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A note on integrable solutions of Hammerstein integral equations

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Abstract. We derive a set of sufficient conditions for the existence of solutions of a Hammerstein integral equation.

Keywords. Hammerstein integral equation; Caratheodory condition; Lusin theorem; Scorza Dragoni theorem; Schauder fixed point theorem.

1. Introduction

One of the most frequently investigated integral equations in nonlinear functional analysis is the Hammerstein equation

$$x(t) = a(t) + \int_0^1 k(t,s) f(t, x(s))\,\mathrm{d}s \quad t \in [0,1]. \tag{1}$$

Such an equation has been studied in several papers and monographs [1–6]. Existence theorems for eq. (1) can be obtained by applying various fixed point principles. In [2] Banas proved an existence theorem for (1) using the measure of weak noncompactness. On the other hand Emmanuele [5] established an existence theorem for the same equation using Schauder's fixed point theorem. In this paper we shall prove the existence of solutions of the following nonlinear Hammerstein equation

$$x(t) = g(t, x(t)) + \int_0^1 k(t,s) f(t, x(\sigma(s)))\,\mathrm{d}s \quad t \in [0,1] \tag{2}$$

by suitably adopting the technique of [5]. The result generalizes the result of [5].

2. Existence theorem

In order to prove existence theorem for (2) we shall first prove the following theorem:

Theorem 1. *Assume that*

(i) $a_1 \in L^1[0,1]$ *and* $a_1(t) \geqslant 0$ *for all* $t \in [0,1]$.
(ii) $f:[0,1] \times R \to R$ *satisfies Caratheodory condition and there exist* $a_2 \in L^1[0,1]$ *and* $b_2 > 0$ *such that*

$$|f(t, x(t))| \leqslant a_2(t) + b_2 |x(t)|$$

for a.e. $t \in [0,1]$ *and all* $x \in R$.

(iii) $k:[0,1]\times[0,1]\to R^+$ *is measurable with respect to both variables and is such that the integral* operator

$$Kx(t)=\int_0^1 k(t,s)x(s)\,\mathrm{d}s \text{ maps } L^1[0,1] \text{ into itself.}$$

(iv) $\sigma:[0,1]\to[0,1]$ *is absolutely continuous and there exists a constant* $M>0$ *such that* $\sigma'(t)\geqslant M$ *for all* $t\in[0,1]$.

(v) $b_1+\dfrac{b_2\|K\|}{M}<1.$

Then there exists a unique a.e. non-negative function $\varphi\in L^1[0,1]$ *such that*

$$\varphi(t)=\frac{a_1(t)}{1-b_1}+\frac{1}{1-b_1}\int_0^1 k(t,s)[a_2(s)+b_2\varphi(\sigma(s))]\,\mathrm{d}s.$$

Proof. Define a function $\psi:[0,1]\to R$ by

$$\psi(t)=a_1(t)+\int_0^1 k(t,s)a_2(s)\,\mathrm{d}s.$$

Put $B_r=\{x\in L^1[0,1]:\|x\|\leqslant r\}$ where $r=\dfrac{M\|\psi\|}{M-b_1M-b_2\|K\|}$.

Define an operator $F:L^1[0,1]\to L^1[0,1]$ by

$$Fx(t)=\frac{a_1(t)}{1-b_1}+\frac{1}{1-b_1}\int_0^1 k(t,s)[a_2(s)+b_2x(\sigma(s))]\,\mathrm{d}s.$$

From our assumptions for $x\in B_r$ we have

$$\begin{aligned}
\|Fx\| &= \int_0^1 |Fx(t)|\,\mathrm{d}t\\
&\leqslant \frac{1}{1-b_1}\int_0^1 a_1(t)\,\mathrm{d}t+\frac{1}{1-b_1}\int_0^1\left|\int_0^1 k(t,s)[a_2(s)+b_2x(\sigma(s))]\,\mathrm{d}s\right|\mathrm{d}t\\
&\leqslant \frac{1}{1-b_1}\int_0^1\left[a_1(t)+\int_0^1 k(t,s)a_2(s)\,\mathrm{d}s\right]\mathrm{d}t\\
&\quad+\frac{1}{(1-b_1)M}\int_0^1\int_0^1 k(t,s)b_2|x(\sigma(s))|\sigma'(s)\,\mathrm{d}s\,\mathrm{d}t\\
&\leqslant \frac{1}{1-b_1}\|\psi\|+\frac{1}{(1-b_1)M}b_2\|K\|\,\|x\|\\
&\leqslant \frac{1}{1-b_1}\|\psi\|+\frac{1}{(1-b_1)M}b_2\|K\|r=r.
\end{aligned}$$

Thus we have $F(B_r)\subset B_r$. If we define $B_r^+=\{x\in B_r:x(t)\geqslant 0 \text{ a.e.}\}$ then $F(B_r^+)\subset B_r^+$. Also B_r^+ is a complete metric space, since B_r^+ is a closed subset of $L^1[0,1]$.

Now for any two elements $x, y \in B_r^+$ we have

$$\|Fx - Fy\| \leqslant \frac{1}{1-b_1}\int_0^1 \left| \int_0^1 k(t,s) b_2 [x(\sigma(s)) - y(\sigma(s))] \, \mathrm{d}s \right| \mathrm{d}t$$
$$\leqslant \frac{1}{(1-b_1)M} b_2 \|K\| \, \|x-y\| \, \|x-y\|.$$

On applying contraction fixed point theorem we get a fixed point for F. This proves Theorem 1.

Theorem 2. *Assume that*

(i) $g:[0,1]\times R \to R$ *satisfies Caratheodory conditions and there exist* $a_1 \in L^1[0,1]$ *and* $b_1 > 0$ *such that*

$$|g(t, x(t))| \leqslant a_1(t) + b_1 |x(t)|$$

for a.e. $t \in [0,1]$ *and for* $x \in R$ *and*

$$|g(t, x(t) \cdot g(s, x(s))| \leqslant \omega(|t-s|)$$

where $\omega(|t-s|) \to 0$ *as* $t \to s$.

(ii) $f:[0,1]\times R \to R$ *satisfies Caratheodory condition and there exist* $a_2 \in L^1[0,1]$ *and* $b_2 > 0$ *such that*

$$|f(t, x(t))| \leqslant a_2(t) + b_2 |x(t)|$$

for a.e. $t \in [0,1]$ *and all* $x \in R$.

(iii) $k:[0,1]\times[0,1] \to R^+$ *satisfies Caratheodory condition and is measurable with respect to the second variable. Also the integral operator*

$$Kx(t) = \int_0^1 k(t,s) x(s) \, \mathrm{d}s \text{ maps } L^1[0,1] \text{ into itself.}$$

(iv) $\sigma:[0,1]\to[0,1]$ *is absolutely continuous and there exists a constant* M *such that* $\sigma'(t) \geqslant M$ *for all* $t \in [0,1]$.

(v) $b_1 + \dfrac{b_2 \|K\|}{M} < 1.$

Then (2) has a solution in $L^1[0,1]$.

Proof. Since all the assumptions of Theorem 1 are satisfied, there exists a unique a.e. non-negative function φ such that

$$\varphi(t) = \frac{1}{1-b_1}\left\{ a_1(t) + \int_0^1 k(t,s)[a_2(s) + b_2 \varphi(\sigma(s))] \, \mathrm{d}s \right\}.$$

First let us assume $\varphi = 0_{L^1[0,1]}$ in $L^1[0,1]$. In this case, if we take

$$y(t) = g(t, \varphi(t)) + \int_0^1 k(t,s) f(t, \varphi(\sigma(s))) \, \mathrm{d}s$$

then

$$|y(t)| \leqslant a_1(t) + b_1\varphi(t) + \int_0^1 k(t,s)[a_2(s) + b_2\varphi(\sigma(s))]\,\mathrm{d}s = \varphi(t)$$

and so $y(t) = 0$. Therefore $\varphi = 0_{L^1[0,1]}$ is the solution of (2). Now, assume that $\varphi \neq 0_{L^1[0,1]}$. Define a set Q in $L^1[0,1]$ by

$$Q = \{x \in L^1[0,1] : |x(t)| \leqslant \varphi(t) \text{ a.e.}\}.$$

Then clearly Q is nonempty, bounded, closed and convex set in $L^1[0,1]$. Define an operator $H : L^1[0,1] \to L^1[0,1]$ by

$$Hx(t) = g(t, x(t)) + \int_0^1 k(t,s) f(t, x(\sigma(s)))\,\mathrm{d}s.$$

Then according to our assumptions H is continuous and for $x \in Q$, we have

$$\begin{aligned} |Hx(t)| &\leqslant a_1(t) + b_1|x(t)| + \int_0^1 k(t,s)[a_2(s) + b_2|x(\sigma(s))|]\,\mathrm{d}s \\ &\leqslant a_1(t) + b_1\varphi(t) + \int_0^1 k(t,s)[a_2(s) + b_2\varphi(\sigma(s))]\,\mathrm{d}s \\ &= \varphi(t). \end{aligned}$$

Therefore $H(Q) \subset Q$. Now we shall prove that $H(Q)$ is relatively compact. Using Lusin's and Scorza-Dragoni's theorems [see 5] for each positive integer n there exists a closed set $A_n \subset [0,1]$ such that $m(A_n^c) < (1/n)$ and $a_1|_{A_n}, \varphi|_{A_n}, k|_{A_n \times [0,1]}$ are uniformly continuous. Now let (y_k) be a sequence in Q. For $t', t'' \in A_n$ we have

$$\begin{aligned} |Hy_k(t') - Hy_k(t'')| &\leqslant |g(t', y_k(t')) - g(t'', y_k(t''))| \\ &\quad + \int_0^1 |k(t',s) - k(t'',s)|[a_2(s) + b_2\varphi(\sigma(s))]\,\mathrm{d}s \\ &\leqslant \omega(|t' - t''|) + \int_0^1 |k(t',s) - k(t'',s)|[a_2(s) + b_2\varphi(\sigma(s))]\,\mathrm{d}s. \end{aligned}$$

This proves that (Hy_k) is a sequence of equicontinuous functions on A_n. Also for every $t \in A_n$ we have

$$|Hy_k(t)| \leqslant a_1(t) + b_1\varphi(t) + \int_0^1 k(t,s)[a_2(s) + b_2\varphi(\sigma(s))]\,\mathrm{d}s.$$

Because of the continuity of a_1 and φ on the compact set A_n and k on the compact set $A_n \times [0,1]$ the sequence (Hy_k) is equibounded on A_n. By applying the Ascoli–Arzela theorem we get for each n there exists a subsequence $(y_{k(h)})$ of (y_k) such that $(Hy_{k(h)})$ is a Cauchy sequence in the space $C^\circ(A_n)$ of all equicontinuous and equibounded functions on A_n. Now, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_A \varphi(s)\,\mathrm{d}s < (\varepsilon/4)$ whenever $m(A) < \delta$. Choose a positive integer N such that $(1/N) < \delta$. Then $m(A_N^c) < \delta$. Therefore

$$\int_{A_N^c} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)|\,\mathrm{d}t \leqslant \int_{A_N^c} \varphi(t)\,\mathrm{d}t + \int_{A_N^c} \varphi(t)\,\mathrm{d}t < \frac{\varepsilon}{2}.$$

Also

$$\int_{A_N} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)|\,\mathrm{d}t < \frac{\varepsilon}{2}$$

for sufficiently large h' and h'' since (Hy_k) is a Cauchy sequence in $C^\circ(A_N)$. Hence

$$\begin{aligned}\|Hy_{k(h')} - Hy_{k(h'')}\|_{L^1[0,1]} &= \int_{A_N^c} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)|\,\mathrm{d}t \\ &\quad + \int_{A_N} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)|\,\mathrm{d}t \\ &< \varepsilon\end{aligned}$$

for sufficiently large h' and h''. Therefore $(Hy_{k(h)})$ is a convergent subsequence of the sequence (Hy_k) in $L^1[0,1]$. This proves the relative compactness of $H(Q)$. Applying the Schauder fixed theorem we get a fixed point for H. This proves our theorem.

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