H} = \{u \in \mathscr{H}_0 \otimes L_2(G) : (I \otimes L_{\rho \cdot p})u = (id \otimes \rho)\pi \otimes I)u$ for all continuous linear functionals ρ on $C(H)\}$. Then $I \otimes \mathfrak{R}^G$ keeps $\mathscr{H}$ invariant; the restriction of $I \otimes \mathfrak{R}^G$ to $\mathscr{H}$ is therefore an isometric comodule, so that $\Psi^{-1}((I \otimes \mathfrak{R}^G)|_{\mathscr{H}})$ is a unitary representation of G acting on $\mathscr{H}$. We call this *the representation induced by* $\hat{\pi}$, and denote it by $\mathrm{ind}_H^G \hat{\pi}$ or simply by $\mathrm{ind}\,\hat{\pi}$ when there is no ambiguity about G and H.

Let $\hat{\pi}_1$ and $\hat{\pi}_2$ be two unitary representations of H. Then clearly we have

(i) $\mathrm{ind}\,\hat{\pi}_1$ and $\mathrm{ind}\,\hat{\pi}_2$ are equivalent whenever $\hat{\pi}_1$ and $\hat{\pi}_2$ are equivalent, and
(ii) $\mathrm{ind}(\hat{\pi}_1 \oplus \hat{\pi}_2)$ and $\mathrm{ind}\,\hat{\pi}_1 \oplus \mathrm{ind}\,\hat{\pi}_2$ are equivalent.

Before going to the Frobenius reciprocity theorem, let us briefly describe what we mean by restriction of a representation to a subgroup. Let $\hat{\pi}^G$ be a unitary representation of G acting on a Hilbert space $\mathscr{H}_0$. We call $(id \otimes p)\hat{\pi}^G$ the restriction of $\hat{\pi}^G$ to H and denote it by $\hat{\pi}^{G|H}$. To see that it is indeed a unitary representation, observe that $\Psi((id \otimes p)\hat{\pi}^G) = (I \otimes p)\Psi(\hat{\pi}^G)$ which is clearly an isometric comodule. Therefore by 2.3, $\hat{\pi}^{G|H}$ is a unitary representation of H acting on $\mathscr{H}_0$. Denote $\Psi(\hat{\pi}^G)$ by π^G and $\Psi(\hat{\pi}^{G|H})$ by $\pi^{G|H}$.

3.3 **Theorem.** *Let $\hat{\pi}^G$ and $\hat{\pi}^H$ be irreducible unitary representations of G and H respectively. Then the multiplicity of $\hat{\pi}^G$ in $\mathrm{ind}_H^G \hat{\pi}^H$ is the same as that of $\hat{\pi}^H$ in $\hat{\pi}^{G|H}$.*

Proof. Let $\mathscr{I}(\hat{\pi}^{G|H}, \hat{\pi}^H)$ (respectively $\mathscr{I}(\hat{\pi}^G, \mathrm{ind}\,\hat{\pi}^H)$) denote the space of intertwiners between $\hat{\pi}^{G|H}$ and $\hat{\pi}^H$ (respectively $\hat{\pi}^G$ and $\mathrm{ind}\,\hat{\pi}^H$). Assume that $\hat{\pi}^G$ and $\hat{\pi}^H$ act on $\mathscr{K}_0$ and $\mathscr{H}_0$ respectively. $\mathscr{K}_0 \otimes C(G)$ can be regarded as a subspace of $\mathscr{K}_0 \otimes L_2(G)$ and hence π^G, as a map from $\mathscr{K}_0$ into $\mathscr{K}_0 \otimes L_2(G)$. Since $\pi^G = \Psi(\hat{\pi}^G)$ is unitary, we have for $u, v \in \mathscr{K}_0$,

$$\langle \pi^G(u), \pi^G(v) \rangle_{\mathscr{K}_0 \otimes L_2(G)} = h(\langle \pi^G(u), \pi^G(v) \rangle_{\mathscr{K}_0 \otimes C(G)}) = h(\langle u, v \rangle I) = \langle u, v \rangle.$$

Thus $\pi^G:\mathscr{K}_0\to\mathscr{K}_0\otimes L_2(G)$ is an isometry. Let $S:\mathscr{K}_0\to\mathscr{H}_0$ be an element of $\mathscr{I}(\hat{\pi}^{G|H},\hat{\pi}^H)$. $(S\otimes I)\pi^G$ is then a bounded map from $\mathscr{K}_0$ into $\mathscr{H}_0\otimes L_2(G)$. Denote it by $f(S)$. It is not too difficult to see that $f(S)$ actually maps $\mathscr{K}_0$ into $\mathscr{H}$, and intertwines $\hat{\pi}^G$ and ind $\hat{\pi}^H$. $f:S\mapsto f(S)$ is thus a linear map from $\mathscr{I}(\hat{\pi}^{G|H},\hat{\pi}^H)$ to $\mathscr{I}(\hat{\pi}^G,\text{ind}\,\hat{\pi}^H)$.

We shall now show that f is invertible by exhibiting the inverse of f. Take a $T:\mathscr{K}_0\to\mathscr{H}$ that intertwines $\hat{\pi}^G$ and ind $\hat{\pi}^H$. For any $u\in\mathscr{H}_0$, $T^u:=(\langle u|\otimes I)T$ is a map from $\mathscr{K}_0$ to $L_2(G)$ intertwining $\hat{\pi}^G$ and the right regular representation $\mathfrak{R}^G$ of G, i.e. $\mathfrak{R}^G\,T^u=(T^u\otimes id)\pi^G$. Now, π^G is finite-dimensional, so that $\pi^G(\mathscr{K}_0)\subseteq\mathscr{K}_0\otimes_{alg}A(G)$. Hence $\mathfrak{R}^G\,T^u(\mathscr{K}_0)\subseteq L_2(G)\otimes_{alg}A(G)$. By 2.5, $T^u(\mathscr{K}_0)\subseteq A(G)$. Since this is true for all $u\in\mathscr{H}_0$, $T(\mathscr{K}_0)\subseteq\mathscr{H}_0\otimes_{alg}A(G)$. Therefore $(I\otimes\varepsilon_G)T$ is a bounded operator from $\mathscr{K}_0$ to $\mathscr{H}_0$. Denote it by $g(T)$.

For a comodule π and a linear functional ρ, denote $(id\otimes\rho)\pi$ by π_ρ. Let ρ be a linear functional on $C(H)$. Then $\pi^H_\rho g(T)=\pi^H_\rho(I\otimes\varepsilon_G)T=(I\otimes\varepsilon_G)(\pi^H_\rho\otimes id)T=(I\otimes\varepsilon_G)(I\otimes L_{\rho\cdot p})\,T=(I\otimes\rho\circ p)T$. On the other hand, since T intertwines $\hat{\pi}^G$ and ind $\hat{\pi}^H$, we have $g(T)(\pi^{G|H})_\rho=g(T)(I\otimes\rho)\pi^{G|H}=g(T)(I\otimes\rho)(I\otimes p)\pi^G=(I\otimes\varepsilon_G)\;T\pi^G_{\rho\cdot p}=(I\otimes\varepsilon_G)(I\otimes\mathfrak{R}^G_{\rho\cdot p})T=(I\otimes\rho\circ p)T$. Thus $\pi^H_\rho g(T)=g(T)(\pi^{G|H})_\rho$ for all continuous linear functionals ρ on $C(H)$, which implies $g(T)\in\mathscr{I}(\hat{\pi}^{G|H},\hat{\pi}^H)$. The map $T\mapsto g(T)$ is the inverse of f. Therefore $\mathscr{I}(\hat{\pi}^{G|H},\hat{\pi}^H)\cong\mathscr{I}(\hat{\pi}^G,\text{ind}\,\hat{\pi}^H)$, which proves the theorem. ■

COROLLARY 1.

For any unitary representation $\hat{\pi}^G$ of G and $\hat{\pi}^H$ of H, the spaces $\mathscr{I}(\hat{\pi}^{G|H},\hat{\pi}^H)$ and $\mathscr{I}(\hat{\pi}^G,\text{ind}\,\hat{\pi}^H)$ are isomorphic.

COROLLARY 2.

Let H be a subgroup of G and K be a subgroup of H. Suppose $\hat{\pi}$ is a unitary representation of K. Then $\text{ind}^G_K\hat{\pi}$ and $\text{ind}^G_H(\text{ind}^H_K\hat{\pi})$ are equivalent.

3.4 Action of $SU_q(2)$ on the sphere S^2_{q0} has been decomposed by Podleś (see [5]). Here we give an alternative way of doing it using the Frobenius reciprocity theorem.

Let us start with a few observations. Let u be the function $z\mapsto z$, $z\in S^1$, where S^1 is the unit circle in the complex plane. Then u is unitary, and generates the C^*-algebra $C(S^1)$ of continuous functions on S^1. Let α and β be the two elements that generate the algebra $C(SU_q(2))$ and obey the following relations:

$$\alpha^*\alpha+\beta^*\beta=I=\alpha\alpha^*+q^2\beta\beta^*,$$

$$\alpha\beta-q\beta\alpha=0=\alpha\beta^*-q\beta^*\alpha,\quad \beta^*\beta=\beta\beta^*.$$

The map $p:\alpha\mapsto u$, $\beta\mapsto 0$ extends to a C^*-homomorphism from $C(SU_q(2))$ onto $C(S^1)$. It is in fact a quantum group homomorphism. By 2.2, S^1 is a subgroup of $SU_q(2)$.

For any $n\in\{0,1/2,1,3/2,\ldots\}$, if we restrict the right-regular comodule $\mathfrak{R}$ of $SU_q(2)$ to the subspace $\mathscr{H}_n$ of $L_2(SU_q(2))$ spanned by

$$\{\alpha^{*i}\beta^{2n-i}:i=0,1,\ldots,2n\},\tag{3.1}$$

then we get an irreducible isometric comodule. Denote it by $u^{(n)}$. It is a well-known fact ([6], [7]) that these constitute *all* the irreducible comodules of $SU_q(2)$. If we take the basis of $\mathscr{H}_n$ to be (3.1) with proper normalization, the matrix entries of $u^{(n)}$ turn out to be

$$u_{ij}^{(n)} = (d_i^{(n)}/d_j^{(n)})^{1/2} \sum_{r=(i-j)\vee 0}^{(2n-j)\wedge i} \binom{i}{r}_{q^{-2}} \binom{2n-i}{r+j-i}_{q^{-2}} (-1)^r q^{r(2i-r+1)+(j-i)(2n-j)} \times \alpha^{*i-r}\alpha^{2n-j-r}\beta^{r+j-i}\beta^{*r},$$

where

$$d_k^{(n)} = \sum_{r=0}^{k} \binom{k}{r}_{q^{-2}} (-1)^r q^{r(2k-r+1)} \frac{1-q^2}{1-q^{4n+2r-2k+2}};$$

$$\binom{r}{s}_{q^{-2}} := \frac{(r)_{q^{-2}}(r-1)_{q^{-2}}\ldots(1)_{q^{-2}}}{(s)_{q^{-2}}(s-1)_{q^{-2}}\ldots(1)_{q^{-2}}(r-s)_{q^{-2}}(r-s-1)_{q^{-2}}\ldots(1)_{q^{-2}}};$$

$$(k)_{q^{-2}} := 1 + q^{-2} + q^{-4} + \cdots + q^{-2k+2}.$$

Since $u^{(n)}|^{S^1} = (I \otimes p)u^{(n)}$, matrix entries of $u^{(n)}|^{S^1}$ are given by

$$(u^{(n)}|^{S^1})_{ij} = \begin{cases} u^{2(n-i)} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.2}$$

Therefore if n is an integer then the trivial representation occurs in $u^{(n)}|^{S^1}$ with multiplicity 1, and does not occur otherwise.

Consider now the action of $SU_q(2)$ on S^2_{q0}. Recall ([5]) that $C(S^2_{q0}) = \{a \in C(SU_q(2)): (p \otimes id)\mu(a) = I \otimes a\}$ and the action is the restriction of μ to $C(S^2_{q0})$. From the above description, $C(S^2_{q0})$ can easily be shown to be equal to $\{a \in C(S^2_{q0}): L_{\rho \cdot p}(a) = \rho(I)a$ for all continuous linear functionals ρ on $C(S^1)\}$. Therefore when we take the closure of $C(S^2_{q0})$ with respect to the invariant inner product that it carries and extend the action there as an isometry, what we get is the restriction of the right-regular comodule $\mathfrak{R}$ of $SU_q(2)$ to the subspace $\mathscr{H} = \{u \in L_2(SU_q(2)): L_{\rho \cdot p}(u) = \rho(I)u$ for all continuous linear functionals ρ on $C(S^1)\}$, which is nothing but the representation $\hat{\pi}$ of $SU_q(2)$ induced by the trivial representation of S^1 on $\mathbb{C}$. Hence the multiplicity of $u^{(n)}$ in $\hat{\pi}$ is same as that of the trivial representation of S^1 in $u^{(n)}|^{S^1}$ which is, from (3.2), 1 if n is an integer and 0 if n is not. Thus the action splits into a direct sum of all the integer-spin representations.

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