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Two remarkable doubly exponential series transformations of Ramanujan

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Dedicated to the memory of Professor K G Ramanathan

Abstract. The purpose of this note is to prove two doubly exponential series transformations found in Ramanujan's second notebook.

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Ramanujan's notebooks [2] contain many fascinating theorems, which, it would seem, would never have been discovered by any other person. As excellent illustrations of this, Ramanujan offers transformation formulas for two doubly exponential series. It is very surprising that such elegant transformations exist. Although beautiful by themselves, we think that they will be useful in other investigations. These two remarkable series identities are stated without proof by Ramanujan on page 279 in his second notebook [2] and are numbered 4) and 5) on that page. The purpose of this note is to provide the missing proofs.

Ramanujan's two formulas can be stated as follows. First, if $\alpha\beta = 2\pi$, then

$$\alpha \sum_{k=0}^{\infty} \exp(-ne^{k\alpha}) = \alpha\left\{\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} n^k}{k!(e^{k\alpha}-1)}\right\} - \gamma - \log n + 2\sum_{k=1}^{\infty} \varphi(k\beta), \quad (1)$$

where

$$\varphi(\beta) = \sqrt{\frac{\pi}{\beta \sinh(\pi\beta)}} \cos\left(\beta \log\frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1\cdot 2\beta} + \dots\right). \quad (2)$$

Second, if $\alpha\beta = \pi/2$ then

$$\alpha \sum_{k=0}^{\infty} (-1)^k \exp(-ne^{(2k+1)\alpha}) = \alpha\left\{\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k n^k}{k!(e^{k\alpha}+e^{-k\alpha})}\right\}$$

$$+ \sum_{k=0}^{\infty} (-1)^k \psi((2k+1)\beta), \quad (3)$$

where

$$\psi(\beta) = \sqrt{\frac{\pi}{\beta \sinh(\pi\beta)}} \sin\left(\beta \log\frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1\cdot 2\beta} + \frac{B_4}{3\cdot 4\beta^3} - \dots\right). \quad (4)$$

In (1), γ denotes Euler's constant, and in (2) and (4), $B_k, k \geqslant 0$, denotes the kth Bernoulli number defined by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, \quad |z| < 2\pi.$$

We emphasize that we have altered Ramanujan's notation in (1)–(4). In particular, in Ramanujan's convention, all even indexed Bernoulli numbers are positive.

The definitions of $\varphi(\beta)$ and $\psi(\beta)$ given in (2) and (4), respectively, are decidedly enigmatic. Appearing in the arguments of the trigonometric functions are apparently asymptotic series as β tends to ∞. Thus the definitions of $\varphi(\beta)$ and $\psi(\beta)$ are imprecise, and so Ramanujan's claims are unclear. Nonetheless, we shall show that (1) and (3) are correct, if (2) and (4) are properly interpreted.

We begin by defining functions $G(\beta)$ and $B(\beta)$ by

$$\begin{aligned}\Gamma(i\beta + 1) &= (i\beta)^{i\beta + 1/2} \exp(-i\beta)\sqrt{2\pi}\, G(\beta) \\ &= (i\beta)^{i\beta + 1/2} \exp(-i\beta)\sqrt{2\pi} \exp\{-iB(\beta)\}. \qquad (5)\end{aligned}$$

Then [4, pp. 252–253], as β tends to ∞,

$$B(\beta) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_{2k}}{(2k-1)(2k)\beta^{2k-1}}$$

and

$$G(\beta) \sim 1 + \frac{1}{12 i\beta} - \frac{1}{288\beta^2} - \frac{139}{51840(i\beta)^3} - \frac{571}{2488320\beta^4} + \dots. \qquad (6)$$

Ramanujan less explicitly gives the asymptotic expansion for $B(\beta)$ in the argument to the trigonometric functions in (2) and (4).

We can now state our main results. Immediately after this, we will derive Ramanujan's identities as consequences.

Theorem 1. *Let n, α, and β be positive with $\alpha\beta = 2\pi$. Then* (1) *holds, where*

$$\begin{aligned}\varphi(\beta) &= \frac{1}{\beta} \operatorname{Im}\{n^{-i\beta}\Gamma(i\beta + 1)\} \\ &= \sqrt{\frac{2\pi}{\beta}} \exp(-\pi\beta/2)\left\{\sin\left(\beta \log\frac{\beta}{n} - \beta + \frac{\pi}{4}\right)\operatorname{Re}\{G(\beta)\}\right. \\ &\quad \left. + \cos\left(\beta\log\frac{\beta}{n} - \beta + \frac{\pi}{4}\right)\operatorname{Im}\{G(\beta)\}\right\} \\ &\sim \sqrt{\frac{2\pi}{\beta}} \exp(-\pi\beta/2)\left\{\sin\left(\beta\log\frac{\beta}{n} - \beta + \frac{\pi}{4}\right)\left\{1 - \frac{1}{288\beta^2} + \dots\right\}\right. \\ &\quad \left. - \cos\left(\beta\log\frac{\beta}{n} - \beta + \frac{\pi}{4}\right)\left\{\frac{1}{12\beta} + \dots\right\}\right\}, \qquad (7)\end{aligned}$$

as β tends to ∞.

Theorem 2. *Let n, α, and β be positive with αβ = π/2. Then* (3) *holds, where*

$$\begin{aligned}\psi(\beta) &= \frac{1}{\beta}\mathrm{Re}\{n^{-i\beta}\Gamma(i\beta+1)\}\\ &= -\sqrt{\frac{2\pi}{\beta}}\exp(-\pi\beta/2)\left\{\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\mathrm{Re}\{G(\beta)\}\right.\\ &\quad\left.-\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\mathrm{Im}\{G(\beta)\}\right\}\\ &\sim -\sqrt{\frac{2\pi}{\beta}}\exp(-\pi\beta/2)\left\{\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\left\{1-\frac{1}{288\beta^2}+\dots\right\}\right.\\ &\quad\left.+\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\left\{\frac{1}{12\beta}+\dots\right\}\right\}, \qquad (8)\end{aligned}$$

as β tends to ∞.

We first show that Ramanujan's definitions of (2) and (4) are compatible with the far right sides of (7) and (8), respectively. As β tends to ∞,

$$\begin{aligned}&\sqrt{\frac{\pi}{\beta\sinh(\pi\beta)}}\cos\left(\beta\log\frac{\beta}{n}-\beta-\frac{\pi}{4}-B(\beta)\right)\\ &= \sqrt{\frac{2\pi}{\beta}}\exp(-\pi\beta/2)(1-\exp(-2\pi\beta))^{-1/2}\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}-B(\beta)\right)\\ &\sim \sqrt{\frac{2\pi}{\beta}}\exp(-\pi\beta/2)\left\{\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\cos B(\beta)\right.\\ &\quad\left.-\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\sin B(\beta)\right\}.\end{aligned}$$

Thus (2) and (7) are in agreement. The argument showing that (4) and (8) agree is similar. This justifies Ramanujan's claims. We now proceed to prove the two main theorems.

Proof (*Theorem* 1). First,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k}{k!(e^{k\alpha}-1)} &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k}{k!e^{k\alpha}}\sum_{j=0}^{\infty}e^{-kj\alpha}\\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k e^{-kj\alpha}}{k!}\\ &= \sum_{j=1}^{\infty}(1-\exp(-ne^{-j\alpha})).\end{aligned}$$

Thus, the proposed identity may be written in the equivalent form

$$\alpha \sum_{k=1}^{\infty} (e^{-ne^{k\alpha}} + e^{-ne^{-k\alpha}} - 1) - \frac{1}{2}\alpha + \alpha e^{-n}$$
$$= -\gamma - \log n + 2\sum_{k=1}^{\infty} \varphi(k\beta). \tag{9}$$

Secondly, we apply Poisson's summation formula [3, p. 60] to the function

$$f(x) := \exp(-ne^{x}) + \exp(-ne^{-x}) - 1. \tag{10}$$

Observing that $f(0) = 2\exp(-n) - 1$, we find that, for $\alpha, \beta > 0$ with $\alpha\beta = 2\pi$,

$$\alpha\left\{\frac{1}{2}(2e^{-n} - 1) + \sum_{k=1}^{\infty} (\exp(-ne^{k\alpha}) + \exp(-ne^{-k\alpha}) - 1)\right\}$$
$$= \int_0^{\infty} f(x)\mathrm{d}x + 2\sum_{k=1}^{\infty} \int_0^{\infty} f(x)\cos(k\beta x)\mathrm{d}x. \tag{11}$$

Comparing (9) and (11), we see that it remains to prove that

$$-\gamma - \log n + 2\sum_{k=1}^{\infty} \varphi(k\beta) = \int_0^{\infty} f(x)\mathrm{d}x + 2\sum_{k=1}^{\infty} \int_0^{\infty} f(x)\cos(k\beta x)\mathrm{d}x. \tag{12}$$

Observe by (10) that $f(x)$ is even. Setting $u = e^{x}$, we find that

$$\int_0^{\infty} f(x)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\mathrm{d}x$$
$$= \frac{1}{2}\int_0^{\infty} (e^{-nu} + e^{-n/u} - 1)\frac{\mathrm{d}u}{u}$$
$$= \frac{1}{2}\left(-\int_0^{1/n} \frac{1 - e^{-nu}}{u}\mathrm{d}u + \int_{1/n}^{\infty} \frac{e^{-nu}}{u}\mathrm{d}u\right.$$
$$\left. + \int_0^{1/n} \frac{e^{-n/u}}{u}\mathrm{d}u - \int_{1/n}^{\infty} \frac{1 - e^{-n/u}}{u}\mathrm{d}u\right)$$
$$= \frac{1}{2}\left(-\int_0^{1} \frac{1 - e^{-x}}{x}\mathrm{d}x + \int_1^{\infty} \frac{e^{-x}}{x}\mathrm{d}x\right.$$
$$\left. + \int_{n^2}^{\infty} \frac{e^{-x}}{x}\mathrm{d}x - \int_0^{n^2} \frac{1 - e^{-x}}{x}\mathrm{d}x\right).$$

Since [1, p. 103]

$$\gamma = \int_0^{1} \frac{1 - e^{-x}}{x}\mathrm{d}x - \int_1^{\infty} \frac{e^{-x}}{x}\mathrm{d}x,$$

we find that

$$\int_0^\infty f(x)\mathrm{d}x = \frac{1}{2}\left(-\gamma + \int_1^\infty \frac{e^{-x}}{x}\mathrm{d}x - \int_0^1 \frac{1-e^{-x}}{x}\mathrm{d}x - \int_0^{n^2}\frac{\mathrm{d}x}{x}\right)$$
$$= \frac{1}{2}(-\gamma-\gamma-\log n^2)$$
$$= -\gamma - \log n. \tag{13}$$

Using (13) in (12), we find that it suffices to prove that

$$\sum_{k=1}^\infty \varphi(k\beta) = \sum_{k=1}^\infty \int_0^\infty f(x)\cos(k\beta x)\mathrm{d}x. \tag{14}$$

Set

$$I := I(\beta) := \int_0^\infty f(x)\cos(\beta x)\mathrm{d}x$$

By (14), it now suffices to prove that $I(\beta) = \varphi(\beta)$, where $\varphi(\beta)$ is defined by (7). Letting $u = e^x$, we find that

$$I = \frac{1}{2}\int_{-\infty}^\infty f(x)\cos(\beta x)\mathrm{d}x$$
$$= \frac{1}{2}\int_0^\infty (e^{-nu} + e^{-n/u} - 1)\cos(\beta\log u)\frac{\mathrm{d}u}{u}.$$

Integrating by parts, we find that

$$I = \frac{n}{2\beta}\int_0^\infty \left(e^{-nu} - \frac{1}{u^2}e^{-n/u}\right)\sin(\beta\log u)\mathrm{d}u$$
$$= \frac{n}{2\beta}\left(\int_0^\infty e^{-nu}\sin(\beta\log u)\mathrm{d}u - \int_0^\infty \frac{e^{-n/u}}{u^2}\sin(\beta\log u)\mathrm{d}u\right)$$
$$= \frac{n}{2\beta}(I_1 - I_2),$$

say. Setting $t = 1/u$ in I_2, we deduce that $I_2 = -I_1$. Hence,

$$I = \frac{n}{\beta}I_1$$
$$= \frac{n}{\beta}\int_0^\infty e^{-nu}\sin(\beta\log u)\mathrm{d}u$$
$$= \frac{n}{2\beta i}\int_0^\infty (e^{-nu}u^{i\beta} - e^{-nu}u^{-i\beta})\mathrm{d}u$$
$$= \frac{1}{2\beta i}\{n^{-i\beta}\Gamma(i\beta+1) - n^{i\beta}\Gamma(-i\beta+1)\}$$

$$= \frac{1}{\beta}\mathrm{Im}\{n^{-i\beta}\Gamma(i\beta+1)\}.$$

Hence, $I = I(\beta) = \varphi(\beta)$, by (7). This completes the proof of (1).

Lastly, from (5),

$$\varphi(\beta) = \frac{1}{\beta}\mathrm{Im}\left((i\beta)^{1/2}\left(\frac{i\beta}{ne}\right)^{i\beta}\sqrt{2\pi}G(\beta)\right)$$

$$= \sqrt{\frac{2\pi}{\beta}}\mathrm{Im}(e^{i(\pi/4+\beta\log(i\beta/(ne)))}G(\beta))$$

$$= \sqrt{\frac{2\pi}{\beta}}e^{-\pi\beta/2}\mathrm{Im}(e^{i(\pi/4+\beta\log(\beta/n)-\beta)}G(\beta)).$$

Hence, the second equality in (7) follows, and the asymptotic formula follows by using (6). □

Proof (*Theorem* 2). First,

$$\sum_{k=1}^{\infty}\frac{(-1)^k n^k}{k!(e^{k\alpha}+e^{-k\alpha})} = \sum_{k=1}^{\infty}\frac{(-1)^k n^k}{k!e^{k\alpha}}\sum_{j=0}^{\infty}(-1)^j e^{-2kj\alpha}$$

$$= \sum_{j=0}^{\infty}(-1)^j(\exp(-ne^{-(2j+1)\alpha})-1).$$

Thus, the proposed identity (3) can be recast in the form

$$\alpha\sum_{k=0}^{\infty}(-1)^k(\exp(-ne^{(2k+1)\alpha})-\exp(-ne^{-(2k+1)\alpha})+1)$$

$$= \frac{1}{2}\alpha + \sum_{k=0}^{\infty}(-1)^k\psi((2k+1)\beta). \tag{15}$$

Next, we apply the Poisson summation formula for Fourier sine transforms [3, p. 66] to the function

$$f(x) := \exp(-ne^{x}) - \exp(-ne^{-x}) + 1.$$

Thus, for $\alpha, \beta > 0$ and $\alpha\beta = \pi/2$,

$$\alpha\sum_{k=0}^{\infty}(-1)^k(\exp(-ne^{(2k+1)\alpha})-\exp(-ne^{-(2k+1)\alpha})+1)$$

$$= \sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}f(x)\sin((2k+1)\beta x)\mathrm{d}x. \tag{16}$$

We recall that

$$\frac{\alpha}{2} = \frac{\pi}{4\beta} = \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}. \tag{17}$$

Using (16) and (17) in (15), we find that it suffices to prove that

$$\sum_{k=0}^{\infty}(-1)^k\psi((2k+1)\beta)$$
$$=\sum_{k=0}^{\infty}(-1)^k\left(\int_0^{\infty}f(x)\sin((2k+1)\beta x)\mathrm{d}x-\frac{1}{(2k+1)\beta}\right). \tag{18}$$

Set

$$I:=I(\beta):=\int_0^{\infty}f(x)\sin(\beta x)\mathrm{d}x-\frac{1}{\beta}.$$

By (18), we now see that it suffices to prove that $I(\beta)=\psi(\beta)$, where $\psi(\beta)$ is defined by (8).

Setting $x=e^u$ and integrating by parts, we find that

$$\begin{aligned}
I&=\int_0^{\infty}(\exp(-ne^x)-\exp(-ne^{-x})+1)\sin(\beta x)\mathrm{d}x-\frac{1}{\beta}\\
&=\int_0^{\infty}(e^{-nu}-e^{-n/u}+1)\sin(\beta\log u)\frac{\mathrm{d}u}{u}-\frac{1}{\beta}\\
&=-\frac{n}{\beta}\int_1^{\infty}\left(e^{-nu}+\frac{1}{u^2}e^{-n/u}\right)\cos(\beta\log u)\mathrm{d}u\\
&=-\frac{n}{\beta}\int_0^{1}\left(e^{-nt}+\frac{1}{t^2}e^{-n/t}\right)\cos(\beta\log t)\mathrm{d}t,
\end{aligned}$$

where we set $u=1/t$. Hence,

$$\begin{aligned}
I&=-\frac{n}{2\beta}\int_0^{\infty}\left(e^{-nt}+\frac{1}{t^2}e^{-n/t}\right)\cos(\beta\log t)\mathrm{d}t\\
&=-\frac{n}{2\beta}(I_1+I_2),
\end{aligned}$$

say. Letting $t=1/u$ in I_2, we easily find that $I_2=I_1$. Consequently,

$$\begin{aligned}
I&=-\frac{n}{\beta}I_1\\
&=-\frac{n}{\beta}\int_0^{\infty}e^{-nt}\cos(\beta\log t)\mathrm{d}t\\
&=-\frac{n}{2\beta}\int_0^{\infty}(e^{-nt}t^{i\beta}+e^{-nt}t^{-i\beta})\mathrm{d}t\\
&=-\frac{1}{2\beta}\{n^{-i\beta}\Gamma(i\beta+1)+n^{i\beta}\Gamma(-i\beta+1)\}\\
&=-\frac{1}{\beta}\mathrm{Re}\{n^{-i\beta}\Gamma(i\beta+1)\}.
\end{aligned}$$

Thus, we have shown that $I(\beta) = \psi(\beta)$, by (8). This completes the proof of (3).

The remaining two claims in (8) follow as in the proof of Theorem 1. □

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