Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 1, February 1994, pp. 295–304.

Hierarchic control

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Distributed control is applied to a system modelled by a parabolic evolution equation. One considers situations where there are two cost (objective) functions. One possible way is to cut the control into 2 parts, one being thought of as "the leader" and the other one as "the follower". This situation is studied in the paper, with one of the cost functions being of the controllability type. Existence and uniqueness is proven. The optimality system is given in the paper.

Keywords. Hierarchic control; stackleberg terminology; optimality system.

1. Introduction

Let us firstly recall briefly what is meant by *controllability* for distributed systems, in the case of the *wave equation.*

We shall then introduce what we mean by "*hierarchic control*" in the framework of controllability. □

Let Ω be a bounded open set of $\mathbb{R}^n$, with smooth boundary Γ.

Let Γ_0 be a subset of Γ. *The control will be applied on* Γ_0, and will depend on t.

The state y of the system is given by

$$\frac{\partial^2 y}{\partial t^2} - \Delta y = 0 \text{ in } \Omega \times (0, T) \tag{1.1}$$

subject to

$$y = \begin{cases} v & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0, T) \\ 0 & \text{on} \quad \Sigma \backslash \Sigma_0, \Sigma = \Gamma \times (0, T) \end{cases} \tag{1.2}$$

and with the initial conditions

$$y|_{t=0} = \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \text{ in } \Omega. \tag{1.3}$$

In order to make things precise, let us assume that

$$v \in L^2(\Sigma_0). \tag{1.4}$$

Then, (1.1), (1.2), (1.3) admit a unique solution, denoted by

$$\left| \begin{array}{l} y(x, t; v) = y(v) \\ y(v) \in L^2(\Omega \times (0, T)). \end{array} \right. \tag{1.5}$$

For the proof of (1.5), one can consult J L Lions [4], Chapter 2, Sections 4.2, 4.4 and [5].

One can prove that – may be after modifying the solution on a set of measure 0 on $(0, T)$–

$$\left| \begin{array}{l} t \to \left\{ y(t; v), \dfrac{\partial y}{\partial t}(t; v) \right\} \text{ is continuous} \\ \text{from } [0, T] \to L^2(\Omega) \times H^{-1}(\Omega) \end{array} \right. \tag{1.6}$$

where

$$H^{-1}(\Omega) = \text{dual of } H_0^1(\Omega),$$

$$H_0^1(\Omega) = \left\{ \varphi \mid \varphi, \frac{\partial \varphi}{\partial x_1}, \cdots, \frac{\partial \varphi}{\partial x_n} \in L^2(\Omega), \varphi = 0 \text{ on } \Gamma \right\}$$

and where $y(t; v)$ denotes the function $x \to y(x, t; v)$.

The problem of *exact controllability* is the following: let y^0, y^1 be given arbitrarily in $L^2(\Omega) \times H^{-1}(\Omega)$, and let $T > 0$ be given.

One wants to find $v \in L^2(\Sigma_0)$ such that

$$y(T; v) = y^0, \quad \frac{\partial y}{\partial t}(T; v) = y^1. \tag{1.7}$$

One says that one has exact controllability if such a control v exists for any couple $\{y^0, y^1\}$. In this case, there are infinitely many v's giving (1.7). One reasonable choice is then to consider

$$\inf \frac{1}{2} \int_{\Sigma_0} v^2 \, d\Sigma, \quad v \text{ subject to (1.7).} \tag{1.8}$$

For a study of this problem we refer to J L Lions [5], Vol. 1. □

Approximate controllability comes next. Let B_0 (resp. B_{-1}) be the unit ball of $L^2(\Omega)$ (resp. $H^{-1}(\Omega)$) and let α_0, α_1 be arbitrarily small positive numbers. We "relax" (1.7) by

$$y(T; v) \in y^0 + \alpha_0 B_0, \frac{\partial y}{\partial t}(T; v) \in y^1 + \alpha_1 B_{-1}, \tag{1.9}$$

If there is a control v satisfying (1.8) for every couple $\{y^0, y^1\}$ and for every $\alpha_0, \alpha_1 > 0$, one says that one has *approximate controllability*.

Of course approximate controllability is equivalent to

$$\left\{ y(T; v), \frac{\partial y}{\partial t}(T; v) \right\} \text{ spans a dense subset of } L^2(\Omega) \times H^{-1}(\Omega). \tag{1.9 bis}$$

One can prove (cf. J L Lions [5], Vol. 1), in particular using Holmgren's uniqueness theorem (cf. loc. cit., Th. 8.2, Chapter 1), that

$$\left| \begin{array}{l} \text{approximate controllability holds true if (and only if)} \\ T > 2d(\Omega, \Gamma_0), \end{array} \right. \tag{1.10}$$

where

$$\left|\begin{array}{l} \mathrm{d}(\Omega, \Gamma_0) = \sup_{x \in \Omega} \mathrm{d}(x, \Gamma_0), \\ \mathrm{d}(x, \Gamma_0) = \text{distance from } x \text{ to } \Gamma_0 \text{ taken in the geodesic sense in } \Omega. \end{array}\right. \tag{1.11}$$

In this case, we are looking for

$$\inf \frac{1}{2} \int_{\Sigma_0} v^2 \mathrm{d}\Sigma, \quad v \text{ subject to (1.9).} \tag{1.12}$$

We refer to J L Lions [5] for this question, where, in particular, we derive the *optimality system*, i.e. the set of P.D.E's (Partial Differential Equation) and of Variational Inequalities which characterize the unique solution of (1.12). □

But there are many situations—it is almost always the case!—where achieving (1.8) is *not the only criteria.*

For instance—and this will be the question we shall address here—one may want to achieve (1.8) but we also want that, during the whole interval $t \in (0, T)$, $y(t; v)$ "does not go too far" from a given state $y_2(x, t)$. In a quantitative manner, we introduce

$$\mathscr{T}(v) = \frac{1}{2} \iint_{\Omega \times (0, T)} (y(x, t; v) - y_2(x, t))^2 \mathrm{d}x \mathrm{d}t + \frac{\beta}{2} \int_{\Sigma_0} v^2 \mathrm{d}\Sigma \tag{1.13}$$

and we would like to "minimize" (1.13).

But of course minimizing at the same time (1.12) and (1.13) does not make sense. □

This is a question of *multicriteria optimization*-a situation which is classical in economy. Precisely possible notions that one can use when dealing with these questions are coming from Economy with Pareto optimal control and with Stackleberg [10] Optimization.

We divide Γ_0 in two parts $\Gamma_1, \Gamma_2 \subset \Gamma$

$$\Gamma_0 = \Gamma_1 \cup \Gamma_2 \tag{1.14}$$

and we consider

$$v = \{v_1, v_2\}, \quad v_i = \text{control function in } L^2(\Sigma_i), \quad i = 1, 2. \tag{1.15}$$

We can also write

$$v = v_1 + v_2 \tag{1.16}$$

with

$$\Gamma_1 = \Gamma_2 = \Gamma_0.$$

We assume that there is a *hierarchy* in our wishes. The main objective is to have (1.8) at "minimum cost". The second priority is to maintain (1.13) as small as possible. In the decomposition (1.14), (1.15) we also establish a hierarchy. We think of v_1 as being the "main" control, *the leader* (in Stackleberg terminology), and we think of v_2 *as the follower*, always in Stackleberg terminology.

Let us set

$$y(v) = y(v_1, v_2). \tag{1.16}$$

We write (1.13) as

$$J_2(v_1, v_2) = \frac{1}{2}\iint_{\Omega \times (0,T)} (y(v_1, v_2) - y_2)^2 \mathrm{d}x\mathrm{d}t + \frac{\beta}{2}\int_{\Sigma_2} v_2^2 \mathrm{d}\Sigma \tag{1.17}$$

(we drop in the last term $\frac{\beta}{2}\int_{\Sigma_1} v_1^2 \mathrm{d}\Sigma$, without changing anything as it will be clear below).

Then given v_1 we consider

$$\inf_{v_2 \in L^2(\Sigma_2)} J_2(v_1, v_2). \tag{1.18}$$

This is a classical type problem in the control of distributed systems (cf. J L Lions [8]). It admits a unique solution

$$v_2 = \mathscr{F}(v_1) \tag{1.19}$$

given by an optimality system that we present in §2 below.

We then consider the state

$$y(v_1, \mathscr{F}(v_1)) \tag{1.20}$$

given by the solution of (compare to (1.1), (1.2), (1.3)):

$$\begin{gathered}
\frac{\partial^2 y}{\partial t^2} - \Delta y = 0 \text{ in } \Omega \times (0, T), \\
y = \begin{cases} v_1 \text{ on } \Sigma_1 = \Gamma_1 \times (0, T) \\ \mathscr{F}(v_1) \text{ on } \Sigma_2 = \Gamma_2 \times (0, T) \\ 0 \text{ on } \Sigma \backslash \Sigma_1 \cup \Sigma_2 = \Sigma \backslash \Sigma_0, \end{cases} \\
y|_{t=0} = \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \text{ in } \Omega.
\end{gathered} \tag{1.21}$$

Then we address the main criteria, i.e. we want to find

$$\inf \frac{1}{2}\int_{\Sigma_1} v_1^2 \mathrm{d}\Sigma, \tag{1.22}$$

v_1 subject to

$$\begin{gathered}
y(T; v_1, \mathscr{F}(v_1)) \in y^0 + \alpha_0 B_0, \\
\frac{\partial y}{\partial t}(T; v_1, \mathscr{F}(v_1)) \in y^1 + \alpha_1 B_{-1}
\end{gathered} \tag{1.23}$$

provided this is possible $\forall y^0, y^1, \alpha_0, \alpha_1$.

Actually we shall prove (§3 below) that in case (1.16)

$$\left| \begin{array}{l} \left\{ y(T; v_1, \mathscr{F}(v_1)), \dfrac{\partial y}{\partial t}(T; v_1, \mathscr{F}(v_1)) \right\} \\ \text{spans a dense subspace of } L^2(\Omega) \times H^{-1}(\Omega) \end{array} \right. \tag{1.24}$$

if and only if

$$T > 2d(\Omega, \Gamma_1). \tag{1.25}$$

□

Remark 1.1. Since we have

$$d(\Omega, \Gamma_1) \geqslant d(\Omega, \Gamma_0) \tag{1.26}$$

condition (1.25) *is stronger* than (1.10), as it is natural (since we want to achieve *more*). □

Remark 1.2. The idea of dividing v in two parts v_1, v_2 in order to apply Stackleberg Optimization has been introduced in [7], for systems governed by *parabolic equations.* □

Remark 1.3. The fact that T should be large enough (cf. (1.10), (1.25)) is due to the finite speed of wave propagation. Conditions of this type do not appear for diffusion type systems. □

Remark 1.4. The problem addressed in this paper arises from applications. For instance, if we consider the controllability of a *flexible structure* (large space structure, sea-platform, flexible robot,...), we want to reach a given state at a given time T, with some constraints in the course of the operations, which *can* be expressed by J_2. □

Remark 1.5. The division of v in 2 parts v_1, v_2, can be made in infinitely many ways. It is quite natural to try to maintain $d(\Omega, \Gamma_1)$ as close as possible from $d(\Omega, \Gamma_0)$ (cf. (1.25), (1.26)). There are no other *mathematical* considerations. □

Remark 1.6. We present the solution in the specific case of the wave equation with Dirichlet's type control. But the method is completely general. Similar considerations could be made for all the situations considered in [5], Vols 1 and 2. □

Remark 1.7. One can use similar notions for *non-linear* controllability problems, but with most of the theoretical questions being then *open.* □

Remark 1.8. Numerical aspects are not considered here. A survey of numerical methods for controllability problems is given in R Glowinski and the author in [2]. □

2. Optimality system for the follower

Let us denote by v_2 the solution of (1.18). We set for a moment

$$y(v_1, v_2) = y.$$

The Euler equation for (1.18) is given by

$$\iint_{\Omega \times (0,T)} (y - y_2)\hat{y}\,dx\,dt + \rho_0 \int_{\Sigma_2} v_2 \hat{v}_2\,d\Sigma = 0, \tag{2.1}$$

for every $\hat{v}_2 \in L^2(\Sigma_2)$, where $\hat{y}$ is given by

$$\frac{\partial^2 \hat{y}}{\partial t^2} - \Delta \hat{y} = 0 \text{ in } \Omega \times (0, T),$$

$$\hat{y} = \begin{cases} 0 \text{ on } \Sigma_1, \\ \hat{v}_2 \text{ on } \Sigma_2, \\ 0 \text{ on } \Sigma \backslash \Sigma_1 \cup \Sigma_2 \end{cases} \tag{2.2}$$

$$\hat{y}|_{t=0} = \left.\frac{\partial \hat{y}}{\partial t}\right|_{t=0} = 0.$$

We introduce p by

$$\left| \begin{array}{l} p'' - \Delta p = y - y_2 \text{ in } \Omega \times (0, T), \\ p(T) = p'(T) = 0, p = 0 \text{ on } \Sigma \end{array} \right. \tag{2.3}$$

where we write from now on, $\varphi' = \dfrac{\partial \varphi}{\partial t}$, $\varphi'' = \dfrac{\partial^2 \varphi}{\partial t^2}$.

If we multiply (2.3) by $\hat{y}$ and if we integrate by parts (the integrations by parts being valid) we obtain

$$\iint_{\Omega \times (0, T)} (y - y_2)\hat{y} \mathrm{d}x \mathrm{d}t = -\int_{\Sigma_2} \frac{\partial p}{\partial v} \hat{v}_2 \mathrm{d}\Sigma, \tag{2.4}$$

so that (2.1) becomes

$$\frac{\partial p}{\partial v} = \beta v_2 \text{ on } \Sigma_2. \tag{2.5}$$

We can summarize as follows. Given v_1 in $L^2(\Sigma_1)$, the follower v_2 is given by

$$v_2 = \mathscr{F}(v_1) = \frac{1}{\beta}\frac{\partial p}{\partial v} \text{ on } \Sigma_2 \tag{2.6}$$

where $\{y, p\}$ is the unique solution of (the optimality system)

$$y'' - \Delta y = 0,$$

$$p'' - \Delta p = y - y_2,$$

$$y(0) = y'(0) = 0, \quad p(T) = p'(T) = 0,$$

$$y = \begin{cases} v_1 \text{ on } \Sigma_1 \\ \dfrac{1}{\beta}\dfrac{\partial p}{\partial v} \text{ on } \Sigma_2, p = 0 \text{ on } \Sigma, \\ 0 \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \tag{2.7}$$

Of course, $\{y, p\}$ depends on v_1:

$$\{y, p\} = \{y(v_1), p(v_1)\}. \tag{2.8}$$

We have now to find the optimal leader.

3. Optimality system for the leader

We can now rewrite (1.21), (1.22), (1.23) as follows, using (2.7), (2.8). We are looking for

$$\inf \frac{1}{2} \int_{\Sigma_1} v_1^2 \, d\Sigma \tag{3.1}$$

where v_1 is subject to

$$y(T; v_1) \in y^0 + \alpha_0 B_0, \quad y'(T; v_1) \in y^1 + \alpha_1 B_{-1}, \tag{3.2}$$

assuming that such v_1's do exist. □

We are now going to show that in case (1.16)

$$\left| \begin{array}{l} \{y(T; v_1), y'(T; v_1)\} \text{ spans a dense (affine) subspace of} \\ L^2(\Omega) \times H^{-1}(\Omega) \text{ when } v_1 \text{ spans } L^2(\Sigma_1), \end{array} \right. \tag{3.3}$$

provided that (1.25) holds true. □

Let us set

$$y = y_0 + z, \quad p = p_0 + q, \tag{3.4}$$

where $\{y_0, p_0\}$ is given by

$$\begin{aligned} & y_0'' - \Delta y_0 = 0, \\ & p_0'' - \Delta p_0 = y_0 - y_2, \\ & y_0(0) = y_0'(0) = 0, \quad p_0(T) = p_0'(T) = 0 \text{ in } \Omega, \\ & y_0 = \begin{cases} 0 \text{ on } \Sigma_1 \\ \dfrac{1}{\beta} \dfrac{\partial p_0}{\partial v} \text{ on } \Sigma_2, p_0 = 0 \text{ on } \Sigma, \\ 0 \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \end{aligned} \tag{3.5}$$

and where $\{z, q\}$ is given by

$$\begin{aligned} & z'' - \Delta z = 0, \\ & q'' - \Delta q = z, \\ & z(0) = z'(0) = 0, q(T) = q'(T) = 0, \\ & z = \begin{cases} v_1 \text{ on } \Sigma_1 \\ \dfrac{1}{\beta} \dfrac{\partial q}{\partial v} \text{ on } \Sigma_2, q = 0 \text{ on } \Sigma, \\ 0 \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \end{aligned} \tag{3.6}$$

We set next

$$L v_1 = \{z'(T; v_1), -z(T; v_1)\} \tag{3.7}$$

which defines

$$L \in \mathscr{L}(L^2(\Sigma_1); H^{-1}(\Omega) \times L^2(\Omega)). \tag{3.8}$$

Then (3.2), using (3.4) and (3.7) can be rewritten as

$$Lv_1 \in \{y^1 - y_0'(T) + \alpha_1 B_{-1}, -y^0 + y_0(T) + \alpha_0 B_0\}. \tag{3.9}$$

We introduce two convex proper functions as follows, firstly

$$F_1(v_1) = \frac{1}{2}\int_{\Sigma_1} v_1^2 \mathrm{d}\Sigma \text{ on } L^2(\Sigma_1), \tag{3.10}$$

the second one being given on $H^{-1}(\Omega) \times L^2(\Omega)$ by

$$F_2(g,h) = \begin{cases} 0 \text{ if } \{g,h\} \in \{y^1 - y_0'(T) + \alpha_1 B_{-1}, -y^0 + y_0(Y) + \alpha_0 B_0\}, \\ +\infty \text{ otherwise.} \end{cases} \tag{3.11}$$

With these notations problem (3.1), (3.2) becomes equivalent to

$$\inf._{v_1} F_1(v_1) + F_2(Lv_1) \tag{3.12}$$

provided we prove that *the range of L is dense in* $H^{-1}(\Omega) \times L^2(\Omega)$, *under condition* (1.25). □

We introduce the "adjoint states" φ, ψ as follows. Let f^0, f^1 be given in $H_0^1(\Omega) \times L^2(\Omega)$. We define φ, ψ as the unique solution of the system

$$\begin{aligned} &\varphi'' - \Delta\varphi = \psi, \\ &\psi'' - \Delta\psi = 0, \\ &\varphi(T) = f^0, \varphi'(T) = f^1, \psi(0) = \psi'(0) = 0, \\ &\varphi = 0 \text{ on } \Sigma,\ \psi = \begin{cases} 0 \text{ on } \Sigma_1 \\ \frac{1}{\beta}\frac{\partial\varphi}{\partial v} \text{ on } \Sigma_2, \\ 0 \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \end{aligned} \tag{3.13}$$

If we multiply the first (resp. second) equation in (3.13) by z (resp. q), we obtain, after integrations by parts,

$$(z'(T), f^0) - (z(T), f^1) = -\int_{\Sigma_1} \frac{\partial\varphi}{\partial v} v_1 \mathrm{d}\Sigma. \tag{3.14}$$

Therefore if $(z'(T), f^0) - (z(T), f^1) = 0 \forall v_1 \in L^2(\Sigma_1)$, then

$$\frac{\partial\varphi}{\partial v} = 0 \text{ on } \Sigma_1. \tag{3.15}$$

Then in case (1.16) $\psi = 0$ on Σ so that $\psi \equiv 0$. Therefore

$$\varphi'' - \Delta\varphi = 0, \varphi = 0 \text{ on } \Sigma \tag{3.16}$$

and satisfies to (3.15). Therefore, according to Holmgren's uniqueness theorem (cf. L Hormander [3], Th. 5.3.3, and for the explicit use of it made here, cf. J L L'ions [5], Chapter 1, Section 8) if (1.25) takes place, $\varphi \equiv 0$, so that $f^0 = 0, f^1 = 0$ and (3.3) is proven. □

We can now apply to (3.12) the duality of Fenchel and T R Rockafellar [9] (cf. also I Ekeland and R Temam [1]). We obtain

$$\inf_{v_1} F_1(v_1) + F_2(Lv_1) =$$

$$- \inf_{f^0, f^1 \in H_0^1(\Omega) \times L^2(\Omega)} F_1^*(L^*\{f^0, f^1\}) + F_2^*(-f^0, -f^1) \tag{3.17}$$

where F_i^* is the conjugate function of F_i and L^* the adjoint of L.

By virtue of (3.14)

$$L^*\{f^0, f^1\} = -\frac{\partial \varphi}{\partial \nu} \text{ on } \Sigma_1. \tag{3.18}$$

We see easily that

$$F_1^* = F_1$$

and

$$F_2^*(f^0, f^1) = (f^0, y^1 - y_0'(T)) + \alpha_1 \|f^0\|_{H_0^1(\Omega)} + (f^1, -y^0 + y_0(T)) + \alpha_0 \|f^1\|_{L^2(\Omega)}.$$

Therefore the (opposite of) right hand side of (3.17) is given by

$$\inf_{f^0, f^1}\left[\frac{1}{2}\int_{\Sigma_1}\left(\frac{\partial \varphi}{\partial \nu}\right)^2 d\Sigma_1 - (f^0, y^1 - y_0'(T)) + (f^1, y^0 - y_0(T)) + \alpha_1 \|f^0\|_{H_0^1(\Omega)} + \alpha_0 \|f^1\|_{L^2(\Omega)}\right]. \tag{3.19}$$

This is the dual problem of (3.1), (3.2). □

We have now two ways to derive the optimality system for the leader control, starting from the primal or from the dual problem. We obtain the following result.

Theorem 3.1. *We assume that* (1.16) *and* (1.25) *hold true. For* $\{f^0, f^1\}$ *in* $H_0^1(\Omega) \times L^2(\Omega)$ *we uniquely define* $\{\varphi, \psi, y, p\}$ *by:*

$$\varphi'' - \Delta\varphi = \psi, \psi'' - \Delta\psi = 0,$$

$$y'' - \Delta y = 0, p'' - \Delta p = y - y_2, \text{ in } \Omega \times (0, T),$$

$$\varphi(T) = f^0, \varphi'(T) = f^1, \psi(0) = \psi'(0) = 0,$$

$$y(0) = y'(0) = 0, p(T) = p'(T) = 0, \text{ in } \Omega$$

$$\varphi = 0 \text{ on } \Sigma, \ \psi = \begin{cases} 0 \text{ on } \Sigma_1 \\ \dfrac{1}{\beta}\dfrac{\partial \varphi}{\partial \nu} \text{ on } \Sigma_2, \\ 0 \text{ on } \Sigma_3 \end{cases} \tag{3.20}$$

$$y = \begin{cases} -\dfrac{\partial\varphi}{\partial\nu} \text{ on } \Sigma_1 \\ \dfrac{1}{\beta}\dfrac{\partial p}{\partial\nu} \text{ on } \Sigma_2, p = 0 \text{ on } \Sigma, \\ 0 \text{ on } \Sigma\backslash\Sigma_0. \end{cases}$$

We uniquely define f^0, f^1 as the solution of the Variational Inequality

$$\begin{gathered}(y''(T) - y^1, \hat{f}^0 - f^0) - (y(T) - y^0, \hat{f}^1 - f^1) \\ + \alpha_1 \|\hat{f}^0\|_{H_0^1(\Omega)} - \alpha_1 \|f^0\|_{H_0^1(\Omega)} + \alpha_0 \|\hat{f}^1\|_{L^2(\Omega)} - \alpha_0 \|f^1\|_{L^2(\Omega)} \geqslant 0 \\ \forall \hat{f}^0, \hat{f}^1 \in H_0^1(\Omega) \times L^2(\Omega). \end{gathered} \tag{3.21}$$

Then the optimal leader is given by

$$v_1 = -\frac{\partial\varphi}{\partial\nu} \text{ on } \Sigma_1 \tag{3.22}$$

where φ corresponds to the solution of (3.21).

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