Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 1, February 1994, pp. 263–267.

Iterations of random and deterministic functions

K B ATHREYA

Department of Mathematics, 400 Carver Hall, Iowa State University, Ames, IA 50010–2066, USA

Dedicated to the memory of Professor K G Ramanathan

Abstract. Let f be a probability generating function on $[0, 1]$. The convergence of its iterates f_n to fixed points is studied in this paper. Results include rates for f and f^{-1}. Also iterates of independent identically distributed stable processes are studied and a trichotomy based on the order of the stability is established.

Keywords. Random maps; generating functions; iteration; stable processes; auto regressive processes.

1. Introduction

Let f be map from an interval I on the real line into itself. Let $f_0(s) \equiv s, f_{n+1}(s) = f(f_n(s))$ for $n \geqslant 0$. The sequence $\{f_n(s)\}_0^\infty$, called *the iterates of f*, arise in many areas of mathematics. The problems that are studied in this connection include finding all the fixed points, the rates of convergence to their fixed points and similar aspects for the iterates of the inverse f^{-1} of f. In this paper we study this for a class of f's that are generating functions of probability distributions on the nonnegative integers. These are motivated by applications to the theory of branching processes (see [2]).

Let $X_0(t, \tilde{\omega})$ be map from $T \times \tilde{\Omega}$ to T where T and $\tilde{\Omega}$ are nonempty sets. Let $\tilde{B}$ be a σ-algebra of subsets of $\tilde{\Omega}$ and $\tilde{P}$ be a probability measure on $\tilde{B}$. Then under appropriate measurability conditions, $\{X_0(t, \tilde{\omega}): t \in T\}$ is called a T-valued stochastic process on $(\tilde{\Omega}, \tilde{B}, \tilde{P})$ with index set T. Let (Ω, B, P) be a probability space on which is defined a sequence $\{X_i(t, \omega)\}\, i = 1, 2, \ldots$ of independent identically distributed copies of the stochastic process X_0. Let $Y_0(t, \omega) \equiv t$ and $Y_{n+1}(t, \omega) = X_{n+1}(Y_n(t, \omega), \omega)$ for $n = 0, 1, 2, \ldots$. The sequence $\{Y_n(t, \omega)\}_0^\infty$, called the *i.i.d iterates of the stochastic process* X_0, arise in many areas of probability theory. The problems studied include convergence of these iterates and rates of convergence. In this paper we study the case when $T = (-\infty, \infty)$ and the process X has a self similarity property of the form $X_0(t, \tilde{\omega})$ has the same distribution as $|t|^{1/\alpha} X_0(1, \tilde{\omega})$. These include Brownian motion and stable processes. We also study the case when X_0 is a random walk on the integers. Finally we indicate some open problems.

2. Iterates of probability generating functions

Let $\{p_j\}_0^\infty$ be a sequence of numbers satisfying $p_j \geqslant 0, \sum_0^\infty p_j = 1$. Then for s real,

$$f(s) \equiv \sum_{0}^{\infty} p_j s^j \tag{1}$$

is called the (probability) generating function of the sequence $\{p_j\}$. It is convergent for $|s| \leqslant 1$. The iterates $\{f_n(s)\}_{n=0}^{\infty}$ have a nice interpretation in terms of a class of stochastic processes known as *branching processes*. Let $T = N^+ \equiv \{0, 1, 2, \ldots\}$ and $X_0(t, \tilde{\omega})$ be a random walk with step distribution $\{p_j\}$. That is, $X_0(0, \tilde{\omega}) = 0$, $X_0(n, \tilde{\omega}) = \sum_{j=1}^{n} \zeta_i(\tilde{\omega})$ for $n \geqslant 1$ where $\{\zeta_i(\tilde{\omega})\}_1^{\infty}$ are independent random variables with distribution $\{p_j\}_0^{\infty}$. Let $\{Y_n(t, \omega)\}_0^{\infty}$ be the i.i.d. iterate sequence of X_0 as defined in the introduction. Then using EZ to denote the expectation, i.e. the integral of a B-measurable function $Z(\omega)$ with respect to the measure P on the space (Ω, B, P), it can be verified that for $0 \leqslant s \leqslant 1$

$$E(s^{Y_n(1,\omega)}) = f_n(s) \text{ for } n \geqslant 1. \tag{2}$$

Indeed, it is not difficult to show that the conditional expectation

$$E(s^{Y_{n+1}(1,\omega)} | Y_n(1, \omega), Y_{n-1}(1, \omega), \ldots, Y_0(1, \omega)) \tag{3}$$

is $f(s)^{Y_n(1,\omega)}$ (using the independence of X_{n+1} and Y_n and the random walk nature of X_{n+1}). This, in turn, implies (2). The sequence $Z_n \equiv Y_n(1, \omega)\, n = 0, 1, 2, \ldots$ is called a Galton–Watson branching process (see [2]). Thus, there is a connection between the deterministic iterates sequence $\{f_n\}_0^{\infty}$ and the stochastic iterates sequence $\{Y_n\}_0^{\infty}$. It can be shown that $f(s)$ has almost two fixed points in [0, 1] and $f_n(s)$ converges to one of these as $n \to \infty$. This in turn is related to the instability of the sequence $\{Z_n = Y_n(1, \omega)\}$ in the sense that Z_n either goes to ∞ or to zero as $n \to \infty$.

The following result is known. See [2], Chapter 1.

Theorem 1. *a) There are atmost two solutions to* $f(s) = s$ in $[0, 1]$. *b) If* $m = \Sigma j p_j > 1$ *there are exactly two solutions,* q *and* 1, *with* $0 \leqslant q < 1$. *c) If* $m \leqslant 1$ *then there is only one solution, namely,* $q = 1$. *d) For* $0 \leqslant s < 1$, $f_n(s) \to q$ *where* q *is the smaller of the two roots of* $f(s) = s$ *in* $[0, 1]$. *e) For any distribution of the initial value* $Y_0(t, \omega)$

$$P(Y_n(t, \omega) \to 0) + P(Y_n(t, \omega) \to \infty) = 1. \tag{4}$$

An interesting question that arises from part d of Theorem 1 is about the rate of convergence of $f_n(s)$ to q (See [2] Chapter 1).

Theorem 2. *Let* $m = \Sigma j p_j \neq 1$ *and* $\gamma = f'(q) \neq 0$. *Then,*

$$\lim_{n \to \infty} \frac{f_n(s) - q}{\gamma^n} \equiv Q(s) \tag{5}$$

exists for $0 \leqslant s < 1$ *and* $Q(\cdot)$ *is the unique solution of the functional equation*

$$Q((s) = \gamma Q(s)) \textit{ for } 0 \leqslant s < \max(q, 1) \tag{6}$$

subject to $Q(q) = 0$. *Further, if* $m > 1$ *then* $\lim_{s \uparrow 1} Q(s) = \infty$.

An application of Theorem 2 yields the following large deviation result for branching processes.

Theorem 3. *Assume* $p_0 = 0, p_1 > 0$ *and* $\Sigma j^{2r+\delta} p_j < \infty$ *for some* $r \geqslant 1$ *and* $\delta > 0$ *such that* $p_1 m^r > 1$. *Then*

$$\lim_{n \to \infty} \frac{1}{p_1^n} P\left(\left| \frac{Z_{n+1}}{Z_n} - m \right| > \varepsilon \right) = \sum_k q_k \phi(k, \varepsilon) < \infty, \tag{7}$$

where $\{q_k\}$ *is the sequence with generating function* $Q(s)$ *as in* (5) *and* (6) *and*

$$Z_n \equiv Y_n(1, \omega), \phi(k, \varepsilon) = P\left(\left| \frac{Y_1(k, \omega)}{k} - m \right| > \varepsilon \right). \tag{8}$$

There is a counterpart to Theorem 2 when $p_1 = 0 = p_0$.

Theorem 4. *Let* $p_0 = 0 = p_1$ *and* $k = \inf\{j : j \geqslant 1, p_j \neq 0\}$. *Then*

$$f_n(s) = s^{k^n} p_k^{\Sigma_0^{n-1} k^j} (R_n(s))^{k^n}, \tag{9}$$

where $\lim_n R_n(s) = R(s)$ *exists uniformly in* $[0, 1]$ *with* $R(0) = 1$ *and* $R(1) < \infty$. *Further,*

$$(f_n(s))^{1/k^n} \to p_k^{1/(k-1)} s R(s) \tag{10}$$

An application of Theorem 4 is the following

Theorem 5. *Let* $p_0 = 0 = p_1$ *and* $k = \inf\{j : j \geqslant 1,\ p_j \neq 0\}$. *Let* $f(s_0) = \Sigma s_0^j p_j < \infty$ *for some* $1 < s_0 < \infty$. *Then for each* $\varepsilon > 0$ *there exists* C_ε *and* λ_ε *such that* $0 < \lambda_\varepsilon < 1$ *and*

$$P\left(\left| \frac{Z_{n+1}}{Z_n} - m \right| > \varepsilon \right) \leqslant C_\varepsilon \lambda_\varepsilon^{k^n}. \tag{11}$$

Theorem 6. *Let* $p_0 = 0$ *and* $g = f^{-1}$ *defined by* $f(g(s)) = s$. *Let* $f(s_0) = \sum_j p_j s_0^j$ *be* $< \infty$ *for some* $s_0 > 1$. *Then for* $1 \leqslant s \leqslant f(s_0), g_n(s) \downarrow 1$ *and*

$$\tilde{Q}_n(s) \equiv \mathrm{m}^n(q_n(s) - 1) \downarrow \tilde{Q}(s). \tag{12}$$

A consequence of Theorem 6 is the following

Theorem 7. *Let* $f(s_0) = \Sigma p_j s_0^j$ *be* $< \infty$ *for some* $s_0 > 1$. *Then there exists a* $\theta_1 > 0$ *such that*

$$\sup_n E(\exp(\theta_1 \ W_n)) < \infty, \tag{13}$$

where $W_n = Z_n m^{-n}$. *Further there exist constants* C *and* $\lambda > 0$ *such that for each* $\varepsilon > 0$

$$P(|W_n - W| \geqslant \varepsilon) \leqslant C \exp - \left(\lambda \varepsilon \frac{2}{3} \left(m \frac{1}{3} \right)^n \right), \tag{14}$$

where $W = \lim_n W_n$.

The proofs of Theorem 3–7 are in Athreya [1].

The extensions of above results to generating functions of probability measures on the nonnegative lattices in Euclidean spaces are contained in the thesis of Vidyashankar [4].

3. Iterations of random processes

Let $\{X_i(t,\omega): t\in T\}_1^\infty$ be a sequence of T-valued stochastic processes with index set T and defined on a probability space (Ω, B, P). Let

$$Y_0(t,\omega)=t,\ Y_{n+1}(t,\omega)=X_{n+1}(Y_n(t,\omega)) \text{ for } n=0,1,2,\ldots. \tag{15}$$

We study the iterate sequence $\{Y_n\}$ when $\{X_i\}$ are independent and identically distributed copies of a stable process of order α on R. Let $\{X(t,\omega); t\in R\}$ be a real valued stochastic process such that:

i) $X(0,\omega)=0$ w.p.1. and
ii) Both $\{X(t,\omega): t\geqslant 0\}$ and $\{X(-t,\omega): t\geqslant 0\}$ be independent copies of a real valued stochastic process with independent and stationary increments.

Let $\{X_n(t,\omega); -\infty<t<\infty\}_1^\infty$ be independent copies of X. Let $\{Y_n(t,\omega)\}_1^\infty$ be as in (15).

Let $\mu(t,\cdot)$ be the probability distribution of $X(t,\omega)$ for $t\geqslant 0$. Then for each fixed t, $\{Y_n(t,\omega)\}n=0,1,\ldots$ is a Markov chain with state space R, stationary transition probability function $P(x,A)=\mu(|x|,A)$ for $x\in R, A\in B(R)$ the Borel σ-algebra of R.

Theorem 8. *Let* $\mu(t,A)=\Phi\left(\frac{A}{\sqrt{t}}\right)$ *where*

$$\Phi(A)=\int_A \frac{1}{\sqrt{2\pi}}\exp-\left(\frac{x^2}{2}\right)\mathrm{d}x \text{ for } A\in B(R).$$

Then for any $t\neq 0$, $Y_n(t,\omega)$ *converges in distribution and the limit is independent of* t.

More generally let $\mu_\alpha(t,\cdot)$ be the distribution of a symmetric stable process of order α.

Theorem 9. *If* $\mu(|t|,\cdot)=\mu_\alpha(t,\cdot)$ *and* $1\leqslant\alpha\leqslant 2$ *then for any* $t\neq 0$, $Y_n(t,\omega)$ *converges in distribution and the limits is independent of* t *and its distribution function is given by*

$$F(y)=\int_0^\infty G(y|x|^{-\alpha})\mathrm{d}H(x) \text{ where } G(x)=P(X_1(1)\leqslant x)$$

Theorem 10. *If* $\mu(|t|,\cdot)=\mu_\alpha(t,\cdot)$ *and* $0<\alpha<1$. *Then for any* $t\neq 0$, $\lim_n |Y_n(t,\omega)|^{\alpha^n}$ *exists w.p.* 1 *and equals* $\exp(|t|+Z(t,\omega))$ *where* $Z(t,\omega)$ *is a random variable with distribution that is identical to that of* $\sum_1^\infty \alpha^j\eta_j$ *where* $\{\eta_j, j=1,2,\ldots\}$ *are independent with distribution same as that of* $\log|X(1)|$.

Theorem 11. *If* $\mu(t,\cdot) = \mu_1(|t|,\cdot)$ (i.e. $\alpha = 1$) *then*

$$\lim_n |Y_n(t,\omega)| = \begin{cases} \infty & w.p.1 \text{ if } \mu = E\log(X,(1)| > 0 \\ 0 & w.p.1 \qquad\qquad\qquad < 0 \end{cases}$$

and if $\mu = 0$ *then w.p.*1 *both* 0 *and* ∞ *are limit points for the sequence* $\{|Y_n(t,\omega)|\}$. *Also* $(|Y_n|\exp-(n\mu))^{1/\sqrt{n}}$ *converges in distribution to* $\exp(N\sigma)$ *where* $N \sim N(0,1)$ *and* $\sigma^2 =$ *variance of* $\log|X_1(1)|$.

The proofs of Theorems 8–11 depend on the following representation. From the definition of Y_n it follows that

$$|Y_{n+1}(t,\omega)| = \frac{|X_{n+1}(Y_n(t,\omega))|}{|Y_n(t,\omega)|^{1/\alpha}} |Y_n(t,\omega)|^{1/\alpha}.$$

Taking logarithms we see that $|Y_n(t,\omega)| \equiv Z_n$ is an autoregressive sequence satisfying

$$Z_{n+1} = \rho Z_n + \eta_{\eta+1}$$

where $\rho = (1/\alpha)$, $\{\eta_n)$ are i.i.d with distribution same as $\log|X_1(1)|$. For $1 < \alpha \leqslant 2$, $\frac{1}{2} \leqslant \rho < 1$ and hence $\{Z_n\}$ is nonexplosive. For $0 < \alpha < 1$, $\rho > 1$ and so $\{Z_n\}$ is explosive. Details of the proofs of Theorems 8–11 will appear in a future publication.

4. Some open problems

a) In the stable processes case do the joint distributions of $(Y_n(t_i,\omega), i = 1,2,\dots k)$ converge as $n \to \infty$ where $t_1, t_2, \dots, t_k$ are fixed in t?
b) What happens in the case when the stable processes are replaced by a general additive process with stationary independent increments? Are there reasonable conditions in terms of the Levy measure for convergence? When T is the set of integers and the underlying X process is a random walk the $\{Y_n\}$ sequence yields the self annihilating branching studied by Erickson [3]. It would be worthwhile to extend his results to the Levy process case.
c) Suppose that the $\{X_i\}$ are Markov chains that are positive recurrent. This corresponds to random dynamical systems that are Markov chains in random environments. What happens to $\{Y_n\}$ in this case?

Acknowledgement

Research supported in part by NSF grant DMS 9343932.

References

[1] Athreya K B, Large deviation rates for branching processes – I single type case (1993), Preprint, Dept. of Mathematics, Iowa State University (To appear in the *Ann. Appl. Probab.*)
[2] Athreya K B and Ney P, Branching processes (1972) (New York: Springer-Verlag)
[3] Erickson K B, Self annihilating branching processes, *Ann. Probab.* **1** (1973) 926–946
[4] Vidyashankar A, Large deviation rates for branching processes (1994), Ph.D. Thesis, Dept. of Mathematics, Iowa State University.