

Combinatorial manifolds with complementarity

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Abstract. A simplicial complex is said to satisfy complementarity if exactly one of each complementary pair of nonempty vertex-sets constitutes a face of the complex.

We show that if a d -dimensional combinatorial manifold M with n vertices satisfies complementarity then $d = 0, 2, 4, 8$ or 16 with $n = 3d/2 + 3$ and $|M|$ is a “manifold like a projective plane”. Arnoux and Marin had earlier proved the converse statement.

Keywords. Combinatorial manifolds; complementarity.

1. Introduction

Recall that a *simplicial complex* K is a collection of nonempty sets (sets of *vertices*) such that all nonempty subsets of a member of the collection are again members. A member of K with $i + 1$ vertices is called an *i -face* (or simplex of dimension i). For $\sigma \in K$ $Lk(\sigma) (= link\ of\ \sigma) := \{\gamma \in K; \gamma \cap \sigma = \emptyset, \gamma \cup \sigma \in K\}$. A simplicial complex may be thought of as a prescription for the construction of a topological space by pasting together geometric simplexes. The topological space thus obtained from a simplicial complex K is called a *polyhedron* and is denoted by $|K|$. Let K_1 and K_2 be two simplicial complexes. A map $f: |K_1| \rightarrow |K_2|$ is called PL if there are subdivisions K'_1 and K'_2 of K_1 and K_2 respectively such that $f: K'_1 \rightarrow K'_2$ is simplicial. We write $|K_1| \approx |K_2|$ if $|K_1|$ and $|K_2|$ are PL homeomorphic. A simplicial complex K (respectively $|K|$) is called a *combinatorial d -manifold* (respectively *PL d -manifold*) if for every vertex v in K $Lk(v)$ is a $(d - 1)$ -dimensional combinatorial sphere.

In 1962, Eells and Kuiper [5] proved that a PL manifold M^d with PL Morse number $\mu(M^d) = 3$ has dimension $d = 0, 2, 4, 8$ or 16 . If $d = 0$ M^d consists of three points. If $d = 2$ M^d is the real projective plane. For $d = 4, 8$ or 16 , M^d is a simply connected cohomology projective plane over complex numbers, quaternions or Cayley numbers, respectively. Each of the manifolds of above type is called a *manifold like a projective plane*. This classification turned up in the 1987 paper [3] of Brehm and Kühnel on combinatorial manifolds with few vertices. Specifically, they proved that: Let M_n^d be a combinatorial d -manifold with n vertices,

(BK1) if $n < 3[d/2] + 3$ then $|M_n^d| \approx S^d$,

(BK2) if $n = 3(d/2) + 3$ and $|M_n^d| \not\approx S^d$ then $d = 2, 4, 8$ or 16 and $|M_n^d|$ must be a “manifold like a projective plane”. Moreover for $d = 2$ $M_n^d = \mathbb{R}P_6^2$ and for $d = 4$ $M_n^d = \mathbb{C}P_9^2$.

It is classically known that there exists a unique (up to simplicial isomorphism) 6-vertex triangulation (denoted by $\mathbb{R}P_6^2$) of the real projective plane $\mathbb{R}P^2$. It is also known (see [2], [6] and [7]) that there exists a unique (up to simplicial isomorphism) 9-vertex triangulation (denoted by $\mathbb{C}P_9^2$) of the complex projective plane $\mathbb{C}P^2$.

Implicit in [3] is the result that $\mathbb{C}P_9^2$ satisfies complementarity. This result was made explicit by Arnoux and Marin [1] in 1991. More generally, they proved that any manifold as in (BK2) satisfies complementarity. In this article we prove the converse:

Theorem. *Let M_n^d be a combinatorial d -manifold with n vertices. If M_n^d satisfies complementarity then $d = 0, 2, 4, 8$ or 16 with $n = 3(d/2) + 3$ and $|M_n^d|$ is a "manifold like a projective plane".*

2. Preliminaries

Let K be a triangulation of the sphere S^{p-1} with n vertices. The f -vector of K is $f(K) := (f_0, \dots, f_{p-1})$, where f_i is the number of i -faces in K . Thus $f_0 = n$ and $f_i \leq \binom{n}{i+1}$ for $1 \leq i \leq p-1$. Let \mathbb{N} denote the non-negative integers, and define $H: \mathbb{N} \rightarrow \mathbb{N}$ as follows

$$H(m) = \begin{cases} 1 & \text{if } m = 0 \\ \sum_{i=0}^{p-1} f_i \binom{m-1}{i} & \text{if } m > 0. \end{cases} \tag{1}$$

Then there exists (see [8]) integers h_0, \dots, h_p such that

$$(1-x)^p \sum_{m=0}^{\infty} H(m)x^m = h_0 + h_1x + \dots + h_px^p \tag{2}$$

is an identity in the formal power series ring $\mathbb{C}[[x]]$.

For $k \leq p < n-1$ (equating the coefficients of x^k from both sides of $(1+x)^{-(p-k+1)}(1+x)^p = (1+x)^{p+k-p-1}$) we get

$$\sum_{j=0}^k (-1)^{k-j} \binom{p-j}{p-k} \binom{n}{j} = \binom{n+k-p-1}{k}. \tag{3}$$

By substituting $i-1 = p$ and $l = p+1-k$ we get

$$\binom{n-l}{i-l} = \sum_{j=0}^{i-l} (-1)^{i-l-j} \binom{i-1-j}{l-1} \binom{n}{j} = \sum_{m=i}^l (-1)^{l-m} \binom{n}{i-m} \binom{m-1}{l-1}. \tag{4}$$

Then from (1) and (2) by using (4) we get (see [9])

$$h_l = \sum_{i=0}^l (-1)^{i-l} \binom{p-l}{p-i} f_{i-1}, \tag{5}$$

where we set $f_{-1} = 1$.

If $f_{j-1} = \binom{n}{j}$ for $1 \leq j \leq q \leq p$ then by (3) we have

$$h_i = \binom{n+i-p-1}{i} \text{ for } i \leq q. \tag{6}$$

The Dehn–Sommerville equations, which hold for any triangulation of the sphere S^{p-1} , are equivalent to the statement (see [9]):

$$h_i = h_{p-i} \quad 0 \leq i \leq p. \tag{7}$$

3. Proof of the theorem

Throughout, M is an n -vertex combinatorial d -manifold satisfying complementarity. It is trivial from the definition that, for $d = 0$ M consists of three points, and since clearly there is no 1-manifold satisfies complementarity, we may take $d \geq 2$.

We shall repeatedly use the following obvious consequences of complementarity. Since no set of $\geq d + 2$ vertices constitute a face, $n \leq 2d + 3$ and every set of $\leq n - d - 2$ vertices is a face. That is, for $i \leq n - d - 3$, all i -faces occur in M . More generally the number of i -faces + the number of $(n - i - 2)$ -faces = $\binom{n}{i+1}$. As each vertex forms a 0-face, therefore $n > d + 2$. Thus, $d + 2 < n \leq 2d + 3$.

Throughout this section we put $c = \lfloor d/2 \rfloor$. Thus, $d = 2c - 1$ or $2c$.

If F_i is the number of i -faces in M then we have:

$$\begin{aligned} \sum_{i=0}^{n-3} F_i &= \begin{cases} F_0 + (F_1 + F_{2m-3}) + \dots + (F_{m-2} + F_m) + F_{m-1} & \text{if } n = 2m, \\ F_0 + (F_1 + F_{2m-2}) + \dots + (F_{m-1} + F_m) & \text{if } n = 2m + 1 \end{cases} \\ &= \begin{cases} \binom{2m}{1} + \binom{2m}{2} + \dots + \binom{2m}{m-1} + \frac{1}{2} \binom{2m}{m} & \text{if } n = 2m, \\ \binom{2m+1}{1} + \binom{2m+2}{2} + \dots + \binom{2m+1}{m} & \text{if } n = 2m + 1 \end{cases} \\ &= 2^{n-1} - 1, \end{aligned}$$

which is an odd integer, where we set $F_i = 0$ for $i > d$. Therefore the Euler characteristic of $M = \sum_{i=0}^{n-3} (-1)^i F_i$ is odd.

If $n = d + 3$ then (by (BK1)) M is a sphere.

If $n > d + 3$ then all the i -faces occur in M for $i \leq n - d - 3 \geq 1$. Therefore the link of any vertex in M is an $(n - 1)$ -vertex combinatorial $(d - 1)$ -sphere with f -vector satisfying: $f_i = \binom{n-1}{i+1}$ for $0 \leq i \leq n - d - 4$. Hence by (6), the h -vector of this link satisfies $h_i = \binom{n-d-2+i}{i}$ for $0 \leq i \leq n - d - 3$.

If $d = 2c$ then by (7) for $n > 3c + 3$, we get $\binom{n-c-3}{c-1} = h_{c-1} = h_{c+1} = \binom{n-c-1}{c+1}$. Which gives $n = 2c + 2$, contrary to our assumption in this case.

If $d = 2c - 1$ then for $n \geq 3c + 3$, we get $\binom{n-c-2}{c-1} = h_{c-1} = h_c = \binom{n-c-1}{c}$. Which gives $n = 2c + 1$, a contradiction.

Thus, $n \leq 3c + 3$ if d is even and $n < 3c + 3$ if d is odd. Therefore, by (BK1) and (BK2) M is either a sphere or a “manifold like a projective plane”. But as Euler characteristic of M is odd, M cannot be a sphere. This completes the proof.

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References

- [1] Arnoux P and Marin A, The Kühnel triangulation of complex projective plane from the view-point of complex crystallography (part II), *Memoirs of the Faculty of Sc., Kyushu Univ. Ser. A* 45 (1991) 167–244
- [2] Bagchi B and Datta B, On Kühnel's 9-vertex complex projective plane, *Geometriae Dedicata* (to appear)
- [3] Brehm U and Kühnel W, Combinatorial manifolds with few vertices. *Topology* 26 (1987) 467–473
- [4] Brehm U and Kühnel W, 15-vertex triangulation of an 8-manifold, *Math. Ann.* 294 (1992) 167–193
- [5] Eells Jr J and Kuiper N H, Manifolds which are like projective plane, *Publ. Math. I.H.E.S.* 14 (1962) 181–222
- [6] Kühnel W and Banchoff T F, The 9-vertex complex projective plane, *The Math. Intell.* 5 No. 3 (1983) 11–22
- [7] Kühnel W and Laßmann G, The unique 3-neighbourly 4-manifold with few vertices, *J. Combin. Theory (A)* 35 (1983) 173–184
- [8] Stanley R P, The Upper Bound Conjecture and Cohen-Macaulay Rings, *Stud. Appl. Math.* LIV (1975) 135–142
- [9] Stanley R P, The Number of Faces of a Simplicial Convex Polytope, *Adv. Math.* 35 (1980) 236–238