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Symplectic structures on locally compact abelian groups and polarizations

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Let X be a locally compact abelian group and $\omega(\cdot,\cdot)$ a symplectic structure on it. A polarization for (X,ω) is a pair of totally isotropic closed subgroups G, G^* of X such that $X = G\cdot G^*$ and $\omega(\cdot,\cdot)$ defines a dual pairing of G and G^*. In this paper we describe a class of such groups which always admit a polarization and also discuss their structure.

Keywords. Symplectic structures.

1. Introduction

Let $\mathscr{L}$ denote the class of locally compact Hausdorff, abelian and second countable groups. Fox $X\in\mathscr{L}$, consider an alternating bicharacter ω on X, i.e. (i) For each $x, y\to\omega(x,y)$ is character in $y\in X$, and for each $y, x\to\omega(x,y)$ is character in x and (ii) ω is alternating i.e., $\omega(x,x)=1$ for each $x\in X$. Such ω is known to provide a classification of central extensions of X by the circle group T. For the central extension $1\to T\to E_\omega\to X\to 1$ corresponding to the classifying invariant ω, the analogue of the Stone–von Neumann theorem has been proved under two sets of assumptions. The first one is

$$(X,\omega) \text{ is nondegenerate} \tag{1}$$

i.e., $\omega(x_0,y)=1$ for all y implies $x_0=0$ (the identity element of X) and if χ is any continuous character of X, then there exists an $x_0\in X$ such that $\chi(y)=\omega(x_0,y)$ for all y. This assumption makes it possible to apply Mackey's theory of systems of imprimitivity to E_ω to get Stone–von Neumann theorem (see for example [M]). On the other hand, Weil [W1] used a different kind of assumption on (X,ω) to obtain the same result. This assumption may be succinctly described by saying that (X,ω) admits a polarization i.e., there exist closed subgroups G, G^* of X with the following properties:

(i) $G\cap G^*=\langle 1\rangle$, $X=G\cdot G^*$
(ii) $\omega(G,G)=1$, $\omega(G^*,G^*)=1$ i.e., $\omega(x,y)=1$ whenever $x,y\in G$, and also whenever $x,y\in G^*$
(iii) the mapping $x,y\to\omega(x,y)$ of $G\times G^*$ into T is a dual pairing of G and G^*.

Note the assumption (i) means that X is a direct sum (or direct product) of the groups G and G^* and the assumption (iii) identifies G^* as the character group of G.

Note then

$$\omega(x + x^*, y + y^*) = \omega(x, y^*)\omega(y, x^*)^{-1}. \tag{2}$$

Conversely if $G \in \mathscr{L}$ and G^* is its dual group, then $(G \times G^*, \omega)$ is nondegenerate, where $\omega(x, x^*) = x^*(x)$ and $\omega(\cdot,\cdot)$ in general is defined by (2). The problem we are concerned about in this paper is to find the class of groups X such that any nondegenerate symplectic structure ω on X will admit a polarization. The main result is.

Theorem 1. *Let $\mathscr{L}_0$ denote the class of groups $X \in \mathscr{L}$ possessing the following properties:*

(i) *The maximal compact connected subgroups of both X and its dual X^* are tori*
(ii) $\dim_{\mathbb{Q}_p} \hom(X, \mathbb{Q}_p) < \infty$, *for all primes p*
(iii) *The subgroup $\{x \in X : p \cdot x = 0\}$ is finite for each prime p.*

If $X \in \mathscr{L}_0$ and ω a nondegenerate symplectic structure on X, then (X, ω) admits a polarization.

A proof of this theorem is given in § 3. In § 2, we recall certain basic results on structure of groups in $\mathscr{L}$ and also discuss the structure of groups in class $\mathscr{L}_0$ in some detail. In § 3 we consider symplectic structures (X, ω) and prove various decomposition theorems, leading to a proof of Theorem 1. In a sequel we will discuss applications of these to the metaplectic representation.

2. The structure of the class of groups in $\mathscr{L}_0$

Let $\mathscr{L}$ denote the class of second countable, Hausdorff, locally compact abelian groups. For any $G \in \mathscr{L}$, let $G^* = \hom(G, T)$ be the character group of G. Let G^0 be the connected component of G, and let ${}^0G = \{x \in G : x$ is a compact element of G, i.e., the subgroup generated by x has compact closure$\}$. Then we have.

Lemma 2. *For any $G \in \mathscr{L}$, ${}^0G$ is a closed subgroup and it is the annihilator in G of the connected component of G^*. Moreover*

$${}^0G = \cap \{\ker \xi : \xi \in \hom(G, R)\}.$$

(For proof see [HR] p. 382 and p. 390).

Lemma 3. *For any G in $\mathscr{L}$, let $K = {}^0G \cap {}^0G$—the subgroup of compact elements in G^0. Then K is compact, connected and is the maximal compact connected subgroup of G. Moreover $G^0 \simeq R^n$. K for some n.* (See [HR] page 95).

Lemma 4. *The subgroup ${}^0G \cdot G^0$ is open in G and its annihilator in G^* is the maximal compact connected subgroup of G^*. In particular $G/({}^0G \cdot G^0)$ is discrete and torsion free.* ([HR], § 9.26(a), p. 103).

Using these we can next prove.

PROPOSITION 5

For a group G in $\mathscr{L}$, the following statements are equivalent:

(i) *the maximal compact connected subgroup of G^* is a torus, i.e., is isomorphic to T^m for some m*

(ii) $\dim_R \hom(G, R) < \infty$, *and if* $\xi \in \hom(G, R)$ *is such that* $\xi(G) \subseteq \mathbf{Q}$, *then either* $\xi(G) = (0)$ *or* $\xi(G)$ *is a lattice*
(iii) *For some integers n and k,* $G \simeq {}^0G \times R^n \cdot \mathbb{Z}^k$. *(direct sum)*

Proof. (i)→(iii). From Lemma 4, it follows that $G/({}^0G \cdot G^0)$ is the character group of the maximal compact connected component of G^*. Thus $G/({}^0G \cdot G^0)$ is $\simeq \mathbb{Z}^k$ for some k. This implies that $G \simeq ({}^0G \cdot G^0) \cdot \mathbb{Z}^k$. Since $({}^0G \cdot G^0) \simeq {}^0G \cdot R^n$, the conclusion (iii) follows.

(iii)→(i). For (iii) implies that $G/({}^0G \cdot G^0) \simeq \mathbb{Z}^k$ so that the maximal compact connected subgroup of G^* is $\simeq T^k$.

(iii)→(ii). This is clear.

(ii)→(i). Let $D = G/({}^0G \cdot G^0)$. Note the property (ii) for G implies that the same is valid for D. Let $\dim \hom(D, R) = d$ and let $\xi_1, \ldots, \xi_d$ be a basis for $\hom(D, R)$. Consider the map $\varphi = (\xi_1, \ldots, \xi_d)$ of $D \to R^d$. It is then clear that the subspace spanned by $\xi(D)$ is R^d. Let v_j, $j = 1, 2, \ldots, d$ be a basis of R^d, with $v_j \in \xi(D)$ for all j. Let A be a $d \times d$ matrix, nonsingular such that $Av_j = e_j$, where e_j is the standard basis of R^d. Let $\eta = A\xi$, and $\psi = (\eta_1, \ldots, \eta_d)$. Then $e_j \in \psi(D)$ and so $\psi(D) \supseteq \mathbb{Z}^d$. Also if $\xi_j(x) = 0$ for all j, then $x \in \cap\{\ker \xi : \xi \in \hom(D, R)\} = {}^0D = \langle 0 \rangle$. Thus φ and ψ are both injective. Let $x_j \in D$, be such that $\psi(x_j) = e_j$ and let $D_0 = \Sigma \mathbb{Z} x_j$, be the subgroup generated by x_j, $j = 1, 2, \ldots, d$. From the construction it is clear that D/D_0 is torsion. For if $x \in D$, such that $\pi(x) \in D/D_0$ is an element of infinite order, π being the canonical map $D \to D/D_0$, then there exists a $\lambda \in \hom(D/D_0, R)$, such that $\lambda(\pi(x)) \neq 0$. Consider $\mu = \lambda \circ \pi$. Then $\mu \in \hom(D, R)$ and so $\mu = \Sigma c_j \eta_j$. But then $\mu(x_j) = 0$ for all j implies that $c_j = 0$ for all j, since $\eta_j(x_i) = \delta_{ij}$. Thus $\mu = 0$, contradicting $\lambda(\pi(x)) \neq 0$. Thus D/D_0 is torsion. This implies that $\mathbb{Z}^d \subseteq \psi(D) \subseteq \mathbb{Q}^d$. Thus each η_j is such that $\eta_j(D) \subseteq \mathbb{Q}$. Since D also satisfies (ii), it follows that $\eta_j(D)$ is a lattice. Thus there exists an integer N such that $\mathbb{Z}^d \subseteq \psi(D) \subseteq (1/N)\mathbb{Z}^d$. So $\psi(D) \simeq \mathbb{Z}^d$, or ψ being injective, it follows that $D \simeq \mathbb{Z}^d$. This proves that G has property (i) □

COROLLARY 6

Let $G \in \mathscr{L}$ *be such that the maximal compact connected components of both G and* G^* *are tori. Then there exist closed subgroups* G_∞ *and* G_f *of G such that*

(i) $G = G_\infty \cdot G_f$ *(direct sum)*
(ii) $G_\infty \simeq R^m \times T^n \times \mathbb{Z}^k$
(iii) G_f *is totally disconnected and every element of* G_f *is compact.*

Proof. Let K be the maximal compact connected subgroup of G. Then $K \simeq T^n$. Since subgroups which are isomorphic to tori are direct summands, it follows that there exists a closed subgroup G_f of ${}^0G$, such that ${}^0G \simeq K \cdot G_f$ direct sum. Since K is the connected component of ${}^0G$, it follows that G_f is totally disconnected. The rest follows form Proposition 5.

Remark-Definition 7. From Lemma 1, it follows that G is totally disconnected if and only if every element of G^* is compact. Thus the following statments are equivalent and the class of groups satisfying them is denoted by $\mathscr{C}$.

(i) G is totally disconnected and every element of G is compact. (ii) Both G and G^* are totally disconnected. (iii) Every element of both G and G^* is compact.

To analyze the structure of this class further we recall the notion of topological p-group (see [A] or [B] for details). For $G \in \mathscr{L}$, p a prime, define $G_p = \{x \in G:$

$\lim_{n\to\infty} p^n\cdot x = 0\}$. (Note here we have used the additive notation for G. However for some groups such as T we use the multiplicative notation.) Clearly G_p is a subgroup (not in general closed) of G and is called the topological p-primary component of G. The group is said to be p-primary if $G = G_p$. For example it is known that $T_p =$ the p-primary component of T is the subgroup $=\{\exp 2\pi i(m/p^n)\colon m, n\in\mathbb{Z}\}$ or $T_p \simeq \mathbb{Z}[1/p]/\mathbb{Z}$ as a subgroup. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\mathbb{Z}_p$—the ring of p-adic integers. Then $\mathbb{Z}_p$ is compact and its character group is isomorphic to T_p (with discrete topology).

Lemma 8 (i) *An element $x\in G_p$ if and only if the map $x\to n\cdot x$ of $\mathbb{Z}$ into G extends as a continuous homomorphism of $\mathbb{Z}_p$ into G.*

(ii) *If G is totally disconnected then G_p is closed and is a $\mathbb{Z}_p$-module.*

(iii) *If G is p-primary, so is G^*.* (For proofs and other details see [A] or [B]).

Lemma 9 *Let $G\in\mathscr{C}$ i.e., both G and G^* are totally disconnected. Let K be a compact open subgroup of G and let G_p, K_p denote their p-primary components. Then K_p is a compact open subgroup of G_p and $G\simeq \Pi G_p$ is the restricted direct product with respect to $\{K_p\}$.* (For proof see [B]).

This leads us to introduce the class $\mathscr{C}_p$. Let $\mathscr{C}_p$ consist of all $G\in\mathscr{C}$ which satisfy the following conditions:

(i) G is p-primary,
(ii) $\dim_{\mathbb{Q}_p} \hom(G, \mathbb{Q}_p) < \infty$
(iii) The subgroup $=\{x\in G\colon p\cdot x = 0\}$ of elements of order p is finite.

We then have.

PROPOSITION 10

Let $G\in\mathscr{C}_p$. Then G is isomorphic to a group of the form $G\simeq \mathbb{Q}_p^a\times\mathbb{Z}_p^b\times T_p^c\times B$, where B is a finite p-group, where a, b, c are finite integers.

The proof will be carried out in a number of steps.

Step 1. Let $\Gamma = \cap\{\ker\gamma\colon \gamma\in\hom(G,\mathbb{Q}_p)\}$. Then for some integer a, b, $a+b = d = \dim\hom(G,\mathbb{Q}_p)$, $G/\Gamma\simeq \mathbb{Q}_p^a\times\mathbb{Z}_p^b$.

Proof. Let $\xi_1,\ldots,\xi_d$ be a basis for $\hom(G,\mathbb{Q}_p)$. Let $\varphi\colon G\to\mathbb{Q}_p^d$, $\varphi = (\xi_1,\ldots,\xi_d)$. Then the linear independence of ξ_j's implies that $\xi(G)$ spans $\mathbb{Q}_p^d$. Thus $\xi(G)$ contains a basis of $\mathbb{Q}_p^d$, is a $\mathbb{Z}_p$-module and so $\xi(G)$ is an open $\mathbb{Z}_p$-module or a lattice in $\mathbb{Q}_p^d$. From the known structure of such lattices (see [W2], Chapter 2), it follows that

$$\xi(G) = \sum_{j=1}^{a} \mathbb{Q}_p v_j + \sum_{a+1}^{d} \mathbb{Z}_p v_j \simeq \mathbb{Q}_p^a\times\mathbb{Z}_p^b.$$

Step 2. The subgroup Γ is discrete and $\simeq T_p^c\times B$, for some finite group B.

Proof. Using the earlier notation, let $x_j\in G$ be such that $\varphi(x_j) = v_j$. Let $M = \Sigma\mathbb{Z}_p x_j$. Then M is a compact subgroup of G, $\Gamma\cap M = \langle 0\rangle$ and $\Gamma\cdot M = \varphi^{-1}(\Sigma\mathbb{Z}_p v_j)$ and so $\Gamma\cdot M$ is open in G. Let Γ_0 be a compact open subgroup of Γ. Then $\Gamma_0\cdot M$ is a compact open subgroup of G. Then Γ_0^*—the dual group of Γ_0 is a discrete p-primary group. Also Γ_0^* is reduced or has no divisible subgroup. For if it has, then being p-primary it has subgroups of the form T_p and Γ_0 will then have quotients of the form $\mathbb{Z}_p$ (for $\mathbb{Z}_p$ is the character group of T_p). But such a quotient gives rise to a homomorphism of Γ_0 into $\mathbb{Q}_p$. This can be extended first to $\Gamma_0 M$, since $\Gamma_0 \cap M = \langle 0 \rangle$, and then to G, since $\Gamma_0 M$ is open in G. But this contradicts that $\Gamma_0 \subset \Gamma = \ker \hom(G, \mathbb{Q}_p)$. Thus Γ_0^* is reduced and so has a cyclic direct summand from a general theorem on abelian groups (see [K], page 21, Theorem 9).

Clearly then Γ_0 also has a cyclic direct summand $\Gamma_0 = C \cdot \Gamma_1$, and the argument can be repeated for Γ_1. But the number of cyclic direct summands arising in this way has to be finite since $\{x \in \Gamma_0 : p \cdot x = 0\}$ is finite. Thus Γ_0 itself is finite. This implies that Γ is discrete, since Γ_0 is open. Clearly Γ_d—the maximal divisible subgroup of Γ, is a direct summand and is $\simeq T_p^c$, c being finite integer, since elements of order p in Γ are finite. Thus $\Gamma = \Gamma_d \cdot B$. The subgroup B is then finite by a similar argument.

Finally note that if $L \in \mathscr{L}$ is p-primary and $L_0 \subset L$ is closed subgroup then L_0 is a direct summand either when $L/L_0 \simeq \mathbb{Z}_p^b$ for some finite b or when L_0 is discrete and $\simeq T_p^b$. Here the second part comes out of duality. Applying this to G, we get $G \simeq T_p^c \times \mathbb{Z}_p^b \cdot H$, where H is a closed subgroup, with the property that $B \subset H$, $H/B \simeq \mathbb{Q}_p^a$ and B is finite. If L is the annihilator of B in H^*, then $L \simeq (\mathbb{Q}_p^a)^* \simeq \mathbb{Q}_p^a$ and $H^*/L \simeq B^* \simeq B$. Thus L is open. An open divisible subgroup is a direct summand and thus $H^* \simeq L \cdot B^*$ or $H \simeq \mathbb{Q}_p^a \cdot B$. □

COROLLARY 11

If $G \in \mathscr{C}_p$, so does G^*.

This is clear since character groups of the form $\mathbb{Q}_p^a \times \mathbb{Z}_p^b \times T_p^c \times B_p$ are again of the same form. Also any group of the above form belongs to $\mathscr{C}_p$.

Putting together these results one can describe the structure of the class $\mathscr{L}_0$ as follows. (See Theorem 1 in § 1 for definition of the class $\mathscr{L}_0$).

PROPOSITION 12

If $G \in \mathscr{L}_0$, so does its dual G^*. Moreover there exist closed subgroups G_∞, G_p, such that $G = G_\infty \cdot \Pi_p G_p$—a restricted direct product with respect to $\{K_p\}$, where K_p is a compact open subgroup of G_p. Also the group G_∞ is isomorphics to a group of the form $R^m \cdot T^n \cdot \mathbb{Z}^{k'}$ and $G_p \in \mathscr{C}_p$ for all p.

3. Symplectic spaces (X, ω)

Let $\omega(\cdot,\cdot)$ be u continuous alternating bicharacter of X, $X \in \mathscr{L}$. The alternating property viz: $\omega(x,x) = 1$ for all $x \in X$ implies that $\omega(x,y) = \omega(y,x)^{-1}$, for all x, $y \in X$. If $\omega_x : y \to \omega(x,y)$ is the character of X defined by x, then nondegeneracy is equivalent to the statement that $x \to \omega_x$ is an isomorphism of X with its dual X^*. Thus, in what follows we consider symplectic spaces (X, ω), $X \in \mathscr{L}$, ω nondegenerate, and identify the dual X^* with X itself.

Lemma 13. *Let (X, ω) be symplectic, i.e., $X \in \mathscr{L}$ and ω is nondegenerate. Let Z, U be closed subgroups of X such that* (i) *$Z \subseteq Z_* =$ Annihilator of Z in $X = \{y \in X : \omega(x,y) = 1$*

for all $x \in Z\}$ *and* (ii) $X = (Z_*)\cdot U$ *direct sum. Then* $X_1 = Z\cdot U$ *is closed; and if* $X_2 = (X_1)_*$, *then* $X = X_1\cdot X_2$ *direct sum. Note this gives a decomposition of* X *as orthogonal direct sum of symplectic spaces* (X_j, ω).

Proof. Since ω is nondegenerate, X/Z_* is isomorphic to the character group of Z. In particular the map $x, u \to \omega(x, u)$ is a dual pairing of Z and U. Let $H = U_* \cap Z_*$. Then H is closed. Let $x_0 \in Z_*$ be arbitrary. Consider the character $\omega_{x_0}|U$. Then there exists an element $z_0 \in Z$, such that $w_{x_0} = \omega_{z_0}$ on U. Thus $x_0 = z_0\cdot h$, for some $h \in H$. Since we are working within the class of second countable groups, it follows that $Z_* = Z\cdot H$ direct sum and thus $X = Z\cdot H\cdot U$ direct sum. This implies that $X_1 = Z\cdot U$ is closed, and if we write $X_2 = H$, then $X = X_1\cdot X_2$ is an orthogonal symplectic decomposition.

PROPOSITION 14.

Let (X, ω) *be symplectic. Assume that the maximal compact connected subgroup of* X *is a torus. Then there exists an orthogonal symplectic decomposition* $X = X_1\cdot X_2\cdot X_3$ *(i.e.,* $\omega(X_i, X_j) = 1$ *for* $i \neq j$ *and* (X_j, ω) *are nondegenerate) such that* $X_1 \simeq R^{2n}$, $X_2 \simeq T^k \times \mathbb{Z}^k$ *and* X_3 *is totally disconnected or* $X_3 \in \mathscr{C}$.

Proof. Let K be the maximal compact connected subgroup of X. Then $K = ({}^0X \cap X^0)$ and $K_* = ({}^0X\cdot X^0)$. Also X/K_* is isomorphic to the character group of K and so $\simeq \mathbb{Z}^k$. Thus K_* is direct summand. Or there exists a closed subgroup D, $\simeq \mathbb{Z}^k$ such that $X \simeq K_*\cdot D$. Thus by Lemma 13, $X = X_1\cdot Y$ with $X_1 = K\cdot D$ and $Y = (X_1)_*$. Since (Y, ω) is nondegenerate and Y has no compact connected subgroups, it follows that $Y = Y^0({}^0Y)$ direct sum, $Y^0 \simeq R^m$ and ${}^0Y$ is totally disconnected. Since the character group of ${}^0Y$ is also totally disconnected it follows that $\omega(Y^0, {}^0Y) = 1$. Thus if $X_2 = Y^0$ and $X_3 = {}^0Y$, $X = X_1\cdot X_2\cdot X_3$ is the required decomposition.

PROPOSITION 15.

Let (X, ω) *be nondegenerate. Assume that* X *is totally disconnected. Let* X_p *be the p-primary component of* X. *Then* $\omega(X_p, X_q) = 1$ *where* $p \neq q$, *and* (X_p, ω) *is nondegenerate. If* K *is any compact open subgroup of* X, *such that* $\omega(K, K) = 1$, *then* $X = \Pi_p X_p$—*a restricted direct product with respect to* $\{K_p\}$ *is also an orthogonal symplectic decomposition of* X.

Proof. Note that if χ is a continuous character of a p-primary group G, then $\chi(G) \subset T_p$. Thus if $x \in X_p$ and $y \in X_q$, $p \neq q$, then $\omega(x, y) \in T_p \cap T_q = \langle 1 \rangle$. Thus $\omega(X_p, X_q) = 1$. Next let U be a neighborhood of the identity in T, such that U contains no subgroups other than the trivial one. Consider the open subset $W = \{(x, y) \in X \times X : \omega(x, y) \in U\}$. Since compact open subgroups form a basis of neighborhoods of the identity in X, there exist a compact open subgroup K, such that $K \times K \subset W$. Thus $\omega(K, K) = 1$. The rest follows easily (see Lemma 9).

PROPOSITION 16.

Let (X, ω) *be nondegenerate with* $X \in \mathscr{C}_p$. *Then there exists an orthogonal symplectic decomposition* $X = X_1\cdot X_2\cdot X_3$ *(direct sum) such that* $X_1 \simeq T_p^n \times \mathbb{Z}_p^n$, $X_2 = \mathbb{Q}_p^m$ *and* X_3 *is finite.*

Proof. Since $X \in \mathscr{C}_p$, let $X = H_1 \cdot H_2 \cdot H_3 \cdot H_4$, where $H_1 \simeq \mathbb{Q}_p^m$, $H_2 \cong T_p^n$, $H_3 \simeq \mathbb{Z}_p^k$, H_4 finite. Then $H_2 H_4$ is the closed torsion subgroup X_0 of X and H_2 is the divisible part of X_0. Since $X \simeq X^*$ and the divisible part of the torsion subgroup of $X^* \simeq H_3^* \simeq T_p^k$. Thus $T_p^n \simeq T_p^k$ and so $n = k$. Next since $H_1 H_2$ is divisible, while $H_2 H_4$ is torsion, it follows that $\omega(H_1 H_2, H_2 H_4) = 1$. Thus $(H_2)_* = H_1 \cdot H_2 \cdot D \cdot H_4$, for some closed subgroup D of H_3. So $X/(H_2)_* \simeq H_3/D$ and on the other hand $\simeq H_2^* \simeq \mathbb{Z}_p^n \simeq H_3$. Thus $H_3 \simeq H_3/D$, forces $D = \langle 1 \rangle$, or $(H_2)_* = H_1 H_2 H_4$. If you now use Lemma 13, it follows that $X = X_1 \cdot Y$ is an orthogonal symplectic decomposition with $X_1 = H_2 H_3$ and $Y = (X_1)_*$. Clearly $Y \simeq X/X_1 \simeq H_1 H_4$. Thus Y_d—the maximal divisible subgroup of $Y \simeq H_1 \simeq \mathbb{Q}_p^m$ and the torsion subgroup Y_0 of Y is $\simeq H_4$. Clearly $\omega(Y_d, Y_0) = 1$, and $Y = Y_d \cdot Y_0$. Thus if we take $X_2 = Y_d, X_3 = Y_0$, then $X = X_1 \cdot X_2 \cdot X_3$ is an orthogonal symplectic decomposition with the required properties.

As a final step in the proof of Theorem 1 we need the following.

Lemma 17. *Let* (X, ω) *be nondegenerate. In each of the following cases* (i) $X \simeq R^m$, (ii) $X \simeq T^n \times \mathbb{Z}^n$, (iii) $X \simeq \mathbb{Q}_p^m$, (iv) $X \simeq T_p^n \times \mathbb{Z}_p^n$ *and* (v) X *is finite,* (X, ω) *admits a polarization. Moreover the polarizing groups* G, G^* *are of some type as* X.

Proof. In cases (i) and (iii), $\omega(x, y) = \chi(\Omega(x, y))$ for some nondegenerate symplectic bilinear form on $X (= R^m$ or $\mathbb{Q}_p^m)$, where χ is a non-trivial character on R or $\mathbb{Q}_p$. Existence of symplectic bases for Ω gives polarizations for X. The case (v) is a classical result of Frobenius. The cases (ii) and (iv) are proved similarly. We sketch the argument for (ii). Note $X^0 \simeq T^n$. Choose a closed subgroup $S_1 \subset X^0$, $S_1 \simeq T$. Then $X/(S_1)_* \simeq \mathbb{Z}$. Thus $X = (S_1)_* \cdot D_1$, for some closed subgroup $D_1 \simeq \mathbb{Z}$. By Lemma 13, $X = X_1 \cdot X_2$ is an orthogonal symplectic decomposition, $X_1 = S_1 \cdot D_1$, the subgroups S_1, D_1 defining a polarization for X_1. Note X_2 is again of the same type and so induction works. The case (iv) is handled similarly.

Remark. Although the class of groups $\mathscr{L}_0$-includes most of the examples arising in applications, it is not known whether, for a general $X \in \mathscr{L}$, a nondegenerate symplectic structure always admits a polarization.

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