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Existence theory for linearly elastic shells

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Dedicated to the memory of Professor K G Ramanathan

Abstract. We review existence and uniqueness results, recently obtained for three of the most important linear two-dimensional shell models: Koiter's model, the bending model and the membrane model. They rely on a crucial lemma of J L Lions, used in an essential way for establishing in each case a generalized Korn's inequality, which is then combined with a generalized rigid displacement lemma of a geometrical nature.

Keywords. Linearly elastic shells; bending shell model; membrane shell model; Koiter's model.

1. Geometrical and mechanical preliminaries

In what follows, Greek indices and exponents vary in the set $\{1,2\}$, Latin indices and exponents vary in the set $\{1,2,3\}$, and the repeated index or exponent convention for summation is used. The Euclidean inner product, the vector product and the Euclidean norm, of vectors $\mathbf{u}$, $\mathbf{v}\in\mathbf{R}^3$ are denoted as $\mathbf{u}\cdot\mathbf{v}, \boldsymbol{\mu}\times\mathbf{v}$, and $|\mathbf{u}|$.

Let ω be an open, bounded, connected subset of $\mathbf{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ. Let $y=(y^1,y^2)$ denote a generic point of the set $\bar{\omega}$ and let $\partial_\alpha=\partial/\partial y^\alpha$. We consider a surface S in $\mathbf{R}^3$, of the form $S=\boldsymbol{\varphi}(\bar{\omega})$, where $\boldsymbol{\varphi}:\bar{\omega}\to\mathbf{R}^3$ is a given, injective, smooth enough mapping. We assume that the two vectors $\mathbf{a}_\alpha=\partial_\alpha\boldsymbol{\varphi}$ are linearly independent at all points of $\bar{\omega}$.

The vectors $\mathbf{a}_\alpha$ form the covariant basis of the tangent plane, and the vectors $\mathbf{a}^\alpha$, defined by the relations $\mathbf{a}^\alpha\cdot\mathbf{a}_\beta=\delta^\alpha_\beta$, form its contravariant basis. The three vectors $\mathbf{a}^i$, where $\mathbf{a}^3=\mathbf{a}_3=(\mathbf{a}_1\times\mathbf{a}_2)/|\mathbf{a}_1\times\mathbf{a}_2|$ form the contravariant basis at each point of S. The Christoffel symbols are defined by

$$\Gamma^\rho_{\alpha\beta}=\mathbf{a}^\rho\cdot\partial_\alpha\mathbf{a}_\beta,$$

and the first, second and third fundamental forms of S are defined by

$$a_{\alpha\beta}=\mathbf{a}_\alpha\cdot\mathbf{a}_\beta \text{ or } a^{\alpha\beta}=\mathbf{a}^\alpha\cdot\mathbf{a}^\beta,$$

$$b_{\alpha\beta}=-\mathbf{a}_\alpha\cdot\partial_\beta\mathbf{a}_3,$$

$$c_{\alpha\beta}=b^\rho_\alpha b_{\rho\beta}, \text{ where } b^\rho_\alpha=a^{\rho\sigma}b_{\sigma\alpha}.$$

Note that $\Gamma^\rho_{\alpha\beta}=\Gamma^\rho_{\beta\alpha}$, $a_{\alpha\beta}=a_{\beta\alpha}$, $b_{\alpha\beta}=b_{\beta\alpha}$, $c_{\alpha\beta}=c_{\beta\alpha}$. Finally, we let

$$a=\det(a_{\alpha\beta}).$$

We consider a linearly elastic shell, with middle surface S and thickness 2ε, clamped along a portion of its lateral face, and we let $\lambda > 0$ and $\mu > 0$ denote the Lamé constants of its constituting material. In each one of the two-dimensional shell models considered here, the unknowns are the three covariant components $\zeta_i : \bar{\omega} \to \mathbf{R}$ of the displacement $\zeta_i \mathbf{a}^i$ of the points of S, and we let $\boldsymbol{\zeta} = (\zeta_i) : \bar{\omega} \to \mathbf{R}^3$. With an arbitrary vector field $\boldsymbol{\eta} = (\eta_i) : \bar{\omega} \to \mathbf{R}^3$, we associate the (linearized) strain, or change of metric, tensor and the (linearized) change of curvature tensor, whose covariant components are respectively given by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\rho_{\alpha\beta}\eta_\rho - b_{\alpha\beta}\eta_3,$$

$$\begin{aligned}\Upsilon_{\alpha\beta}(\boldsymbol{\eta}) &= \partial_{\alpha\beta}\eta_3 - \Gamma^\rho_{\alpha\beta}\partial_\rho \eta_3 - c_{\alpha\beta}\eta_3 \\ &\quad + b^\rho_\beta(\partial_\alpha \eta_\rho - \Gamma^\sigma_{\rho\alpha}\eta_\sigma) + b^\rho_\alpha(\partial_\beta \eta_\rho - \Gamma^\sigma_{\rho\beta}\eta_\sigma) \\ &\quad + (\partial_\alpha b^\rho_\beta + \Gamma^\rho_{\alpha\sigma} b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta} b^\rho_\sigma)\eta_\rho.\end{aligned}$$

We shall also use the fourth-order elasticity tensor of a two-dimensional shell, whose contravariant components are given by

$$a^{\alpha\beta\rho\sigma} = \frac{4\lambda\mu}{(\lambda + 2\mu)} a^{\alpha\beta} a^{\rho\sigma} + 2\mu(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}).$$

2. The two-dimensional shell model of W T Koiter

The fundamental work of John [17] has led Koiter [18] to propose the following two-dimensional shell model, called Koiter's model: The unknown $\boldsymbol{\zeta} = (\zeta_i)$ solves the following variational problem:

$$\boldsymbol{\zeta} \in \mathbf{V}(\omega) \text{ and } B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega),$$

where (∂_ν denotes the outer normal derivative along γ, and γ_0 is a subset of the boundary γ):

$$\mathbf{V}(\omega) = \{\boldsymbol{\eta} = (\eta_i);\ \eta_\alpha \in H^1(\omega), \eta_3 \in H^2(\omega), \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$B(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_\omega \left\{\frac{\varepsilon^3}{3} a^{\alpha\beta\rho\sigma}\Upsilon_{\rho\sigma}(\boldsymbol{\zeta})\Upsilon_{\alpha\beta}(\boldsymbol{\eta}) + \varepsilon a^{\alpha\beta\rho\sigma}\gamma_{\rho\sigma}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\right\}\sqrt{a}\,\mathrm{d}y,$$

$$L(\boldsymbol{\eta}) = \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y.$$

The linear form L takes into account the applied forces. The given functions p^i are assumed to be in $L^2(\omega)$.

The symmetric bilinear form B and the linear form L are continuous over the space $\mathbf{V}(\omega)$. Hence the existence and uniqueness of the solution of the above variational problem follow, by the *Lax-Milgram lemma*, from:

Theorem 1. *Assume that* $\boldsymbol{\varphi}\in\mathscr{C}^3(\bar{\omega})$ *and that length* $\gamma_0 > 0$. *There exists a constant* β *such that*

$$\beta > 0 \text{ and } B(\boldsymbol{\eta},\boldsymbol{\eta}) \geqslant \beta \|\boldsymbol{\eta}\|^2_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)}$$

for all $\boldsymbol{\eta}\in\mathbf{V}(\omega)$. □

Theorem 1 was first established by Bernadou and Ciarlet [3]. The proof relied on various equivalences of norms involving covariant derivatives (also due to Rougée [23]), on a rigid displacement lemma (cf. (iii) below), and on technical inequalities combined with weak lower semi-continuity properties of the associated quadratic functional; an outline of this proof was also given in Ciarlet [6]. A notable simplification of this proof was recently proposed by Ciarlet and Miara [10]. It is this proof that we sketch here; the full, detailed, proof is given in Bernadou *et al* [4].

Outline of the proof of Theorem 1:

(i) There exists a constant $C_1 > 0$ such that

$$a^{\alpha\rho}(y)a^{\beta\sigma}(y)t_{\rho\sigma}t_{\alpha\beta} \geqslant C_1 \sum_{\alpha,\beta} |t_{\alpha\beta}|^2$$

for all $y\in\bar{\omega}$ and all symmetric tensors $(t_{\alpha\beta})$. Since

$$a^{\alpha\beta}(y)a^{\rho\sigma}(y)t_{\rho\sigma}t_{\alpha\beta} \geqslant 0$$

on the other hand, it suffices to show that there exists a constant $C_2 > 0$ such that

$$\left\{\sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2}\right\}^{1/2} \geqslant C_2 \|\boldsymbol{\eta}\|_{H^1\times H^1\times H^2},$$

for all $\boldsymbol{\eta}\in\mathbf{V}(\omega)$, where, here and subsequently, we let $L^2 = L^2(\omega)$, $H^m = H^m(\omega)$ at some places, for the sake of conciseness.

(ii) Define the space

$$\mathbf{E}(\omega) = \{\boldsymbol{\eta} = (\eta_i); \eta_\alpha\in L^2, \eta_3\in H^1, \gamma_{\alpha\beta}(\boldsymbol{\eta})\in L^2, \Upsilon_{\alpha\beta}(\boldsymbol{\eta})\in L^2\},$$

where both relations $\gamma_{\alpha\beta}(\boldsymbol{\eta})\in L^2$ and $\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\in L^2$ are to be understood in the sense of distributions. We show that

$$\mathbf{E}(\omega) = H^1(\omega)\times H^1(\omega)\times H^2(\omega).$$

Let $\boldsymbol{\eta} = (\eta_i)$ be an arbitrary element of the space $\mathbf{E}(\omega)$. The relations

$$e_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \Gamma^\rho_{\alpha\beta}\eta_\rho + b_{\alpha\beta}\eta_3$$

imply that the functions $e_{\alpha\beta}(\boldsymbol{\eta})$ belong to the space $L^2(\omega)$. Hence the identities (in the sense of distributions)

$$\partial_{\alpha\beta}\eta_\rho = \partial_\alpha e_{\beta\rho}(\eta) + \partial_\beta e_{\alpha\rho}(\boldsymbol{\eta}) - \partial_\rho e_{\alpha\beta}(\boldsymbol{\eta})$$

show that $\partial_\beta(\partial_\alpha \eta_\rho) \in H^{-1}(\omega)$ (the assumption $\boldsymbol{\varphi} \in \mathscr{C}^3(\bar{\omega})$ is used here). Since $\partial_\alpha \eta_\rho \in H^{-1}(\omega)$ (recall that $\eta_\rho \in L^2(\omega)$), a *lemma of J L Lions* (first mentioned by Magenes and Stampacchia [19] and proved in Duvaut and Lions ([15], p. 110), then extended to Lipschitz-continuous boundaries in Borchers and Sohr [5] and Amrouche and Girault [2] implies that the distributions $\partial_\alpha \eta_\rho$ are in the space $L^2(\omega)$; hence $\eta_\rho \in H^1(\omega)$. The definition of $\Upsilon_{\alpha\beta}(\eta_3)$ then implies that $\eta_3 \in H^2(\omega)$, and the inclusion

$$\mathbf{E}(\omega) \subset H^1(\omega) \times H^1(\omega) \times H^2(\omega)$$

is thus established; the other inclusion clearly holds.

When equipped with the norm $\|\cdot\|_{\mathbf{E}}$ defined by

$$\|\boldsymbol{\eta}\|_{\mathbf{E}} = \left\{ \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2} + \sum_{\alpha} \|\eta_\alpha\|^2_{L^2} + \|\eta_3\|^2_{H^1} \right\}^{1/2},$$

the space $\mathbf{E}(\omega)$ becomes a Hilbert space. Since the identity mapping from the space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ into the space $\mathbf{E}(\omega)$ is continuous and onto (we just showed that the two spaces are identical), and since both spaces are complete, the open mapping theorem implies that the identity mapping from $\mathbf{E}(\omega)$ onto $H^1(\omega) \times H^1(\omega)$ $H^2(\omega)$ is also continuous. Hence there exists a constant $C_3 > 0$ such that the following generalized Korn's inequality holds:

$$\|\boldsymbol{\eta}\|_{\mathbf{E}} \geqslant C_3 \left\{ \sum_{\alpha} \|\eta_\alpha\|^2_{H^1} + \|\eta_3\|^2_{H^2} \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

In other words, the norm $\|\cdot\|_{\mathbf{E}}$ is a norm over the space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, equivalent to its product norm.

(iii) We next show that the semi-norm $|\cdot|_{\mathbf{E}}$ defined by

$$|\boldsymbol{\eta}|_{\mathbf{E}} = \left\{ \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2} \right\}^{1/2},$$

is a norm over the space $\mathbf{V}(\omega)$. To this end, it suffices to show that

$$\boldsymbol{\eta} \in \mathbf{V}(\omega) \text{ and } |\boldsymbol{\eta}|_{\mathbf{E}} = 0 \Rightarrow \boldsymbol{\eta} = 0.$$

The *generalized displacement lemma* (cf. Bernadou and Ciarlet [[3], th. 5.1–1] or Bernadou, *et al* [[4], lemma 2.5]) shows that, if an element $\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ satisfies $\Upsilon_{\alpha\beta}(\boldsymbol{\eta}) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω, there exist two vectors $\mathbf{a} \in \mathbf{R}^3$ and $\mathbf{b} \in \mathbf{R}^3$ such that

$$\eta_i(y)\mathbf{a}^i(y) = \mathbf{a} + \mathbf{d} \times \boldsymbol{\varphi}(y) \text{ for all } y \in \omega.$$

The conclusion then follows by taking into account the boundary conditions satisfied by the functions η_i along γ_0 (the assumption length $\gamma_0 > 0$ is needed here).

(iv) We finally show that, over the space $\mathbf{V}(\omega)$, the norm $|\cdot|_{\mathbf{E}}$ is in fact equivalent to the product norm $\|\cdot\|_{H^1 \times H^1 \times H^2}$, i.e., that there exists a constant C_2 such that the inequality announced in (i) holds. Otherwise, there exists a sequence $(\boldsymbol{\eta}^k)$ of elements in $\mathbf{V}(\omega)$ such that

$$|\boldsymbol{\eta}^k|_{\mathbf{E}} \to 0 \text{ as } k \to \infty, \|\boldsymbol{\eta}^k\|_{H^1 \times H^1 \times H^2} = 1 \text{ for all } k.$$

By the Rellich–Kondrašov theorem, there exists a subsequence $(\boldsymbol{\eta}^l)$ that converges in the space $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$. Since $|\boldsymbol{\eta}^l|_{\mathbf{E}} \to 0$ as $l \to \infty$, the subsequence $(\boldsymbol{\eta}^l)$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathbf{E}}$, whence also to $\|\cdot\|_{H^1 \times H^1 \times H^2}$ by (ii). Let $\boldsymbol{\eta} = \lim_{l \to \infty} \boldsymbol{\eta}^l$ in the space $\mathbf{V}(\omega)$. On the one hand, $|\boldsymbol{\eta}|_{\mathbf{E}} = \lim_{l \to \infty} |\boldsymbol{\eta}^l|_E = 0$; on the other, $\|\boldsymbol{\eta}\|_{H^1 \times H^1 \times H^2} = \lim_{l \to \infty} \|\boldsymbol{\eta}^l\|_{H^1 \times H^1 \times H^2} = 1$. Hence we have reached a contradiction, and the proof is complete. □

Remarks. (1) No geometrical assumption on the middle surface S is needed here (by contrast, an assumption of uniform ellipticity will be introduced to handle the membrane model; cf. § 4). (2) The same analysis can be applied to the two-dimensional shell model of Naghdi [22]; cf. Bernadou *et al* ([3], th. 3.1). □

3. The two-dimensional bending shell model

As observed in Ciarlet [9], Koiter's model is *not* a limit model, i.e., one that can be obtained in a rational fashion as a limit of the three-dimensional equations as $\varepsilon \to 0$. Indeed, Sanchez–Palencia [26] has shown that the solution of the three-dimensional shell equations has two essentially different behaviors as the thickness approaches zero, according to the geometry of the middle surface and to the boundary conditions: It converges either to the solution of the bending shell model, or to the solution of the membrane shell model, which are described in this and the next sections (for more details about this limit behavior, the relations between these models, and the differences between shells and plates, see also Destuynder [14], Sanchez–Palencia [24, 25], Ciarlet [7, 8] Miara and Sanchez–Palencia [20]).

More specifically, let

$$\mathbf{V}_0(\omega) = \{\boldsymbol{\eta} \in \mathbf{V}(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

denote the space of *inextensional displacements*, where the strain tensor $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$ and the space $\mathbf{V}(\omega)$ are defined as in § 1 and § 2, respectively.

If $\mathbf{V}_0(\omega) \neq \{\mathbf{0}\}$ (there exist such instances), the first non-zero term $\boldsymbol{\zeta} = (\zeta_i)$ of a formal asymptotic expansion as powers of ε of the covariant components of the three-dimensional displacement is independent of the transverse variable, and it solves the following two-dimensional shell model, called the bending model:

$$\boldsymbol{\zeta} \in \mathbf{V}_0(\omega) \text{ and } B_0(\boldsymbol{\zeta}, \boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}_0(\omega),$$

where the space $\mathbf{V}_0(\omega)$ is defined as above,

$$B_0(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \int_\omega \frac{\varepsilon}{3} a^{\alpha\beta\rho\sigma} \Upsilon_{\rho\sigma}(\boldsymbol{\zeta}) \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y,$$

and the linear form L has the same expression as in § 2 (the tensors $(a^{\alpha\beta\rho\sigma})$ and $(\Upsilon_{\alpha\beta}(\boldsymbol{\eta}))$ are defined as in § 1).

The existence and uniqueness of the solution of the above variational equations are consequences of the following theorem (note that $\mathbf{V}_0(\omega)$ is a closed subspace of $\mathbf{V}(\omega)$):

Theorem 2. *Assume that* $\boldsymbol{\varphi}\in\mathscr{C}^3(\bar{\omega})$, length $\gamma_0 > 0$, *and* $\mathbf{V} \neq \{\mathbf{0}\}$. *Then there exists a constant* β_0 *such that*

$$\beta_0 > 0 \text{ and } B_0(\boldsymbol{\eta}, \boldsymbol{\eta}) \geqslant B_0 \|\boldsymbol{\eta}\|^2_{H^1(\omega) \times H^1(\omega) \times H^2(\omega)}$$

for all $\boldsymbol{\eta}\in\mathbf{V}_0(\omega)$.

Proof. By part (i) of the proof of Theorem 1, there exists a constant $C_4 > 0$ such that

$$B_0(\boldsymbol{\eta}, \boldsymbol{\eta}) \geqslant C_4 \left\{ \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2} \right\}^{1/2} \text{ for all } \boldsymbol{\eta}\in\mathbf{V}(\omega).$$

In the same proof, we have seen that the semi-norm $|\cdot|_{\mathbf{E}}$ is a norm over $\mathbf{V}(\omega)$, equivalent to the product norm $\|\cdot\|_{H^1 \times H^1 \times H^2}$ (cf. (iv)). The conclusion thus follows, since

$$\left\{ \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2 \right\}^{1/2} = |\boldsymbol{\eta}|_{\mathbf{E}} \text{ for all } \boldsymbol{\eta}\in\mathbf{V}_0(\omega). \qquad \square$$

4. The two-dimensional membrane shell model

Let the space $\mathbf{V}_0(\omega)$ of inextensional displacements be defined as in § 3. If $\mathbf{V}_0(\omega) = \{\mathbf{0}\}$ (there exist such instances), the first non-zero term $\boldsymbol{\zeta} = (\zeta_i)$ of a formal asymptotic expansion as powers of ε of the covariant components of the three-dimensional displacement is independent of the transverse variable, and it solves the following two-dimensional shell model, called the membrane model:

$$\boldsymbol{\zeta}\in\mathbf{V}_1(\omega) \text{ and } B_1(\boldsymbol{\zeta}, \boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta}\in\mathbf{V}_1(\omega),$$

where

$$\mathbf{V}_1(\omega) = \{\boldsymbol{\eta} = (\eta_i); \eta_\alpha \in H^1(\omega), \eta_3 \in L^2(\omega), \eta_\alpha = 0 \text{ on } \gamma_0\},$$

$$B_1(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \int_\omega \varepsilon a^{\alpha\beta\rho\sigma} \gamma_{\rho\sigma}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y,$$

and the linear form L has the same expression as in § 2 (the tensors $(a^{\alpha\beta\rho\sigma})$ and $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$ are defined as in § 1).

The existence and uniqueness of the solution of the above variational equations are consequences of the following result:

Theorem 3. *Assume that the boundary* γ *is of class* $\mathscr{C}^3$ *and that* $\gamma_0 = \gamma$. *Assume further that* $\boldsymbol{\varphi}$ *is analytic in an open set* ω' *containing* $\bar{\omega}$. *Assume finally that the surface S is uniformly elliptic, in the sense that there exists a constant b such that*

$$b > 0 \text{ and } b_{\alpha\beta}(y)\xi^\alpha\xi^\beta \geqslant b|\boldsymbol{\xi}|^2$$

for all $y\in\bar{\omega}$ *and all* $\boldsymbol{\xi} = (\xi^\alpha)\in\mathbf{R}^2$, *where* $(b_{\alpha\beta})$ *denotes the second fundamental form of S. Then there exists a constant* β_1 *such that*

$$\beta_1 > 0 \text{ and } B_1(\boldsymbol{\eta}, \boldsymbol{\eta}) \geqslant \beta_1 \|\boldsymbol{\eta}\|^2_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)}$$

for all $\boldsymbol{\eta}\in\mathbf{V}_1(\omega) = H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$. $\square$

Two different proofs of Theorem 3 are given in Ciarlet and Sanchez–Palencia [12, 13], and in Ciarlet and Lods [11]. It is the latter proof that we sketch here.

Outline of the proof of Theorem 3

(i) By part (i) of the proof of Theorem 1, there exists a constant $C_5 > 0$ such that

$$B_1(\boldsymbol{\eta}, \boldsymbol{\eta}) \geqslant C_5 \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in H^1 \times H^1 \times L^2$. It thus suffices to show that there exists a constant $C_6 > 0$ such that

$$\left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2}^2 \right\}^{1/2} \geqslant C_6 \|\boldsymbol{\eta}\|_{H^1 \times H^1 \times L^2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H_0^1 \times H_0^1 \times L^2$ (recall that we assume $\gamma_0 = \gamma$).

(ii) Using the same arguments as in part (ii) of the proof of Theorem 1, i.e., in particular the lemma of J L Lions and the open mapping theorem, one successively shows that

$$\{\boldsymbol{\eta} = (\eta_i); \eta_i \in L^2, \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2\} = H^1 \times H^1 \times L^2,$$

and that there exists a constant $C_7 > 0$ such that the following *generalized Korn's inequality* holds:

$$\left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2}^2 + \sum_i \|\eta_i\|_{L^2}^2 \right\}^{1/2} \geqslant C_7 \left\{ \sum_\alpha \|\eta_\alpha\|_{H^1}^2 + \|\eta_3\|_{L^2}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in H^1 \times H^1 \times L^2$.

(iii) One next establishes another rigid displacement lemma: If the surface S is uniformly elliptic (this assumption is needed from now on), the space

$$\mathbf{R}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H_0^1 \times H_0^1 \times L^2; \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

is finite-dimensional. To this end, one first observes that, if $\boldsymbol{\zeta} = (\zeta_i) \in \mathbf{R}(\omega)$ then $\tilde{\boldsymbol{\zeta}} := (\zeta_\alpha) \in H_0^1 \times H_0^1$ solves the variational equations

$$A_0(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\eta}}) + A_1(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\eta}}) = 0 \text{ for all } \tilde{\boldsymbol{\eta}} := (\eta_\alpha) \in H_0^1 \times H_0^1,$$

where

$$A_0(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\eta}}) = \int_\omega \left\{ \frac{b_{22}}{b_{11}} \partial_1 \zeta_1 \partial_1 \eta_1 + \left(\partial_2 \zeta_1 - 2 \frac{b_{12}}{b_{11}} \partial_1 \zeta_1 \right) \partial_2 \eta_1 \right\} \mathrm{d}y + \int_\omega \left\{ \frac{b_{11}}{b_{22}} \partial_2 \zeta_2 \partial_2 \eta_2 + \left(\partial_1 \zeta_2 - 2 \frac{b_{12}}{b_{22}} \partial_2 \zeta_2 \right) \partial_1 \eta_2 \right\} \mathrm{d}y,$$

$$A_1(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\eta}}) = \int_\omega \left\{ \left(\Gamma_{22}^\rho - \frac{b_{22}}{b_{11}} \Gamma_{11}^\rho \right) \zeta_\rho \partial_1 \eta_1 + 2 \left(\frac{b_{12}}{b_{11}} \Gamma_{11}^\rho - \Gamma_{12}^\rho \right) \zeta_\rho \partial_2 \eta_1 \right\} \mathrm{d}y + \int_\omega \left\{ \left(\Gamma_{11}^\rho - \frac{b_{11}}{b_{22}} \Gamma_{22}^\rho \right) \zeta_\rho \partial_2 \eta_2 + 2 \left(\frac{b_{12}}{b_{22}} \Gamma_{22}^\rho - \Gamma_{12}^\rho \right) \zeta_\rho \partial_1 \eta_2 \right\} \mathrm{d}y.$$

One next shows that there exists a constant $C_9 > 0$ such that

$$A_0(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\eta}}) \geqslant C_8 \|\tilde{\boldsymbol{\eta}}\|^2_{H^1 \times H^1} \text{ for all } \tilde{\boldsymbol{\eta}} = (\eta_\alpha) \in H^1_0 \times H^1_0.$$

Since there exists a constant $C_g > 0$ such that

$$A_1(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\eta}}) \geqslant -2C_8 C_9 \|\tilde{\boldsymbol{\eta}}\|_{L^2 \times L^2} \|\tilde{\boldsymbol{\eta}}\|_{H^1 \times H^1} \text{ for all } \tilde{\boldsymbol{\eta}} \in H^1 \times H^1,$$

one easily deduces that the operator T from $L^2 \times L^2$ into $H^1_0 \times H^1_0$ defined by the relations

$$A_0(T\mathbf{q}, \boldsymbol{\eta}) + A_1(T\mathbf{q}, \boldsymbol{\eta}) + \lambda \int_\omega (T\mathbf{q})^\alpha \eta_\alpha \mathrm{d}y = \int_\omega q^\alpha \eta_\alpha \mathrm{d}y, \text{ where } \lambda = 2C_8(C_9)^2,$$

for all $\boldsymbol{\eta} = (\eta_\alpha) \in H^1_0 \times H^1_0$, is compact.

Since $\tilde{\boldsymbol{\zeta}} \in \operatorname{Ker}(I - \lambda T)$ and $\operatorname{Ker}(I - \lambda T)$ is finite-dimensional (T is compact), the assertion is proved.

(iv) By refining the regularity assumptions on the boundary γ (which was so far assumed to be only Lipschitz-continuous) and on the mapping $\boldsymbol{\varphi}$ (which was so far assumed to be of class $\mathscr{C}^2$ on $\bar{\omega}$), we can strengthen the result of part (iii). More specifically, we show that, if γ is of class $\mathscr{C}^3$, and $\boldsymbol{\varphi}$ is analytic in an open set ω' containing $\bar{\omega}$, the space $\mathbf{R}(\omega)$ (as defined in (iii)) reduces to $\{\mathbf{0}\}$.

Consider the boundary-value problem:

$$[\gamma_{11}(\boldsymbol{\zeta}) :=]\partial_1 \zeta_1 - \Gamma^\rho_{11}\zeta_\rho - b_{11}\zeta_3 = 0 \text{ in } \omega,$$

$$[\gamma_{12}(\boldsymbol{\zeta}) :=]\frac{1}{2}\partial_2 \zeta_1 + \frac{1}{2}\partial_1 \zeta_2 - \Gamma^\rho_{12}\zeta_\rho - b_{12}\zeta_3 = 0 \text{ in } \omega,$$

$$[\gamma_{22}(\boldsymbol{\zeta}) :=]\partial_2 \zeta_2 - \Gamma^\rho_{22}\zeta_\rho - b_{22}\zeta_3 = 0 \text{ in } \omega, \ \zeta_1 = 0 \text{ on } \gamma.$$

This first-order system is a uniformly elliptic system (the assumption of uniform ellipticity of S is needed here) that satisfies the supplementary condition on L, and $\zeta_1 = 0$ on γ is a complementing boundary condition, in the sense of Agmon *et al* [1] (this was first observed by Geymonat and Sanchez-Palencia [16]).

We thus need to show that $\boldsymbol{\zeta} = \mathbf{0}$ is the only solution to the boundary value problem:

$$\begin{cases} \gamma_{\alpha\beta}(\boldsymbol{\zeta}) = 0 \text{ in } \omega, \\ \zeta_\alpha = 0 \text{ on } \gamma. \end{cases}$$

Since the boundary γ is not a characteristic curve for the reduced Cauchy problem where ζ_3 has been eliminated, Holmgren's uniqueness theorem shows that $\boldsymbol{\zeta} = \mathbf{0}$ is the only solution in a small enough neighborhood of any point of γ (the coefficients are analytic in ω' because $\boldsymbol{\varphi}$ is analytic in ω').

By a result of Morrey and Nirenberg [21], any solution of a uniformly elliptic system whose coefficients are analytic in ω is analytic in ω. Therefore $\boldsymbol{\zeta} = \mathbf{0}$ is the only solution in ω, by the analytic continuation theorem for analytic functions of several real variables.

(v) In order to conclude, it suffices to show that there exists a constant $C_{10} > 0$ such that

$$\left\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2}^2\right\}^{1/2} \geqslant C_{10}\left\{\sum_i \|\eta_i\|_{L^2}^2\right\}^{1/2} \text{ for all } \boldsymbol{\eta}\in H_0^1 \times H_0^1 \times L^2,$$

as the desired inequality (that involving the constant C_6, cf. part (i)) will then follow from the inequality established in part (ii).

If this assertion is false, there exists a sequence $(\boldsymbol{\eta}^k)$ of elements in $H_0^1 \times H_0^1 \times L^2$ such that

$$\left\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta}^k)\|_{L^2}^2\right\}^{1/2} \to 0 \text{ as } k\to\infty \text{ and } \left\{\sum_i \|\eta_i^k\|_{L^2}^2\right\}^{1/2} = 1 \text{ for all } k.$$

Hence there exists a subsequence $(\boldsymbol{\eta}^l)$ and an element $\boldsymbol{\eta} = (\eta_i)\in H_0^1 \times H_0^1 \times L^2$ such that ($\to$ and $\rightleftharpoons$ denote strong and weak convergences, respectively):

$$\eta_\alpha^l \rightharpoonup \eta_\alpha \text{ in } H_0^1,\ \eta_\alpha^l \to \eta_\alpha \text{ in } L^2,\ \eta_3^l \rightharpoonup \eta_3 \text{ in } L^2.$$

Since $\gamma_{\alpha\beta}(\boldsymbol{\eta}^l) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{\eta})$ in L^2 on the one hand and $\gamma_{\alpha\beta}(\boldsymbol{\eta}^l)\to 0$ in L^2 on the other, we first conclude that $\boldsymbol{\eta} = \mathbf{0}$, by (iv).

The convergences $\gamma_{\alpha\beta}(\boldsymbol{\eta}^l)\to 0$ in L^2 and $\eta_\alpha^l \to 0$ in L^2, combined with the definition of the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ imply that ($b_{11}\in C^0(\bar{\omega})$ does not vanish in $\bar{\omega}$, by the assumed uniform ellipticity of S)

$$\partial_2\eta_1^l + \partial_1\eta_2^l - 2\frac{b_{12}}{b_{11}}\partial_1\eta_1^l \to 0 \text{ in } L^2,$$

$$\partial_2\eta_2^l - \frac{b_{22}}{b_{11}}\partial_1\eta_1^l \to 0 \text{ in } L^2.$$

From these convergences and the relations

$$\int_\omega \partial_2\eta_1^l\partial_1\eta_2^l \mathrm{d}y = \int_\omega \partial_1\eta_1^l\partial_2\eta_2^l \mathrm{d}y,$$

we then infer that

$$\int_\omega \left\{\left(\partial_2\eta_1^l - \frac{b_{12}}{b_{11}}\partial_1\eta_1^l\right)^2 + \frac{1}{(b_{11})^2}(b_{11}b_{22} - (b_{12})^2)(\partial_1\eta_1^l)^2\right\}\mathrm{d}y \to 0.$$

This last convergence, combined with the uniform ellipticity of S, then implies that

$$\eta_3^l = \frac{1}{b_{11}}\partial_1\eta_1^l - \frac{1}{b_{11}}(\partial_1\eta_1^l - b_{11}\eta_3^l) \to 0 \text{ in } L^2.$$

Hence $\boldsymbol{\eta}^l \to \mathbf{0}$ in $L^2 \times L^2 \times L^2$, which contradicts $\left\{\sum_i \|\eta_i^l\|_{L^2}^2\right\}^{1/2} = 1$, and the proof of Theorem 3 is complete. □

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