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On subsemigroups of semisimple Lie groups

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Abstract. In this paper we classify the subsemigroups of any connected semisimple Lie group G which are K-bi-invariant, where $G = KAN$ is an Iwasawa decomposition of G.

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In a recent investigation of the support behaviour of certain Gauss measures on a connected semisimple Lie group (see [KM]), we encountered the following question.

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$ having a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (in the usual notation of Helgason [He]), and let K be the analytic subgroup corresponding to $\mathfrak{k}$. Can one classify the subsemigroups S of G such that $K \subseteq S$? Here "subsemigroup" means only a subset of G which is closed under the group multiplication. In this note we show that this problem has a very simple answer.

To describe this, we let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$$

be the decomposition of $\mathfrak{g}$ into its simple ideals $\mathfrak{g}_j$, $1 \leqslant j \leqslant n$, and we recall that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for all $1 \leqslant i < j \leqslant n$, and $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal w.r.t. the Killing form on $\mathfrak{g}$. If $\mathfrak{g}_j = \mathfrak{k}_j + \mathfrak{p}_j$ is a Cartan decomposition for $\mathfrak{g}_j$, then $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_2 + \cdots + \mathfrak{k}_n$ and $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2 + \cdots + \mathfrak{p}_n$.

Let N be a subset of $\{1, 2, \ldots, n\}$ and form $\mathfrak{g}_N := \mathfrak{k} + \sum_{j \in N} \mathfrak{p}_j$. It is easy to see that $\mathfrak{g}_N$ is a reductive subalgebra containing $\mathfrak{k}$, and if G_N is the corresponding analytic subgroup, then $G_N = (\prod_{j \in N} G_j)(\prod_{j \notin N} K_j)$, where G_j, K_j are the analytic subgroups determined by $\mathfrak{g}_j$ and $\mathfrak{k}_j$, respectively.

Our question raised above is now answered by the following result.

Theorem. *Let G be a connected semisimple Lie group and let S be any subsemigroup of G bi-invariant under K. Then $S = G_N$ for some subset N of $\{1, 2, \ldots, n\}$.*

In the special case when G is simple (and noncompact) this theorem tells us that K is a maximal proper subsemigroup of G. This special case therefore implies the observation of Hilgert and Hofmann that SO(2) is a maximal proper subsemigroup of SL(2, $\mathbb{R}$) ([Hi H], Corollary 4.20, p. 49) and extends the theorem of Brun (see [B] or [He], Exercise A.3, p. 275) that K is a maximal proper subgroup of G, in the simple case.

PROPOSITION 1

Any subgroup of a connected semisimple Lie group G which contains K is of the form G_N, for some $N \subseteq \{1,2,\ldots,n\}$.

Proof. (i) Let $x \in G \backslash K$ and let H_x denote the subgroup of G generated by K and x. We may write $x = x_1 x_2 \ldots x_n$, where $x_j \in G_j$ for $1 \leqslant j \leqslant n$, and set $N_x = \{1 \leqslant j \leqslant n : x_j \notin K_j\}$. Since x determines x_j up to translation by a central element, and the centre of G lies inside K, x determines N_x uniquely.

For each $j \in N_x$, $H_x \cap G_j$ contains K_j and $x_j K_j x_j^{-1}$. As G_j is simple, Brun's theorem implies that the normaliser of K_j in G_j is K_j, hence $H_x \cap G_j \neq K_j$, so by Brun's theorem again, $H_x \cap G_j = G_j$. It follows that H_x contains G_{N_x}. As G_{N_x} clearly contains K and x, we conclude that $H_x = G_{N_x}$.

(ii) Now let H be an arbitrary subgroup of G containing K and with $H \neq K$. Then

$$H = \bigcup_{x \in H} G_{N_x} = G_N,$$

where $N = \bigcup_{x \in H} N_x$. □

Given semisimple $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we choose a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$ and denote by Σ the set of all roots of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$ (see [He], p. 263, and note that we follow the notation there except that $\mathfrak{a}_\mathfrak{p}$ replaces $\mathfrak{h}_{\mathfrak{p}_0}$ and the subscript 0 on $\mathfrak{g}$ and the subspaces of $\mathfrak{g}$ is dropped). We write $N_K(\mathfrak{a}_\mathfrak{p})$ for the normaliser of $\mathfrak{a}_\mathfrak{p}$ in K.

PROPOSITION 2

There exists $k \in N_K(\mathfrak{a}_\mathfrak{p})$, and some $m \geqslant 1$, such that for all $X \in \mathfrak{a}_\mathfrak{p}$,

$$-X = \sum_{j=1}^{m-1} \mathrm{Ad}(k^j)(X).$$

Proof. For each $\alpha \in \Sigma$, let $r_\alpha : \mathfrak{a}_\mathfrak{p} \to \mathfrak{a}_\mathfrak{p}$ denote the reflection in the hyperplane $\{Y \in \mathfrak{a}_\mathfrak{p} : \alpha(Y) = 0\}$ w.r.t. the restriction to $\mathfrak{a}_\mathfrak{p}$ of the Killing form on $\mathfrak{g}$. We choose a basis of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ from Σ, and set

$$s = r_{\alpha_1} \circ r_{\alpha_2} \circ \cdots \circ r_{\alpha_l},$$

which is a Coxeter element of the Weyl group W of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. Since $\alpha_1, \ldots, \alpha_l$ are linearly independent in $\mathfrak{a}_\mathfrak{p}^*$, we have $s(Y) = Y$ for $Y \in \mathfrak{a}_\mathfrak{p}$ if and only if $r_{\alpha_j}(Y) = Y$ for all $1 \leqslant j \leqslant l$ (c.f. [Ca], Proposition 10.5.6, p. 165). Hence $s(Y) = Y$ if and only if $\alpha_j(Y) = 0$ for all $1 \leqslant j \leqslant l$, which is equivalent to $Y = 0$ since $\alpha_1, \ldots, \alpha_l$ span $\mathfrak{a}_\mathfrak{p}^*$. Hence the linear map $I - s : \mathfrak{a}_\mathfrak{p} \to \mathfrak{a}_\mathfrak{p}$ is invertible.

Let the order of s be m, then from the identity

$$(I + s + \cdots + s^{m-1})(I - s) = I - s^m = 0$$

and the invertibility of $I - s$, it follows that on $\mathfrak{a}_\mathfrak{p}$,

$$I + s + \cdots + s^{m-1} = 0. \tag{1}$$

Because W can also be realised as $N_K(\mathfrak{a}_\mathfrak{p})/C_K(\mathfrak{a}_\mathfrak{p})$, where $C_K(\mathfrak{a}_\mathfrak{p})$ is the centraliser of $\mathfrak{a}_\mathfrak{p}$ in K, we can find $k \in N_K(\mathfrak{a}_\mathfrak{p})$ such that $s = \mathrm{Ad}\, k|_{\mathfrak{a}_\mathfrak{p}}$. Then (1) gives that for all $X \in \mathfrak{a}_\mathfrak{p}$,

$$X + \mathrm{Ad}(k)(X) + \mathrm{Ad}(k^2)(X) + \cdots + \mathrm{Ad}(k^{m-1})(X) = 0,$$

which gives the result. □

COROLLARY 1

There exists $k \in N_K(\mathfrak{a}_p)$ *and* $m \geq 1$ *such that for each* $a \in A = \exp \mathfrak{a}_p$,

$$a^{-1} = (ka)^{m-1} k^{-m+1}$$ □

COROLLARY 2

In any connected semisimple Lie group G, any K-bi-invariant subsemigroup of G is a subgroup containing K.

Proof. Let S be a K-bi-invariant subsemigroup and let $x \in S$, then $x = k_1 a k_2$ for some $a \in A$ and $k_1, k_2 \in K$. Hence $x^{-1} = k_2^{-1} a^{-1} k_1^{-1} \in S$ by Corollary 1. Also $1 \in S$ and so $K \subseteq S$. □

The proof of the theorem stated earlier is now immediate by Propositions 1 and 2, Corollary 2.

Remark. We note the following consequence of the theorem. If G is a connected semisimple Lie group and C is any K-bi-invariant subset of G, then there is some $r \in \mathbb{N}$ such that C^r is a neighbourhood of the identity in $G(C)$, the subgroup of G generated by C.

For, by the theorem above,

$$G(C) = \bigcup_{s=1}^{\infty} C^s,$$

so if λ is Haar measure on $G(C)$, there exists $n \in \mathbb{N}$ such that $\lambda(C^n) > 0$. But we may write $C = KDK$ for $D \subseteq A$, and by Proposition 2, Corollary 2,

$$D^{-1} \subseteq (KDK)^{m-1}$$

and so

$$C^n C^{-n} \subseteq C^{nm}.$$

The result now follows because $C^n C^{-n}$ is a neighbourhood of the identity in $G(C)$, by [W], bottom of page 50.

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