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On composition of some general fractional integral operators

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Abstract. In the present paper we derive three interesting expressions for the composition of two most general fractional integral operators whose kernels involve the product of a general class of polynomials and a multivariable H-function. By suitably specializing the coefficients and the parameters in these functions we can get a large number of (new and known) interesting expressions for the composition of fractional integral operators involving classical orthogonal polynomials and simpler special functions (involving one or more variables) which occur rather frequently in problems of mathematical physics. We have mentioned here two special cases of the first composition formula. The first involves product of a general class of polynomials and the Fox's H-functions and is of interest in itself. The findings of Buschman [1] and Erdélyi [4] follow as simple special cases of this composition formula. The second special case involves product of the Jacobi polynomials, the Hermite polynomials and the product of two multivariable H-functions. The present study unifies and extends a large number of results lying scattered in the literature. Its findings are general and deep.

Keywords. Fractional integral operator; general class of polynomials; multivariable H-function.

1. Introduction

Fractional integral operators play an important role in the theory of integral equations and in problems of mathematical physics. We shall study in this paper the composition of fractional integral operators defined by means of the following equations

$$R^{\eta,\alpha;m,n,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},v_1,\ldots,v_r}_{x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\ldots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}[f(x)]$$

$$= x^{-\eta-\alpha-1}\int_0^x t^{\eta}(x-t)^{\alpha} S_n^m\left[e\left(1-\frac{t}{x}\right)^{v}\right] H\left[z_1\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(1-\frac{t}{x}\right)^{v_r}\right] f(t)\mathrm{d}t \tag{1}$$

$$W^{\eta,\alpha;m,n,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},v_1,\ldots,v_r}_{x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\ldots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}[f(x)]$$

$$= x^{\eta}\int_x^{\infty} t^{-\eta-\alpha-1}(t-x)^{\alpha} S_n^m\left[e\left(1-\frac{x}{t}\right)^{v}\right] H\left[z_1\left(1-\frac{x}{t}\right)^{v_1},\ldots,z_r\left(1-\frac{x}{t}\right)^{v_r}\right] f(t)\mathrm{d}t \tag{2}$$

Here $S_n^m[x]$ denotes the general class of polynomials introduced by Srivastava

[15, p. 1, eqn. (1)]

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \qquad n = 0, 1, 2, \ldots, \tag{3}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}(n, k \geqslant 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [19, pp. 158–161].

The H-function of r complex variables $z_1, \ldots, z_r$ occurring in (1) and (2) was introduced by Srivastava and Panda [18]. We use here the following contracted form [17, p. 251, eqn. (C.1)] to denote it

$$H[z_1, \ldots, z_r] = H\,{}^{0,N:M',N';\ldots;M^{(r)},N^{(r)}}_{P,Q:P',Q';\ldots;P^{(r)},Q^{(r)}}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P}:(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P^{(r)}} \\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}:(d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,Q^{(r)}} \end{matrix}\right] \tag{4}$$

The defining integral and other details about this function can be found in the references given above.

It may be remarked here that all the Greek letters occurring in the right-hand side of (4) are assumed to be positive real numbers for standardization purposes; the definition of this function will, however, be meaningful even if some of these quantities are zero. Again, it is assumed throughout the present work that this function always satisfies the appropriate existence and convergence conditions of its defining integral [17, pp. 252–253, eqs (C.4–C.6)].

On account of the importance of the fractional integral operators in the theory of integral equations and other allied topics, these operators have been studied from time to time by a number of authors notably Riemann–Liouville [6], Weyl [6], Erdélyi [2, 3], Kober [11], Lowndes [12], Goyal and Jain [8], Gupta [9], Kalla [10], Saxena [13], Saxena and Kumbhat [14], Srivastava *et al* [16]. The importance of our study lies in the fact that the kernels (1) and (2) used here are most general in character.

To be specific, we shall assume throughout this paper that

$$f(x) = \begin{cases} O(|x|^{\gamma}), & |x| \to 0 \\ O(|x|^{\delta} e^{-\lambda|x|}), & |x| \to \infty \end{cases}$$

It is easy to verify that the operator defined by (1) exists if

(i) The quantities $v, v_1, \ldots, v_r$ are all positive (some of them may decrease to zero provided that the resulting operator has a meaning).

(ii) $\operatorname{Re}(\alpha) + \sum_{i=1}^{r} v_i \min_{1 \leqslant j \leqslant M^{(i)}} [\operatorname{Re}(d_j^{(i)}/\delta_j^{(i)})] + 1 > 0.$

(iii) $\operatorname{Re}(\eta + \gamma + 1) > 0.$

and the operator defined by (2) exists if
$\mathrm{Re}(\lambda) > 0$ or $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Re}(\eta - \delta) > 0$, and the set of conditions (i) and (ii) specified for the existence of the operator (1) are satisfied.

2. Compositions of fractional integral operators

From the definition (1), we have $R_x^{\eta,\alpha}\{R_x^{\theta,\beta} f(x)\}$

$$= R^{\eta,\alpha;m,n,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},v_1,\ldots,v_r}_{x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\ldots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}$$
$$\left\{R^{\theta,\beta;m',n',v';\ N_1,P_1,Q_1,M^{(r+1)},N^{(r+1)},P^{(r+1)},Q^{(r+1)},\ldots,M^{(2r)},N^{(2r)},P^{(2r)},Q^{(2r)},v_{r+1},\ldots,v_{2r}}_{x;e';\ z_{r+1},\ldots,z_{2r},a'_j,\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},b'_j,\beta_j^{(r+1)},\ldots,\beta_j^{2r},c_j^{(r+1)},\gamma_j^{(r+1)},d_j^{(r+1)},\delta_j^{(r+1)},\ldots,c_j^{(2r)},\gamma_j^{(2r)},d_j^{(2r)},\delta_j^{(2r)}} f(x)\right\}$$
$$= x^{-\eta-\alpha-1}\int_0^x (x-t)^\alpha S_n^m\left[e\left(1-\frac{t}{x}\right)^v\right] H\left[z_1\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(1-\frac{t}{x}\right)^{v_r}\right] t^{\eta-\theta-\beta-1}\times$$
$$\left\{\int_0^t (t-\zeta)^\beta S_{n'}^{m'}\left[e'\left(1-\frac{\zeta}{t}\right)^{v'}\right] H\left[z_{r+1}\left(1-\frac{\zeta}{t}\right)^{v_{r+1}},\ldots,z_{2r}\left(1-\frac{\zeta}{t}\right)^{v_{2r}}\right]\zeta^\theta f(\zeta)\mathrm{d}\zeta\right\}\mathrm{d}t \tag{5}$$

On changing the order of ζ,t-integrals in the right-hand side of (5), we arrive at the following result after a little simplification

$$R_x^{\eta,\alpha}\{R_x^{\theta,\beta} f(x)\} = x^{-\eta-\alpha-1}\int_0^x \zeta^\theta f(\zeta)\Delta \mathrm{d}\zeta \tag{6}$$

where

$$\Delta = \int_\zeta^x (x-t)^\alpha (t-\zeta)^\beta t^{\eta-\theta-\beta-1} S_n^m\left[e\left(1-\frac{t}{x}\right)^v\right] S_{n'}^{m'}\left[e'\left(1-\frac{\zeta}{t}\right)^{v'}\right]$$
$$H\left[z_1\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(1-\frac{t}{x}\right)^{v_r}\right] H\left[z_{r+1}\left(1-\frac{\zeta}{t}\right)^{v_{r+1}},\ldots,z_{2r}\left(1-\frac{\zeta}{t}\right)^{v_{2r}}\right]\mathrm{d}t \tag{7}$$

The change in the order of integration in the right-hand side of (5) is justified by the well-known Fubini's theorem, provided that

$$\mathrm{Re}(\eta-\theta-\beta) > 0,\quad \mathrm{Re}(\theta+\gamma+1) > 0,\quad \mathrm{Re}(\alpha) + \sum_{i=1}^{r} v_i \min_{1\le j\le M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})] + 1 > 0,$$

$$\mathrm{Re}(\beta) + \sum_{i=r+1}^{2r} v_i \min_{1\le j\le M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})] + 1 > 0.$$

Now to evaluate the value of Δ, we express the general class of polynomials involved in the right-hand side of (7) in the series form and the multivariable H-functions in terms of their well known Mellin-Barnes contour integrals [17, pp. 251–252, eqns. (C.1)–(C.3)] and interchange the order of summation and integration in the result thus obtained (which is permissible under the conditions stated), we get the following result after a little simplification

$$\Delta = \sum_{k=0}^{[n/m]} \sum_{k'=0}^{[n'/m']} \frac{(-n)_{mk}(-n')_{m'k'}}{k!\ \ k'!} A_{n,k} A'_{n',k'} e^k e'^{k'}$$

$$\frac{1}{(2\pi\omega)^{2r}} \int_{L_1} \cdots \int_{L_{2r}} \phi_1(\xi_1)\ldots\phi_{2r}(\xi_{2r})\psi(\xi_1,\ldots,\xi_r)\psi'(\xi_{r+1},\ldots,\xi_{2r}) z_1^{\xi_1}\ldots z_{2r}^{\xi_{2r}} \mathrm{d}\xi_1\ldots\mathrm{d}\xi_{2r}$$

$$\int_\zeta^x (x-t)^\alpha (t-\zeta)^\beta t^{\eta-\theta-\beta-1} \left(1-\frac{t}{x}\right)^{vk+v_1\xi_1+\ldots+v_r\xi_r} \left(1-\frac{\zeta}{t}\right)^{v'k'+v_{r+1}\xi_{r+1}+\ldots+v_{2r}\xi_{2r}} \mathrm{d}t \tag{8}$$

The t-integral occurring in the right-hand side of (8) is now evaluated by making the substitution $t = x - (x-\zeta)w$ in it, further ${}_2F_1$ thus obtained is expressed in terms of its well known Mellin-Barnes contour integral and the result thus obtained reinterpreted in terms of the H-function of $2r+1$ variables to yield the value of Δ. Now upon substituting the value of Δ thus obtained in (6), we finally arrive at the following result after a little simplification

$$R_x^{\eta,\alpha}\left\{R_x^{\theta,\beta} f(x)\right\} = x^{-\theta-\alpha-\beta-2} \sum_{k=0}^{[n/m]} \sum_{k'=0}^{[n'/m']} \frac{(-n)_{mk}(-n')_{m'k'}}{k!\ \ k'!} A_{n,k} A'_{n',k'} e^k e'^{k'}$$

$$x^{-(vk+v'k')} \int_0^x \zeta^\theta (x-\zeta)^{\alpha+\beta+vk+v'k'+1} H^{0,N+N_1+3\ :M',N';\ldots;M^{(2r)},N^{(2r)};1,0}_{P+P_1+3,Q+Q_1+2:P',Q';\ldots;P^{(2r)},Q^{(2r)};0,1}$$

$$\left[\begin{array}{c|l} z_1\left(1-\frac{\zeta}{x}\right)^{v_1} & \\ \vdots & \left(\eta-\theta-\beta-v'k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},1\right), \\ z_{2r}\left(1-\frac{\zeta}{x}\right)^{v_{2r}} & (-1-\alpha-\beta-vk-v'k';v_1,\ldots,v_{2r},1), \\ -\left(1-\frac{\zeta}{x}\right) & \end{array}\right.$$

$$\left(-\alpha-vk;v_1,\ldots,v_r,\frac{0,\ldots,0}{r},1\right),\left(-\beta-v'k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},0\right),$$

$$\left(\eta-\theta-\beta-v'k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},0\right),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,N},\left(a'_j;\frac{0,\ldots,0}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},0\right)_{1,P_1},\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{N+1,P}:$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,Q},\left(b'_j;\frac{0,\ldots,0}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(2r)},0\right)_{1,Q_1}\qquad :$$

$$\left.\begin{array}{l}(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(2r)},\gamma_j^{(2r)})_{1,P^{(2r)}};\text{—} \\ (d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(2r)},\delta_j^{(2r)})_{1,Q^{(2r)}};(0,1)\end{array}\right] f(\zeta)\mathrm{d}\zeta \tag{9}$$

where the number occurring below the line at any place on the right-hand side of (9) and throughout the paper indicates the total number of zeros covered by it. Thus $\frac{0,\ldots,0}{r}$ would mean r zeros, and so on, and the conditions of the existence of the operator (1) and the conditions necessary for the change of order of integration stated earlier are satisfied.

On following the procedure adopted by Erdélyi [4] and on making the use of well-known transformation formula for the Gauss' hypergeometric function [5, p. 64, eqn. (23)], the following commutative property of the operator given by (1) can be easily established

$$R_x^{\eta,\alpha}\{R_x^{\theta,\beta}f(x)\} = R_x^{\theta,\beta}\{R_x^{\eta,\alpha}f(x)\}.$$

The expression for the composition of the fractional integral operator defined by (2), can be easily derived in a similar manner. We have

$$W^{\eta,\alpha;m,n,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},v_1,\ldots,v_r}_{x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\ldots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}$$
$$\left\{W^{\theta,\beta;m',n',v';\;N_1,P_1,Q_1,M^{(r+1)},N^{(r+1)},P^{(r+1)},Q^{(r+1)},\ldots,M^{(2r)},N^{(2r)},P^{(2r)},Q^{(2r)},v_{r+1},\ldots,v_{2r}}_{x;e';\;z_{r+1},\ldots,z_{2r},a'_j,\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},b'_j,\beta_j^{(r+1)},\ldots,\beta_j^{(2r)},c_j^{(r+1)},\gamma_j^{(r+1)},d_j^{(r+1)},\delta_j^{(r+1)},\ldots,c_j^{(2r)},\gamma_j^{(2r)},d_j^{2r},\delta_j^{(2r)}}f(x)\right\}$$
$$= W^{\theta,\beta;m',n',v';N_1,P_1,Q_1,M^{(r+1)},N^{(r+1)},P^{(r+1)},Q^{(r+1)},\ldots,M^{(2r)},N^{(2r)},P^{(2r)},Q^{(2r)},v_{r+1},\ldots,v_{2r}}_{x;e';z_{r+1},\ldots,z_{2r},a'_j,\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},b'_j,\beta_j^{(r+1)},\ldots,\beta_j^{(2r)},c_j^{(r+1)},\gamma_j^{(r+1)},\mathrm{d}_j^{(r+1)},\delta_j^{(r+1)},\ldots,c_j^{(2r)},\gamma_j^{(2r)},d_j^{(2r)},\delta_j^{(2r)}}$$
$$\left\{W^{\eta,\alpha;m,n,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},v_1,\ldots,v_r}_{x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\ldots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)}d_j^{(r)},\delta_j^{(r)}}f(x)\right\}$$
$$= x^{\eta}\sum_{k=0}^{[n/m]}\sum_{k'=0}^{[n'/m']}\frac{(-n)_{mk}(-n')_{m'k'}}{k!\;k'!}A_{n,k}A'_{n',k'}\,e^k e'^{k'}$$
$$\int_x^{\infty}\zeta^{-\eta-\alpha-\beta-vk-v'k'-2}(\zeta-x)^{\alpha+\beta+vk+v'k'+1}$$
$$H^{0,N+N_1+3:M',N';\ldots;M^{(2r)},N^{(2r)};1,0}_{P+P_1+3,Q+Q_1+2:P',Q';\ldots;P^{(2r)},Q^{(2r)};0,1}\left[\begin{matrix} z_1\left(1-\frac{x}{\zeta}\right)^{v_1}\\ \vdots\\ z_{2r}\left(1-\frac{x}{\zeta}\right)^{v_{2r}}\\ -\left(1-\frac{x}{\zeta}\right)\end{matrix}\right.$$
$$\left(\theta-\eta-\alpha-vk;v_1,\ldots,v_r,\frac{0,\ldots,0}{r},1\right),\left(-\beta-v'k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},1\right),$$

$$(-1-\alpha-\beta-vk-v'k';v_1,\ldots,v_{2r},1),$$

$$\left(-\alpha-vk;v_1,\ldots,v_r,\frac{0,\ldots,0}{r+1}\right),$$

$$\left(\theta-\eta-\alpha-vk;v_1,\ldots,v_r,\frac{0,\ldots,0}{r+1}\right),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,N},\left(a'_j;\frac{0,\ldots,0}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},0\right)_{1,P_1},\left(a_j;\alpha'_p,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{N+1,P}:$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,Q},\left(b'_j;\frac{0,\ldots,0}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(2r)},0\right)_{1,Q_1}\quad:$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(2r)},\gamma_j^{(2r)})_{1,P^{(2r)}};\text{—}\\(d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(2r)},\delta_j^{(2r)})_{1,Q^{(2r)}};(0,1)\end{matrix}\right]f(\zeta)\mathrm{d}\zeta \tag{10}$$

provided that the composite operator defined by the left-hand side of (10) exists and the following conditions are satisfied

$$\mathrm{Re}(\lambda)>0 \text{ or } \mathrm{Re}(\lambda)=0 \text{ and } \min[\mathrm{Re}(\eta-\theta,\theta-\delta,\theta+\beta-\delta)]>0,$$

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r}v_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0,$$

$$\mathrm{Re}(\beta)+\sum_{i=r+1}^{2r}v_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0.$$

Finally, we obtain the expression for the composition of the fractional integral operators defined by (1) and (2), we have $R_x^{\eta,\alpha}\{W_x^{\theta,\beta}f(x)\}$

$$=R\begin{matrix}\eta,\alpha;m,n,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},v_1,\ldots,v_r\\x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\ldots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}\end{matrix}$$

$$\left\{W\begin{matrix}\theta,\beta;m',n',v';\\x;e'\qquad;\end{matrix}\right.$$
$$N_1,P_1,Q_1,M^{(r+1)},N^{(r+1)},P^{(r+1)},Q^{(r+1)},$$
$$z_{r+1},\ldots,z_{2r},a'_j,\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},b'_j,\beta_j^{(r+1)},$$
$$\left.\begin{matrix}\ldots,M^{(2r)},N^{(2r)},P^{(2r)},Q^{(2r)},v_{r+1},\ldots,v_{2r}\\\ldots,\beta_j^{(2r)},c_j^{(r+1)},\gamma_j^{(r+1)},d_j^{(r+1)},\delta_j^{(r+1)},\ldots,c_j^{(2r)},\gamma_j^{(2r)},d_j^{(2r)},\delta_j^{(2r)}\end{matrix}f(x)\right\}$$

$$=x^{-\eta-\alpha-1}\int_0^x(x-t)^\alpha S_n^m\left[e\left(1-\frac{t}{x}\right)^v\right]H\left[z_1\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(1-\frac{t}{x}\right)^{v_r}\right]t^{\eta+\theta}$$

$$\left\{\int_t^\infty(\zeta-t)^\beta S_{n'}^{m'}\left[e'\left(1-\frac{t}{\zeta}\right)^{v'}\right]H\left[z_{r+1}\left(1-\frac{t}{\zeta}\right)^{v_{r+1}},\ldots,z_{2r}\left(1-\frac{t}{\zeta}\right)^{v_{2r}}\right]\zeta^{-\theta-\beta-1}f(\zeta)\mathrm{d}\zeta\right\}\mathrm{d}t \tag{11}$$

Now changing the order of ζ,t-integrals in the right-hand side of (11), we get after a little simplification.

$$R_x^{\eta,\alpha}\{W_x^{\theta,\beta}f(x)\} = x^{-\eta-\alpha-1}\left[\int_0^x \zeta^{-\theta-\beta-1}f(\zeta)\Delta_1 \mathrm{d}\zeta + \int_x^\infty \zeta^{-\theta-\beta-1}f(\zeta)\Delta_2 \mathrm{d}\zeta\right] \tag{12}$$

where

$$\Delta_1 = \int_0^\zeta (x-t)^\alpha(\zeta-t)^\beta t^{\eta+\theta} S_n^m\left[e\left(1-\frac{t}{x}\right)^v\right] S_{n'}^{m'}\left[e'\left(1-\frac{t}{\zeta}\right)^{v'}\right]$$
$$H\left[z_1\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(1-\frac{t}{x}\right)^{v_r}\right] H\left[z_{r+1}\left(1-\frac{t}{\zeta}\right)^{v_{r+1}},\ldots,z_{2r}\left(1-\frac{t}{\zeta}\right)^{v_{2r}}\right]\mathrm{d}t \tag{13}$$

and

$$\Delta_2 = \int_0^x (x-t)^\alpha(\zeta-t)^\beta t^{\eta+\theta} S_n^m\left[e\left(1-\frac{t}{x}\right)^v\right] S_{n'}^{m'}\left[e'\left(1-\frac{t}{\zeta}\right)^{v'}\right]$$
$$H\left[z_1\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(1-\frac{t}{x}\right)^{v_r}\right] H\left[z_{r+1}\left(1-\frac{t}{\zeta}\right)^{v_{r+1}},\ldots,z_{2r}\left(1-\frac{t}{\zeta}\right)^{v_{2r}}\right]\mathrm{d}t \tag{14}$$

The change in the order of integration in the right-hand side of (11) is justified by the well-known Fubini's theorem, provided that

$\mathrm{Re}(\eta+\theta+1)>0, \mathrm{Re}(\gamma-\theta-\beta)>0, \mathrm{Re}(\lambda)>0$ or $\mathrm{Re}(\lambda)=0$ and

$$\min[\mathrm{Re}(\theta-\delta,\theta+\beta-\delta)]>0,\ \mathrm{Re}(\alpha)+\sum_{i=1}^{r} v_i \min_{1\le j\le M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0,$$

$$\mathrm{Re}(\beta)+\sum_{i=r+1}^{2r} v_i \min_{1\le j\le M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0.$$

Now evaluating the values of Δ_1 and Δ_2 by the usual procedure, we finally obtain the following result after a little simplification

$$R_x^{\eta,\alpha}\{W_x^{\theta,\beta}f(x)\} = \Gamma(\eta+\theta+1)\sum_{k=0}^{[n/m]}\sum_{k'=0}^{[n'/m']}\frac{(-n)_{mk}(-n')_{m'k'}}{k!\ k'!}A_{n,k}A'_{n',k'}e^k e'^{k'}$$

$$\left\{x^{-\eta-1}\int_0^x \zeta^\eta\left(1-\frac{\zeta}{x}\right)^{\alpha+\beta+vk+v'k'+1}\right.$$

$$H^{0,N+N_1+2:M',N';\ldots;M^{(2r)},N^{(2r)};1,0}_{P+P_1+2,Q+Q_1+2:P',Q';\ldots;P^{(2r)},Q^{(2r)};0,1}\left[\begin{matrix} z_1\left(1-\frac{\zeta}{x}\right)^{v_1} \\ \vdots \\ z_{2r}\left(1-\frac{\zeta}{x}\right)^{v_{2r}} \\ -\left(\frac{\zeta}{x}\right)\end{matrix}\right|$$

$$(-1-\eta-\theta-\alpha-\beta-vk-v'k';v_1,\ldots,v_{2r},1),\left(-\beta-v'k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},1\right),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,N},\left(a'_j;\frac{0,\ldots,0}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},0\right)_{1,P_1},\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{N+1,P}:$$

$$\left(-1-\eta-\theta-\beta-v'k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},1\right),(-1-\eta-\theta-\alpha-\beta-vk-v'k';$$

$$v_1,\ldots,v_{2r},0),\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,Q},\left(b'_j;\frac{0,\ldots,0}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(2r)},0\right)_{1,Q_1}:$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(2r)},\gamma_j^{(2r)})_{1,p}(2r);-\\(d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(2r)},\delta_j^{(2r)})_{1,Q}(2r);(0,1)\end{matrix}\right]f(\zeta)\mathrm{d}\zeta$$

$$+x^{\theta}\int_x^{\infty}\zeta^{-\theta-1}\left(1-\frac{x}{\zeta}\right)^{\alpha+\beta+vk+v'k'+1}$$

$$H^{0,N+N_1+2:M',N';\ldots;M^{(2r)},N^{(2r)};1,0}_{P+P_1+2,Q+Q_1+2:P',Q';\ldots;P^{(2r)},Q^{(2r)};0,1}\left[\begin{matrix}z_1\left(1-\frac{x}{\zeta}\right)^{v_1}\\\vdots\\z_{2r}\left(1-\frac{x}{\zeta}\right)^{v_{2r}}\\-\left(\frac{x}{\zeta}\right)\end{matrix}\right|$$

$$(-1-\eta-\theta-\alpha-\beta-vk-v'k';v_1,\ldots,v_{2r},1),\left(-\alpha-vk;v_1,\ldots,v_r,\frac{0,\ldots,0}{r},1\right),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,N},\left(a'_j;\frac{0,\ldots,0}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},0\right)_{1,P_1},\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{N+1,P}:$$

$$\left(-1-\eta-\theta-\alpha-vk;v_1,\ldots,v_r,\frac{0,\ldots,0}{r},1\right),(-1-\eta-\theta-\alpha-\beta-vk'-v'k';v_1,\ldots,v_{2r},0),$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,Q},\left(b'_j;\frac{0,\ldots,0}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(2r)},0\right)_{1,Q_1}:$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(2r)},\gamma_j^{(2r)})_{1,p^{(2r)}};-\\(d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(2r)},\delta_j^{(2r)})_{1,Q^{(2r)}};(0,1)\end{matrix}\right]f(\zeta)\mathrm{d}\zeta\Bigg\}\tag{15}$$

where the conditions of the existence of the operators and the conditions necessary for the change of order of integration stated earlier are satisifed.

The other composition of a mixed type can be handled similarly, indeed we have

$$R_x^{\eta,\alpha}\{W_x^{\theta,\beta}f(x)\}=W_x^{\theta,\beta}\{R_x^{\eta,\alpha}f(x)\}.$$

3. Special cases

(i) If we reduce the multivariable H-functions occurring in the left-hand side of the composition formula given by (9), to the Fox's H-functions [7] in the usual way, we get the following result after a little simplification

$$R^{\eta,\alpha;m,n,v;M,N,P,Q,v}_{x;e;z,c_j,\gamma_j,d_j,\delta_j}\left\{R^{\theta,\beta;m',n',v';M',N',P',Q',v'}_{x;e';z',c'_j,\gamma'_j,d'_j,\delta'_j} f(x)\right\}$$

$$= x^{-\theta-\alpha-\beta-2}\sum_{k=0}^{[n/m]}\sum_{k'=0}^{[n'/m']}\frac{(-n)_{mk}(-n')_{m'k'}}{k!\ \ k'!}A_{n,k}A'_{n',k'}e^k e'^{k'}$$

$$x^{-(vk+v'k')}\int_0^x \zeta^{\theta}(x-\zeta)^{\alpha+\beta+vk+v'k'+1}$$

$$H^{0,2:M,N;M',N'+1;1,0}_{2,1:P,Q;P'+1,Q'+1;0,1}\left[\begin{matrix} z\left(1-\frac{\zeta}{x}\right)^{v} \\ z'\left(1-\frac{\zeta}{x}\right)^{v'} \\ -\left(1-\frac{\zeta}{x}\right)\end{matrix}\right.$$

$$(\eta-\theta-\beta-v'k';0,v',1),(-\alpha-vk;v,0,1):$$

$$(-1-\alpha-\beta-vk-v'k';v,v',1)$$

$$\left.\begin{matrix}(c_j,\gamma_j)_{1,P};(-\beta-v'k',v'),(c'_j,\gamma'_j)_{1,P'};- \\ (d_j,\delta_j)_{1,Q};(d'_j,\delta'_j)_{1,Q'},(\eta-\theta-\beta-v'k',v');(0,1)\end{matrix}\right]f(\zeta)\mathrm{d}\zeta \tag{16}$$

where

$$R^{\eta,\alpha;m,n,v;M,N,P,Q,v}_{x;e;\ \ z,c_j,\gamma_j,d_j,\delta_j}[f(x)] = x^{-\eta-\alpha-1}\int_0^x t^{\eta}(x-t)^{\alpha}S_n^m\left[e\left(1-\frac{t}{x}\right)^{v}\right]$$

$$H^{M,N}_{P,Q}\left[z\left(1-\frac{t}{x}\right)^{v}\middle|\begin{matrix}(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right]f(t)\mathrm{d}t$$

and the conditions of validity of (16) easily obtainable from those of (9) are satisfied.

The composition formula given by (16) is sufficiently general in nature and is of interest by itself. Thus by reducing the general class of polynomials and the Fox's H-functions occurring in it into their respective special cases [19, pp. 158–161; 17, pp. 18–19], we can obtain from it expressions for the composition of several fractional integral operators involving a large number of classical polynomials and simpler special functions. For example, if in the composition formula (16), we take $n = n' = 0$ (the polynomials S_0^m and $S_0^{m'}$ will reduce to $A_{0,0}$ *and* $A'_{0,0}$ respectively and can be taken to be unity without loss of generality) and further put $M = Q = M' = Q' = 1$, $N = P = N' = P' = 0$, $d_j = d'_j = 0$, $\delta_j = \delta'_j = 1$, $v = v' = 0$ and let $z \to 0, z' \to 0$, we get the results which are in essence same as obtained by Buschman [1, p. 100, eq. (2.4)] and Erdélyi [4, p. 166, eqn. (6.1)]. Also, the composition formulae (10) and (15) reduce to the remaining formulae given by Erdélyi [4, pp. 166–167, eqs (6.2) and (6.3)], if we specialize our operators to the simple operators studied by him.

(ii) If in the composition formula given by (9) we reduce the general class of polynomials S_n^m and $S_{n'}^{m'}$ to the Jacobi polynomials and the Hermite polynomials respectively [19, p. 159, eqn. (1.6); p. 158, eqn. (1.4)], we arrive at the following new and interesting result after a little simplification

$$x^{-\eta-\alpha-1}\int_0^x (x-t)^{\alpha} P_n^{(\mu,\rho)}\left[1-2\left(1-\frac{t}{x}\right)\right] H\left[z_1\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(1-\frac{t}{x}\right)^{v_r}\right] t^{\eta-\theta-\beta-1}$$

$$\left\{\int_0^t (t-\zeta)^{\beta}\left(1-\frac{\zeta}{t}\right)^{n'/2} H_{n'}\left[\frac{1}{2\sqrt{\left(1-\frac{\zeta}{t}\right)}}\right] H\left[z_{r+1}\left(1-\frac{\zeta}{t}\right)^{v_{r+1}},\ldots,\right.\right.$$

$$\left.\left. z_{2r}\left(1-\frac{\zeta}{t}\right)^{v_{2r}}\right]\zeta^{\theta} f(\zeta)\mathrm{d}\zeta\right\}\mathrm{d}t$$

$$= x^{-\theta-\alpha-\beta-2}\sum_{k=0}^{n}\sum_{k'=0}^{[n'/2]}\frac{(-n)_k(-n')_{2k'}}{k!\ \ k'!}\binom{\eta+\mu}{n}\frac{(\mu+\rho+n+1)_k}{(\mu+1)_k}(-1)^{k'}$$

$$x^{-(k+k')}\int_0^x \zeta^{\theta}(x-\zeta)^{\alpha+\beta+k+k'+1}$$

$$H^{0,N+N_1+3:M',N';\ldots;M^{(2r)},N^{(2r)};1,0}_{P+P_1+3,Q+Q_1+2:P',Q';\ldots;P^{(2r)},Q^{(2r)};0,1}\left[\begin{matrix} z_1\left(1-\frac{\zeta}{x}\right)^{v_1}\\ \vdots \\ z_{2r}\left(1-\frac{\zeta}{x}\right)^{v_{2r}} \\ -\left(1-\frac{\zeta}{x}\right)\end{matrix}\right|$$

$$\left(\eta-\theta-\beta-k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},1\right),\left(-\alpha-k;v_1,\ldots,v_r,\frac{0,\ldots,0}{r},1\right),$$

$$\left(-1-\alpha-\beta-k-k';v_1,\ldots,v_{2r},1\right),$$

$$\left(-\beta-k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},0\right),$$

$$\left(\eta-\theta-\beta-k';\frac{0,\ldots,0}{r},v_{r+1},\ldots,v_{2r},0\right),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,N},\left(a'_j;\frac{0,\ldots,0}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(2r)},0\right)_{1,P_1},\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{N+1,P}:$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{r+1}\right)_{1,Q},\left(b'_j;\frac{0,\ldots,0}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(2r)},0\right)_{1,Q_1}\quad :$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(2r)},\gamma_j^{(2r)})_{1,P^{(2r)}};\text{---}\\ (d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(2r)},\delta_j^{(2r)})_{1,Q^{(2r)}};(0,1)\end{matrix}\right] f(\zeta)\mathrm{d}\zeta \tag{17}$$

provided that the various integrals involved in the left-hand side of (17) are absolutely convergent.

Several other interesting special cases of (9), (10) and (15) involving a large variety of polynomials (which are special cases of S_n^m and $S_n^{m'}$) and numerous simpler special functions (which are particular cases of the multivariable H-function) can also be worked out but we do not record them here for lack of space.

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