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Gaussian quadrature in Ramanujan's Second Notebook

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Ramanujan's notebooks contain many approximations, usually without explanations. Some of his approximations to series are explained as quadrature formulas, usually of Gaussian type.

Keywords. Gaussian quadrature; series approximations; Ramanujan.

1. Introduction

K G Ramanathan was a gentle man who had a strong sense of duty. Part of his duty was the understanding of Ramanujan and his mathematics, and we can all feel pleased that he helped us understand some of the mathematics Ramanujan did. In light of his work on Ramanujan's work on modular functions, continued fractions, and hypergeometric and basic hypergeometric functions, it is appropriate to dedicate a paper to his memory which deals with material from Ramanujan's Notebooks. The particular questions below deal with orthogonal polynomials, although it is very unlikely Ramanujan knew this. He was just looking for nice approximations he could compute easily, and attractive explicit formulas.

Ramanujan's approximations to certain series which I can explain are:

$$\varphi(0)+\frac{x}{1!}\varphi(1)+\frac{x^2}{2!}\varphi(2)+\frac{x^3}{3!}\varphi(3)+\dots \tag{1.1}$$

$$= e^x\varphi(x) \text{ as the first approximation,} \tag{1.1a}$$

$$= e^x\left\{\frac{\sqrt{1+4x}-1}{2\sqrt{1+4x}}\varphi\left(x+\frac{1+\sqrt{1+4x}}{2}\right)\right.$$
$$\left.+\frac{\sqrt{1+4x}+1}{2\sqrt{1+4x}}\varphi\left(x+\frac{1-\sqrt{1+4x}}{2}\right)\right\} \tag{1.1b}$$

$$= e^x\left\{\frac{2}{3}\varphi(x)+\frac{\sqrt{1+12x}-1}{6\sqrt{1+12x}}\varphi\left(x+\frac{1+\sqrt{1+12x}}{2}\right)\right.$$
$$\left.+\frac{\sqrt{1+12x}+1}{6\sqrt{1+12x}}\varphi\left(x+\frac{1-\sqrt{1+12x}}{2}\right)\right\}. \tag{1.1c}$$

These appear on page 352 of [4]. The rest appear on page 349 in [4].

$$\frac{1}{n}\{\varphi(x-n+1)+\varphi(x-n+3)+\cdots+\varphi(x+n-1)\} \tag{1.2}$$

$$= \varphi(x) \text{ as the first approximation,} \tag{12a}$$

$$= \frac{\varphi\left(x+\sqrt{\dfrac{n^2-1}{3}}\right)+\varphi\left(x-\sqrt{\dfrac{n^2-1}{3}}\right)}{2} \text{ as the second,} \tag{1.2b}$$

$$= \frac{5(n^2-1)\left\{\varphi\left(x+\sqrt{\dfrac{3n^2-7}{5}}\right)+\varphi\left(x-\sqrt{\dfrac{3n^2-7}{5}}\right)\right\}+8(n^2-4)\varphi(x)}{6(3n^2-7)} \tag{1.2c}$$

$$= \left(\frac{1}{4}-\frac{n^2-16}{6\beta}\right)\left\{\varphi\left(x+\sqrt{\frac{\alpha+\beta}{7}}\right)+\varphi\left(x-\sqrt{\frac{\alpha+\beta}{7}}\right)\right\}$$
$$+\left(\frac{1}{4}+\frac{n^2-16}{6\beta}\right)\left\{\varphi\left(x+\sqrt{\frac{\alpha-\beta}{7}}\right)+\varphi\left(x-\sqrt{\frac{\alpha-\beta}{7}}\right)\right\} \tag{1.2d}$$

where $\alpha = 3n^2 - 13$ and $\beta = \sqrt{\frac{4}{5}(6n^4 - 45n^2 + 164)}$.

He also included some examples

$$u_1 + u_2 + \cdots + u_{13} = \frac{13}{25}(7u_2 + 11u_7 + 7u_{12}) \tag{1.3}$$

$$u_1 + u_2 + \cdots + u_{22} = \frac{11}{289}(161u_3 + 256u_{11\frac{1}{2}} + 161u_{20}) \tag{1.4}$$

$$u_1 + u_2 + \cdots + u_7 = \frac{7}{2}(u_2 + u_6) \tag{1.5}$$

$$u_1 + u_2 + \cdots + u_{26} = 13(u_6 + u_{21}) \tag{1.6}$$

$$\varphi(1)+\varphi(2)+\cdots+\varphi(21) = \frac{7}{958}\left[506\{\varphi(2)+\varphi(20)\}+931\left\{\varphi(1)+\varphi\left(11+2\sqrt{\frac{22}{7}}\right)+\varphi\left(11-2\sqrt{\frac{22}{7}}\right)\right\}\right]. \tag{1.7}$$

2. Gaussian quadrature

Let $f(t)$ be a continuous function on an interval $[a, b]$, and $d\alpha(t)$ a non-negative measure on $[a, b]$. The problem of Gaussian quadrature is to approximate

$$\int_a^b f(t)\mathrm{d}\alpha(t) \tag{2.1}$$

by a finite sum which is exact for all polynomials of as high a degree as possible. When $a < t_1 < \cdots < t_k < b$, set

$$w_k(t) = \prod_{i=1}^{k} (t - t_i) \tag{2.2}$$

and

$$w_{j,k}(t) = \frac{w_k(t)}{w_k'(t_j)(t - t_j)}. \tag{2.3}$$

Then

$$L_k^f(t) = \sum_{j=1}^{n} f(t_j)w_{j,k}(t) \tag{2.4}$$

is a polynomial of degree at most $(k - 1)$, and

$$L_k^f(t_j) = f(t_j), \quad j = 1, 2, \ldots, k. \tag{2.5}$$

When $f(t)$ is a polynomial of degree $(k - 1)$, then

$$\int_a^b f(t)\mathrm{d}\alpha(t) = \sum_{j=1}^{k} f(t_j) \int_a^b w_{j,k}(t)\mathrm{d}\alpha(t) \tag{2.6}$$

since

$$f(t) = L_k^f(t) \tag{2.7}$$

for all t. The degree $(k - 1)$ can be increased by an appropriate choice of the points t_j. If $f(t)$ is a polynomial of degree $(2k - 1)$, then

$$f(t) - L_k^f(t) = w_k(t)r_{k-1}(t) \tag{2.8}$$

with $r_{k-1}(t)$ a polynomial of degree $(k - 1)$. If $w_k(t)$ is orthogonal to all polynomials of degree less than k, using the measure $\mathrm{d}\alpha(x)$ to define the inner product, then

$$\int_a^b f(t)\mathrm{d}\alpha(t) - \int_a^b L_k^f(t)\mathrm{d}\alpha(t) = \int_a^b w_k(t)r_{k-1}(t)\mathrm{d}\alpha(t) = 0. \tag{2.9}$$

If

$$\lambda_j = \lambda_{j,k} = \int_a^b w_{j,k}(t)\mathrm{d}\alpha(t) \tag{2.10}$$

then the Gaussian quadrature approximation to (2.1) is

$$\sum_{j=1}^{k} \lambda_j f(t_j). \tag{2.11}$$

There are other expressions for λ_j defined in (2.10). See Theorem 3.42 in Szegö [6] for three other expressions. Two of these expressions show immediately that $\lambda_j > 0$.

3. Ramanujan's claims

To obtain Ramanujan's claims, it is first necessary to identify the measure $d\alpha(t)$, and then to locate the points where the interpolation is done. Finally, the weighting coefficients must be obtained.

In example (1.1), the measure Ramanujan is using is obtained by multiplying both sides of the identities by e^{-x}. The measure is the Poisson distribution

$$\frac{e^{-x}x^j}{j!}, \quad j=0,1,\ldots, \tag{3.1}$$

so $a=0$, $b=\infty$ and the measure is the sum of infinitely many multiples of a shifted delta function.

The orthogonal polynomials for this measure are called Charlier polynomials, and they can be given as a hypergeometric series. The polynomials in (1.2) are also hypergeometric functions, so we recall the definition of a generalized hypergeometric series. This is a series whose term ratio is a rational function.

If the shifted factorial is defined by

$$\begin{aligned}(a)_n &= a(a+1)\cdots(a+n-1), & n&=1,2,\ldots,\\ &\;1, & n&=0,\end{aligned} \tag{3.2}$$

then the hypergeometric series is

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}; y\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n n!}y^n. \tag{3.3}$$

This usually requires that $p\leqslant q+1$, for the series diverges if $p>q+1$ and it does not terminate. The Charlier polynomials are defined by

$$C_n(j;x)={}_2F_0\left(\begin{matrix}-n,-j\\ -\end{matrix}; -\frac{1}{x}\right), \quad n=0,1,\ldots. \tag{3.4}$$

Since this series terminates, divergence is not a problem. See [3].

To obtain the zero used in (1.1a)

$$C_1(j;x)=1-\frac{j}{x}$$

so it vanishes when $j=x$. It is easy to check that formula (1.1a) is exact when

$$\varphi(x)=ax+b,$$

for it is clearly exact when $a=0$, and when $b=0$ the calculation is routine.

A second interpretation of this approximation was given by Ramanujan in Chapter 3 of [4]. See Entry 10 in Berndt's version [1].

$$C_2(j;x)=1-\frac{2j}{x}+\frac{j(j-1)}{x^2}$$

and this vanishes when

$$j = \frac{2x + 1 \pm \sqrt{1 + 4x}}{2}$$

as Ramanujan claimed.

The fact that the coefficients are those given by Ramanujan can be checked in two ways. Either one of the standard formulas can be used to derive them, or Ramanujan's formula can be used to check that there is equality for cubic polynomials.

A similar argument can be tried for the third approximation

$$C_3(j; x) = 1 - \frac{3j}{x} + \frac{3j(j-1)}{x^2} - \frac{j(j-1)(j-2)}{x^3}. \tag{3.4}$$

Ramanujan has the interpolation points at

$$j = x,\ x + \frac{1 + \sqrt{1 + 12x}}{2} \text{ and } x + \frac{1 - \sqrt{1 + 12x}}{2}.$$

$C_3(j; x)$ does not vanish at any of these points, so this is not a Gaussian quadrature formula. A Gaussian formula exists, but the zeros of (3.4) cannot be found as a simple expression, so Ramanujan did something else here. He took the value $j = x$ as one interpolation point, which is reasonable for it is the expected value of the Poisson distribution. The remaining two points were chosen so the formula is exact for polynomials of maximal degree, which is four. This is most easily checked by showing there is equality for polynomials of degree 4. This is a tedious calculation which will not be given here.

The remaining formulas are all Gaussian quadrature formulas. In all the cases Ramanujan is using a uniform distribution on an equally spaced set of points. The usual notation for this takes the points at $j = 0, 1, \ldots, N$. The polynomials orthogonal with respect to this distribution were found by Tchebychef [7]. They are given by a hypergeometric series as

$$Q_k(j; N) = {}_3F_2\left(\begin{matrix} -k, k+1, -j \\ 1, -N \end{matrix}; 1\right) \tag{3.5}$$

where $j, k = 0, 1, \ldots, N$. This is the usual method of taking care of the zero which will appear in the denominator because $(-N)_n = 0$ when $n = N + 1, \ldots$. The two factors $(-j)_n$ and $(-k)_n$ both vanish when $(-N)_n$ vanishes, and so the series continues to vanish when one zero in the numerator cancels a zero in the denominator. However, Ramanujan does not restrict his interpolation points to the integers, so we will define

$$Q_k(j; N) = \sum_{n=0}^{k} \frac{(-k)_n (k+1)_n (-j)_n}{(1)_n (-N)_n n!}$$

when $k = 0, 1, \ldots, N$, but j is now allowed to be real or complex.

Again, we need to check the zeros of this function. In the previous case Ramanujan discovered Gaussian quadrature formulas when the polynomial was of degree 2, but for a cubic he did something else. In the present case, he goes up to degree 4, which is possible because the polynomials are even or odd about the midpoint of the interval of orthogonality depending on the parity of the degree. Thus, such polynomials of degree 3 and 4 can be solved by taking square roots.

The examples (1.3)–(1.7) are instances of the general formulas in (1.2), after the step size has been changed. Ramanujan took step size 2 in (1.2) to avoid fractions in the first expression, but went back to the more usual step size of 1 in the examples (1.3)–(1.7).

4. Comments

The polynomials in (3.5) are special cases of more general orthogonal polynomials. These polynomials.

$$Q_n(x;\alpha,\beta,N)=\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k(-x)_k}{(\alpha+1)_k(-N)_k k!}, \tag{4.1}$$

to revert to the more standard use of letters, are called Hahn polynomials. They are orthogonal on $x=0,1,\ldots,N$ with respect to the distribution

$$\binom{x+\alpha}{x}\binom{N-x+\beta}{N-x},\quad x=0,1,\ldots,N. \tag{4.2}$$

See [3]. The functions in (4.1) are multiples of what are called $3-j$ symbols in quantum angular momentum theory. The location of integer zeros of $3-j$ symbols is of some interest in mathematical physics. See [5] for some recent work.

In the introduction, I wrote that it is very unlikely Ramanujan was aware of the orthogonal polynomials which determine Gaussian quadrature. That should not be a surprise, for Gauss did not use orthogonalitly explicitly when he discovered Gaussian quadrature. Jacobi was the first to make this connection. There are a number of instances when Ramanujan seems to come close to orthogonal polynomials. This is especially true in some of his continued fractions, for the three term recurrence relations which generate these are often directly involved with continued fractions. See, for example, [2]. However, there was no good book on orthogonal polynomials when Ramanujan was working, and no one in England knew much about them when Ramanujan was there. Szegö started the serious development of orthogonal polynomials about the time Ramanujan died. It is a shame he was not aware of this subject, for it is a source of many beautiful identities Ramanujan would have loved, and would have given him another tool to find new results.

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References

[1] Berndt B C, Ramanujan's Notebooks, Part I, (1985) (New York: Springer)
[2] Masson D, Wilson polynomials and some continued fractions of Ramanujan, Rocky Mountain *J. Math.*, **21** (1991) 489–499
[3] Nikiforov A F, Suslov S K and Uvarov V B, Classical orthogonal polynomials of a discrete variable (1991) (Berlin: Springer)
[4] Ramanujan S, Notebooks, volume 2, Tata Institute of Fundamental Research, Bombay, 1957
[5] Srinivasa Rao K, Rajeswari V and King R C, Solutions of Diophantine equations and degree-one polynomial zeros of Racah coefficients, *J. Phys.* **A21** (1988) 1959–1070
[6] Szegö G, *Orthogonal polynomials*, fourth edition, *Am. Math. Soc.*, Providence, RI, 1975
[7] Tchebychef P L, Sur une nouvelle série, Oeuvres de P L, Tchebychef, I Chelsea, New York, 1961, 381–384