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Badly approximable p-adic integers

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Abstract. It is known that the p-adic integers that are badly approximable by rationals form a null set with respect to Haar measure. We define a $[0, 1]$-valued dimension function on the p-adic integers analogous to Hausdorff dimension in $\mathbf{R}$ and show that with respect to this function the dimension of the set of badly approximable p-adic integers is 1.

Keywords. Diophantine approximation; p-adic numbers; Hausdorff dimension.

Introduction

A real number x is called *badly approximable* if, roughly speaking, there are no rationals p/q such that $x - p/q$ is small compared with q^{-2}. It is well known (see [5]) that the set of badly approximable real numbers has Lebesgue measure zero and Hausdorff dimension 1. As might be expected, we can in an analogous way define the set of badly approximable p-adic integers. It is known (see [6]) that this set is a null set with respect to Haar measure on the group $\mathbf{Z}_p$ of all p-adic integers. In this paper we describe a natural analog of Hausdorff dimension applicable to the space of p-adic integers and we show that with respect to this dimension the dimension of the set of badly approximable p-adic integers is 1.

The proof of this result makes use of an approximation scheme for p-adic numbers developed by Mahler in [7], the essential features of which are recalled in the course of §3 below. We also exploit a method initiated by Billingsley in [2], and further developed by the author in [1], for comparing Hausdorff-like dimension functions defined with respect to arbitrary non-atomic measures. The basic facts about this method are explained in §4. With the aid of Mahler's scheme we construct a measure with respect to which the set of badly approximable numbers has measure 1. We then apply Billingsley's method to complete the proof.

1. Notation and preliminary remarks

We denote by $\mathbf{N}$ the set of strictly positive integers and write $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For a natural number N we denote by $[N]$ the set

$$\{h \in \mathbf{N} : h \leqslant N\}.$$

If z is a complex number we shall always write $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. For any real y_0 we denote by $\mathbf{U}_{y_0}$ the set

$$\{z \in \mathbf{C} : y > y_0\}.$$

We denote by Γ the modular group $\mathrm{SL}_2(\mathbf{Z})$, and by $\mathbf{I}$ the identity of Γ. As usual, we let Γ act on the upper half-plane $\mathbf{U}_0$ in the following way. For

$$\omega = \begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix}$$

in Γ and z in $\mathbf{U}_0$ we put

$$\omega z = \frac{\alpha z + \alpha'}{\beta z + \beta'}.$$

We denote by R the standard fundamental region for this action of Γ given by $R = R_1 \cup R_2$ where

$$R_1 = \{z \in \mathbf{U}_0 : |z| > 1, -\tfrac{1}{2} \leqslant x < \tfrac{1}{2}\}$$

and

$$R_2 = \{z \in \mathbf{U}_0 : |z| = 1, -\tfrac{1}{2} \leqslant x \leqslant 0\}.$$

It is easy to check that for any ξ in R the expression

$$\frac{2(r - \xi s)(r - \bar{\xi} s)}{|\xi - \bar{\xi}|}$$

is a positive definite quadratic form in r and s. We may therefore define a positive-valued function Φ_ξ on $\mathbf{R} \times \mathbf{R}$ by setting

$$\Phi_\xi(r, s) = \left(\frac{2(r - \xi s)(r - \bar{\xi} s)}{|\xi - \bar{\xi}|}\right)^{1/2}.$$

For a fixed prime p, we denote by $\mathbf{Z}_p$ the ring of p-adic integers with the usual valuation $|\,|_p$. Thus a typical element ρ of $\mathbf{Z}_p$ is a sequence $(\rho_n)_{n \in \mathrm{N}_0}$, where each ρ_n is an element of the additive group $\mathbf{Z}/p^n\mathbf{Z}$, and for each n the natural homomorphism $\mathbf{Z}/p^{n+1}\mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$ sends ρ_{n+1} to ρ_n. Given $\rho = (\rho_n)_{n \in \mathrm{N}_0}$, $\rho' = (\rho'_n)_{n \in \mathrm{N}_0}$ in $\mathbf{Z}_p$ we define

$$\rho + \rho' = (\rho_n + \rho'_n)_{n \in \mathrm{N}_0}$$

and

$$\rho\rho' = (\rho_n \rho'_n)_{n \in \mathrm{N}_0}.$$

We define $|\rho|_p = p^{-v}$ where $v = v(\rho)$ is the least integer in $\mathbf{N}_0$ such that ρ_{v+1} is different from zero.

We say that ρ, ρ' are congruent modulo p^β, and write $\rho \equiv \rho' \pmod{p^\beta}$, if $|\rho - \rho'|_p \leqslant p^{-\beta}$.

We equip $\mathbf{Z}_p$ with the topology induced by the metric $d(\rho, \rho') = |\rho - \rho'|_p$. The space $\mathbf{Z}_p$ is homeomorphic to the topological product $[p]^\omega$, where $[p]$ is equipped with the discrete topology. Therefore $\mathbf{Z}_p$ is compact.

We put

$$B_h(\rho) = \{\rho' \in \mathbf{Z}_p : \|\rho - \rho'\|_p \leqslant p^{-h}\}$$

and

$$\mathfrak{B} = \{B_h(\rho) : \rho \in \mathbf{Z}_p,\ h \in \mathbf{N}_0\}.$$

The set $\mathfrak{B}$ is a basis for $\mathbf{Z}_p$ consisting of closed open sets. An element of $\mathfrak{B}$ will be called a *sphere*. The reader will observe that the sphere $B_h(\rho)$ is the set of all ρ' in $\mathbf{Z}_p$ with $\rho'_h = \rho_h$. In sections 4–5 below we shall persistently abuse notation by writing ρ_h in place of $B_h(\rho)$.

Let $(\mathbf{Z}_p, \mathscr{E}, \mu)$ be a probability space on $\mathbf{Z}_p$, where $\mathscr{E}$ is the σ-algebra generated by $\mathfrak{B}$. Let μ be any probability measure on $\mathbf{Z}_p$ that is non-atomic, i.e. $\mu(\{\rho\}) = 0$ for all $\rho \in \mathbf{Z}_p$. Suppose $\gamma > 0$. For $\theta > 0$ and $M \subset \mathbf{Z}_p$, write

$$\ell^{\gamma}_{\mu,\theta}(M) = \inf \sum (\mu(B_{h_i}(\rho^{(i)})))^{\gamma}.$$

Here the infimum is taken over all coverings of M by subsets of $\mathfrak{B}$ of the form $\{B_{h_i}(\rho^{(i)}) : i \in \mathbf{N}\}$ such that $\mu(B_{h_i}(\rho^{(i)})) < \theta$ for all $i \in \mathbf{N}$. The (not necessarily finite) limit

$$\ell^{\gamma}_{\mu}(M) = \lim_{\theta \to 0} \ell^{\gamma}_{\mu,\theta}(M)$$

exists for all M. For a simple proof the ℓ^{γ}_{μ} as thus defined is an outer measure see [2], pp. 136, 141. It can be shown ([2], pp. 136–137, 141) that for each $M \subset \mathbf{Z}_p$ there exists a unique real number $\Delta = \Delta_\mu(M)$ such that $\ell^{\gamma}_{\mu} = \infty$ for all $\gamma < \Delta$ and $\ell^{\gamma}_{\mu} = 0$ for all $\gamma > \Delta$.

We define $\eta : \mathfrak{B}(\mathbf{Z}_p) \to \mathbf{R}$ by $\eta(B_h(a)) = p^{-h}$ for all $a \in \mathbf{Z}_p$, $h \in \mathbf{N}_0$. Then by the Carathéodory-Hopf extension theorem ([4], § 13, Theorem A), η can be extended to a probability measure on $\mathbf{Z}_p$, also denoted by η. The measure η is clearly translation-invariant and therefore by the Haar uniqueness theorem ([3], pp. 309–310) it coincides with Haar measure on $\mathbf{Z}_p$. We call $\Delta_\eta(M)$ the *Hausdorff dimension* of M. This terminology is appropriate because, as is proved in [2], p. 140, Hausdorff dimension in $\mathbf{R}$ can be defined by the same procedure with Lebesgue measure in place of η.

2. Statement of the result

For each positive real number τ let us say that a p-adic integer ρ is *badly approximable* (τ) and write $\rho \in J(\tau)$ if for all a, b in $\mathbf{Z}$ we have

$$|a + b\rho|_p \geqslant \tau(\max|a|, |b|)^{-2}.$$

Let us say that ρ is *badly approximable* if it is badly approximable (τ) for some $\tau > 0$. We denote the set of badly approximable p-adic integers by J. Thus

$$J = \bigcup_{\tau > 0} J(\tau).$$

It is well known (see for example [6], Th. 4.23) that $\eta(J) = 0$. Thus it is of interest to determine $\Delta_\eta(J)$. Our purpose in this paper is to prove the following:

Theorem 2.1. *We have* $\Delta_\eta(J) = 1$.

We first recast this result in a more convenient form. For ξ in $\mathbf{R}$ and $\tau > 0$ let $J_\xi(\tau)$ denote the set of ρ such that

$$|a + b\rho|_p \geqslant \tau(\Phi_\xi(a, b))^{-2} \tag{2.1}$$

for all a, b in $\mathbf{Z}$, and write

$$J_\xi = \bigcup_{\tau>0} J_\xi(\tau).$$

Since $(\Phi_\xi)^2$ is positive definite, a simple computation shows that J is identical with J_ξ for each ξ. Therefore Theorem 2.1 is a consequence of the following result, which, though more detailed than Theorem 2.1, appears to be no harder to prove.

Theorem 2.2. *There is a constant C depending only on p such that for any ξ in R and any K in* $\mathbf{N}$ *we have*

$$\Delta_\pi(J_\xi(p^{-K-C})) \geqslant 1 - \frac{1}{2K}.$$

Theorem 2.2 is analogous to the following result on Diophantine approximation in $\mathbf{R}$. Call r in $\mathbf{R}$ *badly approximable* if there is a constant τ such that $|a + br| \geqslant \tau b^{-1}$ for all a, b in $\mathbf{Z}$. Then we have:

Theorem 2.3. *The Hausdorff dimension of the set of badly approximable real numbers is* 1.

This was established by V Jarnik in [5], a pioneering paper in which dimension theory was applied for the first time in the study of Diophantine approximation. The proof of Theorem 2.3 depends on a special feature of $\mathbf{R}$, namely the availability of an appropriate continued fraction algorithm. It turns out that badly approximable real numbers are those whose simple continued fractions have bounded partial denominators.

To prove Theorem 2.2 we shall use an approximation scheme for p-adic integers developed by K Mahler in [7]. As Mahler points out, his scheme is a working substitute for a continued fraction algorithm in the sense that it yields all "good" approximations to a p-adic integer ρ, that is all potential counterexamples to (2.1). Lemma 3.2 below is a more precise statement of this fact. As we shall see, the badly approximable p-adic integers can be nicely characterized in the language of Mahler's scheme.

In the early days of research on Hausdorff dimension it was notoriously difficult to find sharp lower bounds for the dimension of sets like $J(\tau)$. It is now, in many cases, much easier, thanks to a method developed by P Billingsley which we review briefly in section 4 before applying it to the present problem.

3. Mahler's approximation scheme

Given a p-adic integer ρ we define, for each n in $\mathbf{N}_0$, an integer $E_n = E_n(\rho)$ by means of the relations

$$0 \leqslant E_n < p^n$$

and

$$|E_n - \rho|_p \leqslant p^{-n}.$$

It is easy to check that exactly one integer E_n satisfies these two relations.

Fix ξ in R, and for each n in $\mathbf{N}_0$ define a complex number $Z_n = Z_n(\rho)$ by setting

$$Z_n = \frac{E_n + \xi}{p^n}.$$

Further for each n in $\mathbf{N}_0$ let

$$z_n = z_n(\rho) = x_n + iy_n = x_n(\rho) + iy_n(\rho)$$

be the unique element of R that is equivalent to Z_n under the action of Γ on $\mathbf{U}_0$.

Suppose that

$$\omega_n = \begin{pmatrix} c_n & c'_n \\ b_n & b'_n \end{pmatrix}$$

is the element of Γ satisfying

$$\omega_n z_n = Z_n,$$

and write

$$a_n = p^n c_n - E_n b_n, \quad a'_n = p^n c'_n - E_n b'_n.$$

Also for each n in $\mathbf{N}_0$ write

$$T_n = \begin{pmatrix} a_n & a'_n \\ b_n & b'_n \end{pmatrix}$$

and for each n in $\mathbf{N}$ write

$$\Omega_n = T_{n-1}^{-1} T_n.$$

We can now state the fundamental results due to Mahler on which our proof of Theorem 2.2 will be based.

Lemma 3.1. ([7], p. 12). *For any ρ in $\mathbf{Z}_p$ and any n in $\mathbf{N}_0$ we have*

$$y_n(\rho) = \frac{p^n}{(\Phi_\xi(a_n, b_n))^2}.$$

Lemma 3.2. ([7], p. 51 (*Theorem* 18)). *Let a, b in $\mathbf{Z}$ satisfy*

$$|a + b\rho|_p \leqslant p^{-n},$$

$$\Phi_\xi(a, b) > 0.$$

Then

$$\Phi_\xi(a, b) \geqslant \Phi_\xi(a_n, b_n).$$

Lemma 3.3. ([7], p. 15). *The subset $M(p)$ of $\mathbf{GL}_2(\mathbf{Z}_p)$ defined by*

$$M(p)=\{\Omega_n(\rho):n\in\mathbf{N},\rho\in\mathbf{Z}_p\}$$

is a finite set and the determinant of each element of $M(p)$ is p. Moreover a matrix Ω is in $M(p)$ if and only if $p\Omega^{-1}$ is in $M(p)$.

Note. It turns out that $M(p)$ is independent of the choice of ξ. However we do not require this fact.

Lemma 3.4. ([7], p. 14). *For each n in $\mathbf{N}_0$ we have*

$$z_{n+1}=\Omega_{n+1}^{-1}z_n.$$

Lemma 3.5. ([7], p. 14). *For each n in $\mathbf{N}_0$ the integers a_n and b_n are relatively prime.*

We now derive some consequences of the preceding lemmas.

Lemma 3.6. *The set of badly approximable p-adic integers coincides with the set of those ρ such that $y_n(\rho)$ remains bounded as n goes to infinity. More precisely, for each $\tau>0$ we have*

$$J_\xi(\tau)=\{\rho\in\mathbf{Z}_p:y_n(\rho)\leqslant\tau^{-1}(\forall n\in\mathbf{N}_0)\}.$$

Proof. It suffices to prove the second statement. Suppose ρ is in $J_\xi(\tau)$. Then by the definition of $J_\xi(\tau)$ we have for each n in $\mathbf{N}_0$ that

$$|a_n+b_n\rho|_p\geqslant\tau(\Phi_\xi(a_n,b_n))^{-2}. \tag{3.1}$$

By the definition of a_n, b_n we have

$$p^{-n}\geqslant|a_n+b_n\rho|_p. \tag{3.2}$$

By Lemma 3.1 we have

$$(\Phi_\xi(a_n,b_n))^{-2}=y_n(\rho)p^{-n}. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) we have

$$1\geqslant\tau y_n(\rho),$$

which proves that $J_\xi(\tau)$ is included in the set of those ρ such that $y_n(\rho)$ never exceeds τ^{-1}.

To prove the reverse inclusion, suppose that ρ satisfies $y_n(\rho)\leqslant\tau^{-1}$ for all n in $\mathbf{N}_0$. Let a, b be any integers and define $h=h(a,b)$ by the relation

$$|a+b\rho|_p=p^{-h}.$$

Then using Lemmas 3.1 and 3.2 we have

$$\begin{aligned}\tau|a+b\rho|_p^{-1}&=\tau p^h\\&\leqslant(y_n(\rho))^{-1}p^h\\&=(\Phi_\xi(a_n,b_n))^2\\&\leqslant(\Phi_\xi(a,b))^2\end{aligned}$$

so that ρ is in $J_\xi(\tau)$ as required.

Lemma 3.7. *There exists a constant C depending only on p such that for any ρ in $\mathbf{Z}_p$, whenever $y_n(\rho) > p^C$ we have*

$$y_{n+1} = p^{\pm 1} y_n \tag{3.4}$$

and if $n \geqslant 1$ we have

$$y_{n-1} = p^{-1} y_n. \tag{3.5}$$

Proof. By Lemmas 3.3 and 3.4 we have for any ρ in $\mathbf{Z}_p$ and any n in $\mathbf{N}_0$ that

$$z_{n+1} = \Omega z_n,$$

where Ω is in $M(p)$. We write

$$\Omega = \begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix}.$$

For a fixed complex number z, the subset $\Upsilon(z)$ of $[0, 2\pi)$ defined by

$$\Upsilon(z) = \{|\arg \Omega z| : \Omega \in M(p)\}$$

is finite, by Lemma 3.3. Now suppose z is in R. We see easily that when y is large enough we have $|\arg \Omega z| < \frac{1}{2}$ for all Ω satisfying $\alpha\beta \neq 0$. Moreover if $\alpha = 0$ we see that $|z| < 1$ whenever y is sufficiently large. Thus there is a constant C such that for z in R, $y > p^C$, and Ω in $M(p)$ we have Ωz in R only when $\beta = 0$.

But then, since by Lemma 3.3 we have $\det \Omega = p$, we find that either $\alpha = p$ or $\beta' = p$. This establishes (3.4), and if $n \geqslant 1$ the same argument with z_{n-1} in place of z_n establishes (3.5).

Lemma 3.8. *Suppose that for some n in $\mathbf{N}_0$ we have*

$$y_n(\rho) > y_{n+1}(\rho) > p^C,$$

where C is the same constant as in Lemma 3.7. *Then we have*

$$y_{n+2} = p^{-1} y_{n+1}.$$

Proof. In view of Lemma 3.7 we need only show that $y_{n+2} \neq p y_{n+1}$. We know, using Lemma 3.7, that $y_n = p y_{n+1}$. Therefore by Lemma 3.4 we can write

$$\Omega_{n+1} = \begin{pmatrix} p & \alpha' \\ 0 & 1 \end{pmatrix},$$

and given z_{n+1} in R there is just one choice of α' such that $z_n = \Omega_{n+1} z_{n+1}$ is in R. Therefore the relation $y_{n+2} = p y_{n+1}$ would imply

$$\Omega_{n+2}^{-1} = p^{-1} \begin{pmatrix} p & \alpha' \\ 0 & 1 \end{pmatrix} = \Omega_{n+1},$$

so that

$$\Omega_{n+1}\Omega_{n+2} = pI,$$

and then each component of T_{n+2} would be divisible by p, which contradicts Lemma 3.5. We conclude that $y_{n+2} \neq py_{n+1}$ as claimed.

4. Billingsley's lower bound for the dimension of a set

A version of the following lemma, relating to subsets of $[0, 1]$, is proved in [2], pp. 144–145, and the proof carries over to the present setting without significant alteration. Recall that we agreed to abuse notation by writing ρ_j in place of $B_j(\rho)$.

Lemma 4.1. *For any non-atomic Borel measures λ, μ on $\mathbf{Z}_p$ and for any $\delta \geqslant 0$, if*

$$M \subset \left\{\rho \in \mathbf{Z}_p : \liminf_{j \to \infty} \frac{\log \lambda(\rho_j)}{\log \mu(\rho_j)} \geqslant \delta\right\}$$

then $\Delta_\mu(M) \geqslant \delta \Delta_v(M)$.

Note: if either of the real numbers a, b is either 0 or 1, then $\log a/\log b$ is defined equal to 0, 1 or ∞ according as $a > b$, $a = b$ or $a < b$. The logarithms can be taken to any positive base except 1, *and in what follows we shall take all logarithms to the base p.*

In order to apply Lemma 4.1 to the problem at hand we need to construct a measure v on $\mathbf{Z}_p$ such that

$$\Delta_v(J_\xi(p^{-K-C})) = 1$$

and such that

$$J_\xi(p^{-K-C}) \subset \left\{\rho \in \mathbf{Z}_p : \liminf_{j \to \infty} \frac{\log v(\rho_j)}{\log \eta(\rho_j)} \geqslant 1 - \frac{1}{2K}\right\}.$$

The construction of such a measure is made possible by the following result, which is a special case of Lemma 5.2 in [1]. If u is any sphere, we denote by $\sigma(u)$ the set of maximal proper subspheres of u.

Lemma 4.2. *Suppose that $\mu' : \mathfrak{B} \setminus \{\mathbf{Z}_p\} \to [0, 1]$ satisfies*

$$\sum_{v \in \sigma(u)} \mu'(v) = 1$$

for all u in $\mathfrak{B}$. Then there is a unique Borel probability measure μ on $\mathbf{Z}_p$ satisfying

$$\mu(u)\mu'(v) = \mu(v)$$

for all u, v in *$\mathfrak{B}$ with v in $\sigma(u)$.*

5. Proof of Theorem 2.2

Let K be a fixed integer greater than 0, and let C be the constant whose existence is guaranteed by Lemma 3.7. For ρ in $\mathbf{Z}_p$ write

$$t_n(\rho) = \#\{\rho'_n \in \sigma(\rho_{n-1}): \log y_n(\rho') \leqslant K + C\}.$$

One checks easily that $y_n(\rho')$ is actually determined by ρ'_n.

We show that if

$$C < \log y_{n-1}(\rho) \leqslant K + C$$

then

$$\#(t_n(\rho)) \geqslant 1. \tag{5.1}$$

Suppose the contrary. Then for every maximal subsphere ρ'_n of ρ_{n-1} we have

$$\log y_n(\rho') > K + C.$$

But there are at least two maximal subspheres ρ'_n contained in ρ_n (in fact there are p of them) and therefore there are at least two Ω in $M(p)$ satisfying

$$\Omega z_{n-1}(\rho) \in R \cap \mathbf{U}_{p^{K+C}}.$$

As in the proof of Lemma 3.8 any Ω satisfying

$$\Omega z_{n-1}(\rho) \in \mathbf{U}_{p^{K+C}}$$

must be of the form

$$\Omega = \begin{pmatrix} p & \alpha' \\ 0 & 1 \end{pmatrix},$$

and there is just one choice of $\alpha' \in \mathbf{Z}$ such that $\Omega z_{n-1}(\rho)$ is in R. Thus we have arrived at a contradiction and must conclude that (5.1) holds as claimed.

We may therefore define a function $v' = v'_K$ on $\mathfrak{B} \backslash \{\mathbf{Z}_p\}$ with values in $[0, 1]$ as follows.

Case (i). If

$$K - 1 + C < \log y_{n-1}(\rho) \leqslant K + C$$

and $\log y_n(\rho) > K + C$, we set

$$v'(\rho_n) = 0.$$

Case (ii). If

$$K - 1 + C < \log y_{n-1}(\rho) \leqslant K + C$$

and $\log y_n(\rho) \leqslant K + C$, we set

$$v'(\rho_n) = (t_n(\rho))^{-1}.$$

Case (iii). If $\log y_{n-1}(\rho)$ lies in the complement of the interval $(K-1+C, K+C]$, we set

$$v'(\rho_n) = p^{-1}.$$

One checks easily (using the definition of $t_n(\rho)$) that v' satisfies the hypotheses of Lemma 4.2, and so there is a probability measure v on $\mathbf{Z}_p$ satisfying (4.1).

To check that v is non-atomic, choose ρ in $\mathbf{Z}_p$, so that

$$\rho = \bigcap_{n \in \mathbf{N}_0} \rho_n.$$

We must show that $v(\rho_n)$ goes to 0 as n goes to ∞. By (4.1) and straightforward induction we have

$$v(\rho_n) = \prod_{1 \leqslant j \leqslant n} v'(\rho_n).$$

Now by Lemma 3.7 we cannot have both

$$K - 1 + C < \log y_{n-1}(\rho) \leqslant K + C$$

and

$$K - 1 + C < \log y_n(\rho) \leqslant K + C.$$

Hence for infinitely many n case (ii) of the definition of v' does not apply and for such n we have

$$v'(\rho_n) \leqslant p^{-1} < 1.$$

Therefore $v(\rho_n) \to 0$ as required, so v is non-atomic.

We now verify that $v(J(p^{-K-C})) = 1$. If ρ is in the complement of $J(p^{-K-C})$ then by Lemma 3.6 for some n in $\mathbf{N}_0$ we have $\log y_n(\rho) > K + C$. Choose N to be the least integer with this property. Then by Lemma 3.7 we have

$$K - 1 + C < \log y_{N-1}(\rho) \leqslant K + C.$$

Therefore case (i) of the definition of v' gives $v'(\rho_N) = 0$, so also $v(\rho_N) = 0$. Thus the complement of $J(p^{-K-C})$ is covered by elements of $\mathfrak{B}$ each of which has measure zero with respect to v. Since $\mathfrak{B}$ is countable we have $v(J(p^{-K-C})) = 1$ as claimed. Our next objective is to show that for all ρ in $J(p^{-K-C})$ we have

$$\liminf_{N \to \infty} \frac{\log v(\rho_N)}{\log \eta(\rho_n)} \geqslant 1 - \frac{1}{2K}. \tag{5.2}$$

For ρ in $J(p^{-K-C})$ let $H = H(\rho)$ be the subset of $\mathbf{N}_0$ consisting of those n for which

$$K - 1 + C < \log y_n(\rho).$$

Let ρ be in $J(p^{-K-C})$, and choose n in $H(\rho)$. By the choice of ρ we have $\log y_n \leqslant K + C$, and also by Lemma 3.6 we have $\log y_{n+1} \leqslant K + C$. Thus by Lemma 3.7 and the fact that $K - 1 + C < \log y_n$ we have

$$\log y_{n+1} = -1 + \log y_n > K - 2 + C.$$

If $K \geqslant 2$ we then have $\log y_{n+1} > C$, and since $\log y_n > \log y_{n+1}$ Lemma 3.8 implies that we have

$$\log y_{n+2} = -2 + \log y_n > K - 3 + C.$$

Continuing in this way we find that

$$\log y_{n+h} = -h + \log y_n > K - h + C \geqslant C$$

for each $h = 0, \ldots, K$. A further application of Lemma 3.7 shows that

$$\log y_{n+K+h} \leqslant K + C$$

for each $h = 0, \ldots, K-1$.

Thus the difference between consecutive elements of $H(\rho)$ is at least $2K$, so we have

$$\#([N] \backslash H(\rho)) > N - 1 - \frac{N}{2K} \tag{5.3}$$

for all N in $\mathbf{N}_0$.

Now if ρ is in $J(p^{-K-C})$, we have for each n in $\mathbf{N}$ either

$$v'(\rho_n) = (t_n(\rho))^{-1}$$

if $n-1$ is in $H(\rho)$, or

$$v'(\rho_n) = p^{-1}$$

otherwise. Therefore for each N in $\mathbf{N}_0$ we have

$$\log v(\rho_N) = -\sum_{n \in [N] \cap H(\rho)} \log t_{n+1}(\rho) - \#([N] \backslash H(\rho)).$$

Also we clearly have

$$\log \eta(\rho_N) = -N,$$

and so using (5.3) and the fact that $t_n(\rho) \geqslant 1$ we have

$$\frac{\log v(\rho_n)}{\log \eta(\rho_n)} > 1 - \frac{1}{N} - \frac{1}{2K}.$$

Therefore letting N go to ∞ we have (5.2).

Since $v(J(p^{-K-C})) = 1$, we certainly have

$$\Delta_v(J(p^{-K-C})) = 1,$$

and now an appeal to Lemma 4.1 with $\lambda = v$, $\mu = \eta$, and $\delta = 1 - (1/2K)$ completes the proof.

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