A note on a generalization of Macdonald's identities for $A_\ell$ and $B_\ell$

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Abstract. Let $\eta(q)$ denote the Dedekind's $\eta$-function. Macdonald obtained identities for $\eta(q)^{\dim g}$ where $g$ is complex simple finite dimensional Lie algebra. The aim of this paper is to obtain generalization of the above identities in the case of $g = A_\ell$ and $B_\ell$. We also get new formulas for the generating functions of the Ramanujan's $r$-function and $\psi_r$-functions.

Keywords. Macdonald's multivariable identities; Dedekind's eta function;

1. Introduction

Let $\eta(q)$ denote Dedekind's $\eta$-function. Macdonald [7] obtained a formula for $\eta(q)^{\dim g}$ for every complex simple Lie algebra $g$ which gives a generalization of the Jacobi's expansion for $\eta(q)^3$. These formulas are some specializations of the Macdonald's multivariable identities [7]. Many other identities involving Dedekind's $\eta$-function were also obtained by Lepowsky [4].

We shall need the following preliminaries. Let $g$ denote the simple finite dimensional Lie algebra of the type $A_\ell$ or $B_\ell$; $\mathfrak{h}$ denotes its Cartan subalgebra; Let $\Delta$, $\Delta_+$ and $\Delta_+$ be the root system, the set of positive roots and the positive dual roots respectively of $g$ and $\rho$ (respectively $\rho'$) be the half-sum of the roots in $\Delta_+$ (resp. $\Delta_+$). $W$ and $M$ are the Weyl group of $g$ and the lattice spanned over $\mathbb{Z}$ by the long roots of $g$ respectively. Let $h$ and $g$ denote the Coxeter and dual Coxeter numbers of $g$ respectively.

Let $\langle \cdot , \cdot \rangle$ denote the pairing of the elements in $\mathfrak{h}$ and $\mathfrak{h}^*$ (cf. [3]). We introduce the following notations:

\[
\Delta_{m} = \{ \alpha \in \Delta_+ \mid \langle \rho, \alpha \rangle \equiv 0 \pmod{m} \}, \\
d_m(\lambda) = \prod_{\alpha \in \Delta_{m}^{\vee}} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}, \quad (\lambda \in \mathfrak{h}^*).
\]

The aim of this paper is to prove the following:

**Theorem 1.** For $\ell \geq 1$, let $m \leq \ell + 1$ be any divisor of $\ell + 1$. Then

\[
\eta(q^{m})^{(\ell + 1)^2/m} \eta(q)^{-1} = \sum_{\alpha \in M} d_m((\ell + 1)\alpha)^{(1/2)(\ell + 1)||p + (\ell + 1)\alpha||^2}
\]
where
\[ d_m(\lambda) = \prod_{\alpha \in \Delta_+^\vee} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}, (\lambda \in \mathfrak{h}^{\ast}). \]

Theorem 2. For \( \ell \geq 3 \), let \( m \) be any divisor of \((2\ell - 1)\). Then
\[
\eta(q)\eta(q^{m})^{(2\ell^2 + \ell - 1)/m} = \sum_{\alpha \in \mathcal{M}} d_m((2\ell - 1)\alpha)q^{(\ell^2(2\ell^2 - 1))/\rho + (2\ell - 1)m^2}.
\]

The above identities are generalizations of Macdonald's formula for \( \eta(q)^{(\ell^2 + 1)/2 - 1} \) and \( \eta(q^{2\ell^2 + \ell}). \)

The consequence of the above theorems are the following corollaries.

**COROLLARY 1**
\[
\eta(q)^{24} = \sum_{r_1, \ldots, r_{12}; r \in \mathbb{Z}} \left\{ (-1)^{r \cdot D_1(r_1, \ldots, r_{12})} q^{(1/44)(6r+1)^2} + \sum_{i=1}^{12} \frac{(13 - 2i + 24r_i)^2}{4}\right\}
\]
where
\[
D_1(r_1, \ldots, r_{12}) = \prod_{i=1}^{6} (2r_i - r_{i+6} + 1).
\]

**COROLLARY 2**
\[
q\{(1 - q^3)(1 - q^6)\ldots\}^8
\]
\[
= \sum_{r_1, \ldots, r_4; r \in \mathbb{Z}} \left\{ (-1)^{r \cdot D_2(r_1, \ldots, r_4)} q^{(1/16)(6r+1)^2} + \sum_{i=1}^{4} (5 - 2i + 8r_i)^2\right\},
\]
where
\[
D_2(r_1, \ldots, r_4) = (2(r_1 - r_3) + 1)(2(r_2 - r_4) + 1).
\]

**COROLLARY 3**
\[
q\{(1 - q^2)(1 - q^4)\ldots\}^{12}
\]
\[
= \sum_{r_1, \ldots, r_6; r \in \mathbb{Z}} \left\{ (-1)^{r \cdot D_3(r_1, \ldots, r_6)} q^{(1/36)(6r+1)^2 + (1/2)} \sum_{i=1}^{6} (7 - 2i + 12r_i)^2\right\},
\]
where
\[
D_3(r_1, \ldots, r_6) = (2(r_1 - r_4) + 1)(2(r_2 - r_5) + 1)(2(r_3 - r_6) + 1).
\]

**COROLLARY 4**

For \( \ell \geq 3 \), we have
\[
\eta(q)\eta(q^{2\ell - 1})^{\ell - 1} = \sum_{\alpha \in \mathcal{M}} d_{2\ell - 1}((2\ell - 1)\alpha)q^{(\ell^2(2\ell - 1))/\rho + (2\ell - 1)m^2}.
\]
Corollary 1 gives a new formula for the generating function $\eta(q)^{24}$ of the Ramanujan's $\tau$-function. This is different from that of Dyson [1] and Lepowsky [4, 5]. Corollaries 2 and 3 give formulas for the generating function $G_k$ of the Ramanujan's $\psi_k$-functions [8].

Now we briefly explain the techniques used to obtain the above results: The substitutions involved in these computations are actually generalizations of the substitutions used by Macdonald [7] and Lepowsky [4, 5].

2. Explanation of the techniques involved

We will use the following form of the Macdonald's identity for the affine Lie algebra of the type $A_{t}^{(1)}$ or $B_{t}^{(1)}$ (the identity is true for any $X_{t}^{(1)}$) (cf. [3]: pp. 168):

$$e\left(-\frac{|\rho|^2}{2g}\right) \prod_{\alpha \neq 1} ((1 - e(-n\delta)) \cdots (1 - e(-n\delta + \rho)))$$

$$= \sum_{\alpha \in \mathfrak{g}} \chi(g\alpha) e\left(-\frac{1}{2g} |\rho + g\alpha|^2\delta\right),$$

(2.1)

where, for $\lambda \in \mathfrak{h}$*

$$\chi(\lambda) = \frac{\sum_{w \in \mathcal{W}} e(w) e(w(\lambda + \rho) - \rho)}{\prod_{\alpha \in \Delta}} (1 - e(-\alpha)).$$

(2.2)

Let $\alpha_1, \ldots, \alpha_{\ell}$ denote the simple roots of $\mathfrak{g}$. For any divisor $m$ of $g$, where $g$ stands for the dual Coxeter number of $\mathfrak{g}$, let $\phi_m$ denote the specialization $\phi_m(e(-\delta)) = q$, $\phi_m(e(\alpha_i)) = w(i = 1, \ldots, \ell)$; here $w$ denotes the primitive root of degree $m$ of unity.

We require the following Lemma.

Lemma. For $\alpha \in \mathcal{M}$,

$$\phi_m(\chi(g\alpha)) = d_m(g\alpha).$$

(2.3)

Proof. Fix $\alpha \in \mathcal{M}$. We define the following homomorphisms:

$$F_1: \mathbb{C}[[e(-\alpha_i): 1 \leq i \leq \ell]] \to \mathbb{C}[[t, t^{-1}]]$$

by

$$F_1(e(-\alpha_i)) = t (i = 1, 2, \ldots, \ell).$$

(Here note that $F_1(e(\alpha_i)) = t^{<\alpha_i, \rho'>}$) and

$$F_2: \mathbb{C}[[e(-\alpha_i^\vee): 1 \leq i \leq \ell]] \to \mathbb{C}[[t, t^{-1}]]$$

by

$$F_2(e(-\alpha_i^\vee)) = t^{<\alpha_i^\vee, g\alpha + \rho>}. $$

Since $\phi_m$ is an homomorphism and since $\lim_{x \to w} F_1(e(-g\alpha)) = 1$, it suffices to prove that

$$\phi_m(e(-g\alpha)\chi(g\alpha)) = d_m(g\alpha).$$
Now,

\[
\phi_m(e(-gx)\chi(gx)) = \phi_m \left( \frac{\sum_{w\in W}\epsilon(w)e(w(gx + \rho) - (\rho + gx))}{\Pi_{\beta \in \Delta_+}(1 - e(-\beta))} \right) \\
= \lim_{t \to \infty} \left\{ \frac{\sum_{w\in W}\epsilon(w)t^{\langle gx + \rho, \rho \rangle} - \langle w(gx + \rho), \rho \rangle}{\Pi_{\beta \in \Delta_+}(1-t^{\langle \rho, \rho \rangle})} \right\} \\
= \lim_{t \to \infty} \left\{ \frac{F_2(\sum_{w\in W}\epsilon(w)e(w(\rho^0) - (\rho^0)))}{F_1(\Pi_{\beta \in \Delta_+}(1 - e(-\beta)))} \right\} \\
= \lim_{t \to \infty} \frac{F_2(\Pi_{\beta \in \Delta_+}(1 - e(-\beta)))}{F_1(\Pi_{\beta \in \Delta_+}(1 - e(-\beta)))}, \quad \text{(by [3], 10.4.4)} \\
= \lim_{t \to \infty} \frac{(\Pi_{\beta \in \Delta_+}(1 - t^{\langle \rho, \rho \rangle}))}{\Pi_{\beta \in \Delta_+}(1 - t^{\langle \rho, \rho \rangle})} \\
= \lim_{t \to \infty} \frac{\Pi_{\beta \in \Delta_+}(1 - t^{\langle \rho, \rho \rangle})}{\Pi_{\beta \in \Delta_+}(1 - t^{\langle \rho, \rho \rangle})} \\
= \lim_{t \to \infty} \frac{\Pi_{\beta \in \Delta_+}(1 - e(-\beta))}{\Pi_{\beta \in \Delta_+}(1 - e(-\beta))} \\
= d_m(gx) \quad \text{(by L' Hospital's rule)}
\]

Let \([x]\) denote the greatest integer contained in \(x\). Let \(\eta_p\) denote the number of roots in \(\Delta_+\) with height \(p\) [5] and for \(0 \leq j \leq m\), let \(N_j(m)\) denote the number of roots in \(\Delta\) with height congruent to \(j\) (mod \(m\)). It is not hard to see that

\[
N_0(m) = 2 \sum_{k=1}^{[h/m]} \eta_{km} \quad \text{(for } j = 0) \quad \text{(2.4)}
\]

and

\[
N_j(m) = \eta_j + \sum_{k=1}^{[h/m]-1} (\eta_{km-j} + \eta_{km+j}) + \eta_{[h/m]m-j}, \quad \text{(for } 1 \leq j < m) \quad \text{(2.5)}
\]

Now applying \(\phi_m\) to (2.1) and using (2.3), (2.4) and (2.5) along with the strange formula of Frendenthal de varies:

\[
\frac{|\rho|^2}{2g} = \frac{\dim g}{24} \quad \text{(cf. [3])}, \quad \text{(2.6)}
\]

we obtain

\[
q^{\dim g/24} \prod_{j=1}^{m} \left( 1 - q^a x_j \right)^{\sum_{j=0}^{m-1} (1 - q^a x_j)^{N_j(m)}} = \sum_{m \in M} d_m(gx)q^{(1/2g)|\rho + \rho||^2}. \quad \text{(2.7)}
\]

Now, using the known facts about \(\eta_p\) (cf. [5; pp. 228]) one can easily compute \(\eta_p\):

\[
\eta_p = \ell + 1 - p \quad \text{for } g = A_\ell
\]

and

\[
\eta_p = \begin{cases} 
\ell - p/2 & \text{if } p \text{ is even} \\
\ell - (p - 1)/2 & \text{if } p \text{ is odd}; \text{ for } g = B_\ell.
\end{cases}
\]

We shall discuss three cases:
Macdonald's identities

Case (i). \( g \) is of type \( A_\ell \). By (2.4) and (2.5) we have:

\[
N_0(m) = ((\ell + 1)^2/m) - (\ell + 1),
\]
and for \( 1 \leq j < m \),

\[
N_j(m) = (\ell + 1)^2/m.
\]

Case (ii). \( g \) is of type \( B_\ell \) and \( m = 1 \).

In this case \( \left[ \frac{h}{m} \right] = 2\ell \). Hence by (2.4) we have

\[
N_0(1) = 2 \left\{ \sum_{k=1}^{2\ell} \eta_k + \sum_{k=1}^{2\ell} \eta_k \right\}
\]

\[
= 2 \left\{ \sum_{k=1}^{2\ell} (\ell - (k - 1)/2) + \sum_{k=1}^{2\ell} (\ell - k/2) \right\}
\]

\[
= (2\ell)(2\ell) - (2\ell)(2\ell + 1)/2 + \ell + \ell^2
\]

\[
= 2\ell^2
\]

Case (iii). \( g \) is of type \( B_\ell \) and \( m > 1 \). In this case \( \left[ \frac{h}{m} \right] = \frac{(2\ell - 1)}{m} \). Hence by (2.4) we have

\[
N_0(m) = 2 \left\{ \sum_{k=1}^{(2\ell - 1)/m} \eta_{km} + \sum_{k=1}^{(2\ell - 1)/m} \eta_{km} \right\}
\]

\[
= 2 \left\{ \sum_{k=1}^{(2\ell - 1)/m} (\ell - (km - 1)/2) + \sum_{k=1}^{(2\ell - 1)/m} (\ell - (km)/2) \right\},
\]

\[
= ((2\ell - 1)/m)(2\ell) - m((2\ell - 1)/m)(((2\ell - 1)/m + 1)/2) + ((2\ell - 1)/m + 1)/2
\]

\[
= (2\ell^2 + \ell - 1)/m - (\ell - 1).
\]

Furthermore, one can easily see that

\[
\eta_{km-j} + \eta_{km+j} = \begin{cases} 
2\ell - km & \text{if } j \text{ is even and } k \text{ is even, or } j \text{ is odd and } k \text{ is odd.} \\
2\ell - km + 1 & \text{if } j \text{ is even and } k \text{ is odd, or } j \text{ is odd and } k \text{ is even.}
\end{cases}
\]

Hence we have by (2.5), that for \( 1 \leq j < m \) and \( j \) even,

\[
N_j(m) = \eta_j + \left\{ \sum_{k=1}^{((2\ell - 1)/m) - 1} (\eta_{km-j} + \eta_{km+j}) + \sum_{k=1}^{((2\ell - 1)/m) - 1} (\eta_{km-j} + \eta_{km+j}) \right\}
\]

\[
+ \eta_{[h/m]m-j}
\]
\[ N_s \sum_{k=1}^{\frac{(2\ell - 1) - (j - 1)}{2}} \sum_{k=1}^{\frac{(2\ell - 1) - (j - 1)}{2}} (2\ell - km + 1) + (\ell - (2\ell - 1) - j - 1)/2 \]
\[ = (2\ell)(2\ell - 1)/m - \frac{1}{2}((2\ell - 1)/m - 1)(2\ell - 1)/m + (2\ell - 1)/m - \frac{1}{2}((2\ell - 1)/m - 1)2 - (\ell - 1), \]
\[ = (2\ell^2 + \ell - 1)/m. \quad (2.12) \]

Similarly, for \(1 \leq j < m\) and \(j\) odd:
\[ N_j = N_j + \sum_{k=1}^{\frac{(2\ell - 1) - (j - 1)}{2}} \sum_{k=1}^{\frac{(2\ell - 1) - (j - 1)}{2}} (2\ell - km + 1) + (\ell - (2\ell - 1) - j)/2 \]
\[ = (2\ell^2 + \ell - 1)/m. \quad (2.13) \]

Now, using (2.4), (2.5) along with (2.9), (2.10), (2.12), (2.13) and the fact that
\[ \prod_{j=1}^{n-1} (1 - a^{(j)}) = (1 - a^s)(1 - a)^{-1}, \quad (a \neq 1) \]
theorems 1 and 2 follow.

Note that for \(m = 1\) the identities of theorems 1 and 2 are precisely the Macdonald's identities for \(\eta(q)^{(\ell+1)^2-1}\) and \(\eta(q)^{2\ell^2+\ell}\) respectively.

Furthermore, using the following identity due to Euler (cf. [6]):
\[ \eta(q) = \sum_{r \in \mathbb{Z}} (-1)^r q^{(1/2)^2(6r + 1)^2} \]
and by replacing \(q\) by \(q^{24\ell^2(\ell + 1)^2}\) in theorem 1, we obtain:
\[ \eta(q^{24\ell^2(\ell + 1)^2})(\ell + 1)^2/m = \sum_{a \in \mathbb{Z} \cap \mathbb{Z}} (-1)^r d_m((\ell + 1)x)q^{(12/\ell^2(\ell + 1)^2)((s + (\ell + 1)a)^2 + 1/\ell + 1)^2(6r + 1)^2}. \quad (2.14) \]

Now the Corollaries 1, 2 and 3 follow by taking \(n = 12\) and \(m = 6, n = 4\) and \(m = 2,\)
and \(n = 6\) and \(m = 3\) respectively. Corollary 4 follows by taking \(m = 2\ell - 1, \) by Theorem 2.

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