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# Modular equations and Ramanujan's Chapter 16, Entry 29

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Dedicated to the memory of my friend, Professor K G Ramanathan

**Abstract.** In this paper we illustrate how some of the classical modular equations can be proved by using only Ramanujan's summation (see (1.1)) and dispensing completely with the Schröter-type methods.

**Keywords.** Modular equations; Rogers-Ramanujan functions.

## 1. Introduction

I first met K G Ramanathan because of our mutual interest in Ramanujan. He came to Penn State in February 1982 at my invitation to give our colloquium. I had been through a particularly trying week and was exhausted to say the least. Ramanathan presented a beautiful lecture explaining and extending work from Ramanujan's Lost Notebook [9]–[13]. I remember few mathematics talks as fondly as I remember that one. The beauty of the work truly revived my spirits.

General interest in Ramanujan's work has been intense in recent years due in no small part to the magnificient edited versions of Ramanujan's Notebooks [2], [3], [4] carefully prepared by Bruce Berndt.

This paper will be devoted to further considerations of modular equations, especially those of degrees 3 and 5. Berndt [4; pp. 6–7] and Hardy [7; Ch. 12] discuss several approaches to modular equations. Succinctly stated they are: (1) the Legendre-Jacobi method using differential equations for elliptic functions [7; §§ 12·4–12·7]; (2) Schröter's method requiring ingenious rearrangements of double theta series [4; p. 73]; (3) the theory of modular forms [4; p. 7], and (4) Ramanujan's method.

Both Hardy and Berndt are uncertain about Ramanujan's method for the excellent reason that he never revealed it. He merely stated his discoveries without proof, and as Berndt puts it "... found more modular equations than all of his predecessors put together."

To prove Ramanujan's formulas both Hardy [7; Ch. 12] and Berndt [4] mix Schröter's method, algebraic manipulation of series and products (what Hardy [7; pp. 220–221] calls "trivial" relations), and Ramanujan's ${}_1\psi_1$-summation [4; p. 32, Entry 17] rewritten as

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n t^n}{(b)_n} = \frac{(b/a, at, q/(at), q; q)_\infty}{(q/a, b/(at), b, t; q)_\infty}, \tag{1.1}$$

where

$$(A)_n = (A;q)_n = \prod_{j=0}^{\infty} \frac{(1-Aq^j)}{(1-Aq^{j+n})} \tag{1.2}$$

$(= (1-A)(1-Aq)\dots(1-Aq^{n-1})$ when $n$ is a positive integer),

$$(A)_\infty = (A;q)_\infty = \lim_{n\to\infty} (A)_n, \tag{1.3}$$

and

$$(A_1 A_2, \dots, A_r; q)_\infty = (A_1)_\infty (A_2)_\infty \dots (A_r)_\infty. \tag{1.4}$$

Actually the only instance of (1.1) required in this regard is the case $b = aq$:

$$\begin{aligned} S(a,t,q) &:= \frac{(at, q/(at), q, q; q)_\infty}{(a, q/a, t, q/t; q)_\infty} \\ &= \sum_{n=-\infty}^{\infty} \frac{t^n}{1-aq^n} \end{aligned} \tag{1.5}$$

(where for convergence we require $|q| < |t| < 1$).

Our object in this paper is to show that the Schröter methods may be entirely dispensed with at least for the standard forms of the modular equations of degrees 3 and 5. I am certainly not suggesting that Ramanujan did not know Schröter's method. However I would stress that the methods given here at each stage suggest very simple combinations of functions (see especially the proof of Entry 29 in §3 below) which translate into complicated and surprising modular equations.

This approach appears to fit in nicely with the first 22 entries considered by Berndt in [5]. This latter paper is devoted to the results on theta-functions and modular equations found in the 100 pages of unorganized material at the end of Ramanujan's second notebook and in the 33 pages of unorganized material comprising the third notebook. The reader's attention is also directed to the recent work of L.-C. Shen [14] who also uses (1.1) to derive Lambert series identities related to modular equations of degrees 3.

## 2. Background

In order to maintain the Ramanujan spirit, we follow Berndt's lead and work with [4; p. 34, p. 36]

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{2.1}$$

$$\psi(q) := \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} q^{n(2n+1)}. \tag{2.2}$$

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.3}$$

If in (1.1) we replace $t$ by $t/a$, set $b = 0$ and then let $a \to \infty$, we obtain Jacobi's triple product [4; p. 35]

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2} t^n = (t, q/t, q; q)_\infty, \tag{2.4}$$

which is equivalent to (with $t = -a, q = ab$)

$$f(a,b) = (-a, -b, ab; ab)_\infty \tag{2.5}$$

Still following Berndt's account of the basics [4; p. 36, Entry 22(i), (ii); p. 37 eq. (22.4); p. 40, Entry 25, (i)–(iv)]

$$\varphi(q) = \frac{(-q, q^2; q^2)_\infty}{(q, -q^2; q^2)_\infty} \tag{2.6}$$

$$\varphi(-q) = \frac{(q)_\infty}{(-q; q)_\infty} \tag{2.7}$$

$$\psi(q) = \frac{(q^2, q^2)_\infty}{(q; q^2)_\infty} \tag{2.8}$$

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8) \tag{2.9}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2) \tag{2.10}$$

$$\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2) \tag{2.11}$$

$$\varphi(q)\psi(q^2) = \psi^2(q). \tag{2.12}$$

An examination of Berndt's clear presentation [4; pp. 36–40] shows the direct derivation of each expression either from (2.5) or from the algebraic manipulation of infinite series and products.

## 3. Entry 29 of Chapter 16

If $ab = cd$, then

$$\begin{aligned} f(a,b)f(c,d) &+ f(-a,-b)f(-c,-d) \\ &= 2f(ac,bd)f(ad,bc), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} f(a,b)f(c,d) &- f(-a,-b)f(-c,-d) \\ &= 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \end{aligned} \tag{3.2}$$

Berndt's proof of these [4; p. 45] is a nice application of the rearrangement of double series. We shall show that these results follow from Ramanujan's ${}_1\psi_1$-summation as stated in (1.5) above.

Obviously

$$S(A,t,q) \pm S(A,-t,q) = \sum_{n=-\infty}^{\infty} \frac{t^n(1 \pm (-1)^n)}{1 - Aq^n}; \tag{3.3}$$

So

$$S(A,t,q)+S(A,-t,q)=2S(A,t^2,q^2), \tag{3.4}$$

$$S(A,t,q)-S(A,-t,q)=2t\,S(Aq,t^2,q^2). \tag{3.5}$$

Equation (3.4) simplifies to (3.1) with $t=c$, $A=-a/c$, $q=ab$, and (3.5) reduces to (3.2) under the same substitution.

For our purposes, Entry 29 given by (3.4) and (3.5) is the most useful. We shall concentrate on the two following specializations. Let

$$\begin{aligned} F_{k,l}(q) &= \frac{(-q^{2l+1},-q^{2k-2l-1},q^{2k},q^{2k};q^{2k})_\infty}{(q^{2l+1},q^{2k-2l-1},-q^{2k},-q^{2k};q^{2k})_\infty} \\ &= \varphi^2(-q^{2k})\frac{(-q^{2l+1},-q^{2k-2l-1};q^{2k})_\infty}{(q^{2l+1},q^{2k-2l-1};q^{2k})_\infty} \\ &= 2S(-1,q^{2l+1},q^{2k}) \\ &= \sum_{n=-\infty}^{\infty}\frac{q^{n(2l+1)}}{1+q^{2kn}} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} G_{k,l}(q) &= \frac{(-q^{2l+k+1},-q^{k-2l-1},q^{2k},q^{2k};q^{2k})_\infty}{(q^{2l+1},q^{2k-2l-1},-q^{k},-q^{k};q^{2k})_\infty} \\ &= \psi^2(-q^{k})\frac{(-q^{2l+k+1},-q^{k-2l-1};q^{2k})_\infty}{(q^{2l+1},q^{2k-2l-1};q^{2k})_\infty} \\ &= S(-q^k,q^{2l+1},q^{2k})_\infty \\ &= \sum_{n=-\infty}^{\infty}\frac{q^{n(2l+1)}}{1+q^{k(2n+1)}}. \end{aligned} \tag{3.7}$$

So by (3.4)

$$F_{k,l}(q)+F_{k,l}(-q)=2F_{k,l}(q^2), \tag{3.8}$$

and by (3.5)

$$F_{k,l}(q)-F_{k,l}(-q)=4q^{2l+1}G_{k,l}(q^2). \tag{3.9}$$

## 4. The modular equation of degree 1

This section is devoted to the case $k=1$, $l=0$ of (3.6) and (3.7). Note that by (2.6)–(2.8):

$$F_{1,0}(q)=\varphi^2(q), \tag{4.1}$$

and

$$G_{1,0}(q)=2\psi^2(q^2). \tag{4.2}$$

Hence by (3.8)

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2), \tag{4.3}$$

and by (3.9)

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4). \tag{4.4}$$

Equations (4.3) and (4.4) are in fact items (v) and (vi) in Entry 25 [4; p. 40]. Lastly, Berndt points out that multiplying them together yields

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{4.5}$$

This identity in the notation of classical elliptic function theory [6; p. 93 eq. (34.32)] is the identity

$$k^2 + k'^2 = 1, \tag{4.6}$$

an identity that could be called (but never is) the modular equation of degree 1.

## 5. The modular equation of degree 3

Now we consider (3.8) with $k = 3$, $l = 0$:

$$F_{3,0}(q) = \frac{\varphi(-q^6)\varphi(-q^2)\varphi(-q^3)}{\varphi(-q)} \tag{5.1}$$

Thus after simplification, (3.8) reduces to

$$\varphi(q)\varphi(-q^3) + \varphi(-q)\varphi(q^3) - 2\varphi(-q^4)\varphi(-q^{12}) = 0. \tag{5.2}$$

This is one of many forms of the modular equation of degree 3. Legendre's standard form of the modular equation of degree 3 [4; p. 232] is

$$\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3) - 4q\psi(q^2)\psi(q^6) = 0. \tag{5.3}$$

To see that (5.2) and (5.3) are equivalent, we apply (2.9) to (5.2):

$$\begin{aligned} 0 &= (\varphi(q^4) + 2q\psi(q^8))(\varphi(q^{12}) - 2q^3\psi(q^{24})) \\ &\quad + (\varphi(q^4) - 2q\psi(q^8))(\varphi(q^{12}) + 2q^3\psi(q^{24})) - 2\varphi(-q^4)\varphi(-q^{12}) \\ &= 2\varphi(q^4)\varphi(q^{12}) - 8\psi(q^8)\psi(q^{24}) - 2\varphi(-q^4)\varphi(-q^{12}). \end{aligned}$$

This reduces to (5.3) upon division by 2 and replacement of $q^4$ by $q$.

## 6. Ramanujan's identities of modulus 5

In this section we consider the cases $k = 5$, $l = 0, 1$ of (3.8) and (3.9). We begin by recalling the Rogers-Ramanujan infinite products [4; pp. 77–78]:

$$g(q) = \frac{1}{(q, q^4; q^5)_\infty}, \tag{6.1}$$

and

$$h(q)=\frac{1}{(q^2,q^3;q^5)_\infty}. \tag{6.2}$$

Clearly from (6.1) and (6.2)

$$g(q)h(q)=\frac{(q^5;q^5)_\infty}{(q)_\infty}, \tag{6.3}$$

$$\frac{g(q)}{h(q^2)}=\frac{1}{(q,q^9;q^{10})_\infty}, \tag{6.4}$$

$$\frac{h(q)}{g(q^2)}=\frac{1}{(q^3,q^7;q^{10})_\infty}, \tag{6.5}$$

$$\frac{g(q)}{h(q^4)}=\frac{(-q^4,-q^6;q^{10})_\infty}{(q,q^9;q^{10})_\infty}, \tag{6.6}$$

and

$$\frac{h(q)}{g(q^4)}=\frac{(-q^2,-q^8;q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}. \tag{6.7}$$

These formulae allow us to make the following identifications from (3.6) and (3.7):

$$F_{5,0}(q)=\frac{\varphi^2(-q^{10})g(q)}{g(-q)}, \tag{6.8}$$

$$F_{5,1}(q)=\frac{\varphi^2(-q^{10})h(q)}{h(-q)}, \tag{6.9}$$

$$G_{5,0}(q)=\frac{\psi^2(-q^5)g(q)}{h(q^4)}, \tag{6.10}$$

$$G_{5,1}(q)=\frac{\psi^2(-q^5)h(q)}{g(q^4)}, \tag{6.11}$$

by (6.3), (6.8) and (6.9)

$$F_{5,0}(q)F_{5,1}(q)=\frac{\varphi^3(-q^{10})\varphi(-q^5)\varphi(-q^2)}{\varphi(-q)}; \tag{6.12}$$

by (6.3), (6.8) and (6.11)

$$F_{5,0}(q)G_{5,1}(q)=\frac{\varphi^2(-q^{10})\psi^2(-q^5)(q^5;q^5)_\infty}{(q)_\infty g(-q)g(q^4)}, \tag{6.13}$$

and by (6.3), (6.9) and (6.10)

$$F_{5,1}(q)G_{5,0}(q)=\frac{\varphi^2(-q^{10})\psi^2(-q^5)(q^5;q^5)_\infty}{(q)_\infty h(-q)h(q^4)}. \tag{6.14}$$

In addition

$$
\begin{aligned}
\frac{h(g)}{g(q)} &= \frac{(q,q^4,q^6,q^9;q^{10})_\infty}{(q^2,q^3,q^7,q^8;q^{10})_\infty} \\
&= \frac{g(q^2)}{h(q^2)^2(q^{10};q^{10})^2_\infty}\cdot\frac{(q,q^9,q^{10},q^{10};q^{10})_\infty}{(q^3,q^7,q^4,q^6;q^{10})_\infty} \\
&= \frac{g(q^2)}{h(q^2)^2(q^{10};q^{10})^2_\infty} S(q^6,q^3,q^{10})_\infty \\
&= \frac{g(q^2)}{h(q^2)^2(q^{10};q^{10})^2_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{1-q^{10n+6}}; \qquad (6.15)
\end{aligned}
$$

We conclude this list of specializations of $S(A,t,q)$ with four which were listed earlier [1] in connection with the mock-theta conjectures:

$$
\begin{aligned}
\varphi(-q)g(q) &= \frac{S(-q,-q^2,q^5)}{(q^{10};q^{10})_\infty} \\
&= \frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q^2)^n}{1+q^{5n+1}}
\end{aligned}
$$

([1; p. 246, eq. (3.15)]), (6.16)

$$
\begin{aligned}
\varphi(-q)h(q) &= \frac{S(-q,-q^3,q^5)}{(q^{10};q^{10})_\infty} \\
&= \frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q^3)^n}{1+q^{5n+1}} \\
&= \frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}q^{2n+1}}{1+q^{5n+4}}
\end{aligned}
$$

([1; p. 246, eq. (3.16)]), (6.17)

$$
\psi(q^2)g(q^4) = \frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{20n+6}}
$$

([1; p. 247, eq. (3.18)]), (6.18)

$$
\psi(q^2)h(q^4) = \frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{20n+14}}
$$

([1; p. 247, eq. (3.17)]). (6.19)

We are now in a position to prove six formulas that are either due to Ramanujan or directly deduced from Ramanujan's work by G N Watson.

$$
\varphi(q)g(-q) - \varphi(-q)g(q) = 2q\psi(q^2)h(q^4)
$$

([16; p. 289, eq. (6)]), (6.20)

$$
\varphi(q)h(-q) + \varphi(-q)h(q) = 2\psi(q^2)g(q^4)
$$

([16; p. 289, eq. (7) corrected]), (6.21)

$$g(q)h(-q)+g(-q)h(q)=\frac{2\psi(q^2)}{(q^2;q^2)_\infty}=2(-q^2;q^2)_\infty^2$$

$$([15;\ \text{p. }60,\ \text{eq. }(5)]), \tag{6.22}$$

$$g(q)h(-q)-g(-q)h(q)=\frac{2q\psi(q^{10})}{(q^2;q^2)_\infty}$$

$$([15;\ \text{p. }60,\ \text{eq. }(6)]), \tag{6.23}$$

$$g(q)g(q^4)+qh(q)h(q^4)=(-q;q^2)_\infty^2,$$

$$([15;\ \text{p. }60,\ \text{eq. }(3)]), \tag{6.24}$$

$$g(q)g(q^4)-qh(q)h(q^4)=\frac{\varphi(q^5)}{(q^2;q^2)_\infty}$$

$$([15;\ \text{p. }60,\ \text{eq. }(4)]). \tag{6.25}$$

By (6.16) and (6.19) we see that

$$\begin{aligned}
&\varphi(q)g(-q)-\varphi(-q)g(q)\\
&\frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\left(\frac{(-q^2)^n}{1-(-1)^n q^{5n+1}}-\frac{(-q^2)^n}{1+q^{5n+1}}\right)\\
&=\frac{1}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\left(\frac{q^{4n}}{1-q^{10n+1}}-\frac{q^{4n}}{1+q^{10n+1}}\right)\\
&=\frac{2}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{14n+1}}{1-q^{20n+2}}\\
&=\frac{2q}{(q^{10};q^{10})_\infty}S(q^2,q^{14},q^{20})\\
&=\frac{2q}{(q^{10};q^{10})_\infty}S(q^{14},q^2,q^{20})\\
&=\frac{2q}{(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{20n+14}}=2q\psi(q^2)h(q^4),
\end{aligned} \tag{6.26}$$

which is (6.20)

In exactly the same way (6.21) follows from (6.17) and (6.18). Now by (6.15), we see that

$$\begin{aligned}
&g(q)h(-q)\pm g(-q)h(q)\\
&=g(q)g(-q)\left(\frac{h(-q)}{g(-q)}\pm\frac{h(q)}{g(q)}\right)\\
&=\frac{g(q)g(-q)g(q^2)}{h(q^2)^2(q^{10};q^{10})_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{3n}((-1)^n\pm 1)}{1-q^{10n+6}}
\end{aligned} \tag{6.27}$$

Therefore the left-hand side of (6.22) is identical with

$$\begin{aligned}
&\frac{2g(q)g(-q)g(q^2)}{h(q^2)^2(q^{10};q^{10})_\infty^2}S(q^6,q^6,q^{20})\\
&=\frac{2g(q)g(-q)g(q^2)(q^{12},q^8,q^{20},q^{20};q^{20})_\infty}{h(q^2)^2(q^{10};q^{10})_\infty^2(q^6,q^{14};q^{20})_\infty^2}\\
&=2(-q^2;q^2)_\infty^2
\end{aligned}\tag{6.28}$$

upon simplification using (6.1), (6.2) and putting all products to the modulus 20.

In exactly the same way, (6.27) implies that the left-hand side of (6.23) is identical with

$$\begin{aligned}
&\frac{-2g(q)g(-q)g(q^2)q^3}{h(q^2)^2(q^{10};q^{10})_\infty^2}S(q^{16},q^6,q^{20})\\
&=\frac{2q\psi(q^{10})}{(q^2;q^2)_\infty}.
\end{aligned}\tag{6.29}$$

To obtain (6.24) we observe that

$$\begin{aligned}
&g(q)g(q^4)+qh(g)h(q^4)\\
&=g(q)\left(\frac{\varphi(q)h(-q)+\varphi(-q)h(q)}{2\psi(q^2)}\right)\\
&\quad+qn(q)\left(\frac{\varphi(q)g(-q)-\varphi(-q)g(q)}{2q\psi(q^2)}\right)\\
&=\frac{\varphi(q)}{2\psi(q^2)}(g(q)h(-q)+h(q)g(-q))\\
&=\frac{\varphi(q)}{2\psi(q^2)}\frac{2\psi(q^2)}{(q^2;q^2)_\infty}=\frac{\varphi(q)}{(q^2;q^2)_\infty}=(-q;q^2)_\infty^2,
\end{aligned}\tag{6.30}$$

as desired.

Finally

$$\begin{aligned}
&\frac{(g(q)g(q^4)-qh(q)h(q^4))\psi^4(-q^5)}{h(q^4)g(q^4)}\\
&=G_{5,0}(q)-qG_{5,1}(q)\\
&=\frac{1}{2}\left(2\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{10n+5}}-2\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}}{1+q^{10n+5}}\right.\\
&\quad\left.+\sum_{n=-\infty}^{\infty}\frac{q^{5n+2}}{1+q^{10n+5}}\right)-\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{q^{5n+2}}{1+q^{10n+5}}\\
&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{q^n+q^{9n+4}-q^{3n+1}-q^{7n+3}+q^{5n+2}}{1+q^{10n+5}}\\
&\quad-\frac{q^2}{2}\sum_{n=-\infty}^{\infty}\frac{q^{5n}}{1+q^{10n+5}}
\end{aligned}$$

$$=\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{2n+1}}-\frac{1}{2}q^2\sum_{n=-\infty}^{\infty}\frac{q^{5n}}{1+q^{10n+5}}$$

$$=\frac{1}{2}G_{1,0}(q)-\frac{1}{2}q^2G_{1,0}(q^5)$$

$$=\frac{1}{2}(\psi(q^2)^2-q^2\psi(q^{10})^2)$$

$$=\frac{1}{2}\frac{(q^{10};q^{10})_\infty^2}{g(q^2)h(q^2)}, \tag{6.31}$$

where to obtain the last expression we applied Entry 10 (v) [4; p. 262]. We note that Berndt's lovely proof of Entry 10 (v) is fully consistent with the object of this paper in that the only result used other than instances of rearrangements of $S(A,t,q)$ is Entry 29 of Chapter 16 which we have shown to be again an application of Ramanujan's ${}_1\psi_1$-summation for $S(A,t,q)$. Equation (6.31) reduces to (6.25) upon product simplification.

## 7. The standard modular equation of degree 5

We shall consider the modular equation of degree 5 in the form given by Watson [16; p. 289]

$$\varphi(q)\varphi(-q^5)-\varphi(-q)\varphi(q^5)=4q(q^4;q^4)_\infty(q^{20};q^{20})_\infty \tag{7.1}$$

The proof goes rapidly using the results of § 6.

$$\varphi(q)\varphi(-q^5)-\varphi(-q)\varphi(q^5)$$

$$=\varphi(q)\varphi(-q)\left(\frac{\varphi(-q^5)}{\varphi(-q)}-\frac{\varphi(q^5)}{\varphi(q)}\right)$$

$$=\frac{\varphi(-q^2)\varphi(-q)}{\varphi^3(-q^{10})}(F_{5,0}(q)F_{5,1}(q)-F_{5,0}(-q)F_{5,1}(-q))$$

$$=\frac{\varphi(-q^2)\varphi(-q)}{2\varphi^3(-q^{10})}\{(F_{5,0}(q)+F_{5,0}(-q))(F_{5,1}(q)-F_{5,1}(-q))$$
$$+(F_{5,0}(q)-F_{5,0}(-q))(F_{5,1}(q)+F_{5,1}(-q))\}$$

$$=\frac{\varphi(-q^2)\varphi(-q)}{2\varphi^3(-q^{10})}\{8q^3G_{5,0}(q^2)F_{5,1}(q^2)+8qG_{5,0}(q^2)F_{5,1}(q^2)\}$$

(by (3.8) and (3.9))

$$=\frac{4q\varphi(-q^2)\varphi(-q)}{\varphi^3(-q^{10})}\left\{\frac{\varphi^2(-q^{20})\psi^2(-q^{10})(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty h(-q^2)h(q^8)}\right.$$
$$\left.+q^2\frac{\varphi^2(-q^{20})\psi^2(-q^{10})(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty g(-q^2)g(q^8)}\right\} \qquad \text{(by (6.8)–(6.11))}$$

$$= \frac{4q\varphi(-q^2)\varphi(-q)\varphi^2(-q^{20})\psi^2(-q^{10})(q^{10};q^{10})_\infty}{\varphi^3(-q^{10})(q^2;q^2)_\infty h(-q^2)h(q^8)g(-q^2)g(q^8)}$$

$$(g(-q^2)g(q^8)-(-q^2)h(-q^2)h(q^8))$$

$$= \frac{4q\varphi(-q^2)\varphi(-q)\varphi^2(-q^{20})\psi^2(-q^{10})(q^{10};q^{10})_\infty}{\varphi^3(-q^{10})(q^2;q^2)_\infty h(-q^2)h(q^8)g(-q^2)g(q^8)} \cdot \frac{\varphi(-q^{10})}{(q^4;q^4)_\infty} \quad \text{(by (6.25))}$$

$$= 4q(q^4;q^4)_\infty(q^{20};q^{20})_\infty. \tag{7.2}$$

The last reduction follows by writing each factor in the penultimate line as an infinite product on the modulus 20 and doing the relevant cancellation.

## 8. Conclusion

The approach described in this paper suggests that further examination of $F_{k,l}(q)$ and $G_{k,l}(q)$ is in order. Obviously higher order modular equations might well be derived from a study of $F_{2k+1,0}(q)$. In addition it is possible that Hickerson's grand treatment of the seventh order mock theta functions [7] may well be related to $F_{7,l}(q)(l = 0, 1, 2)$.

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