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Reduction theory over global fields

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Dedicated to the memory of Professor K G Ramanathan

Abstract. The paper contains an exposition of the basic results on reduction theory in reductive groups over global fields, in the adelic language. The treatment is uniform: number fields and function fields are on an equal footing.

Keywords. Reduction theory; global fields; number fields; function fields.

Introduction

The basic results on reduction theory for a linear algebraic group G over the field of rational numbers were established by Borel and Harish–Chandra in [3]. One of these results is the construction of a fundamental set for an arithmetic subgroup Γ of the real Lie group $G(\mathbf{R})$. For another one, the criterion for compactness of the quotient $G(\mathbf{R})/\Gamma$, a more direct method of proof was given by Mostow and Tamagawa [8]. Godement and Weil [5] showed that this method can also be used to obtain fundamental sets. They used the language of adeles.

Reduction theory for linear algebraic groups over number fields is reduced to groups over $\mathbf{Q}$ by restriction of the ground field. For groups over global fields of positive characteristic, i.e. function fields of dimension one over a finite field of constants, the method of Mostow and Tamagawa can also be used, but only under some restrictions on the characteristic (see [1]). Using another method, involving the study of semi-simple group schemes over complete curves, Harder [6] proved the basic results over function fields without restrictions on the characteristic.

Some 25 years ago, in unpublished seminar notes, I tried to give a uniform treatment of the reduction theory over global fields, by the method of [5], also using Harder's idea to employ Galois descent. This attempt was not successful; there was a gap in the notes. However, they contain a proof of the compactness theorem.

In the meantime, no uniform treatment of the basic results on reduction theory seems to have appeared in the literature. The present note, which is to a large extent expository, attempts to give such a treatment. The method is essentially that of the old notes. But I have abandoned the method of Mostow and Tamagawa altogether. Galois descent is used instead.

In applying the method of [8] (and its extension in [5]) one encounters a somewhat subtle question. This method seems to involve an application of the following strong version of the Hilbert–Mumford theorem. Let G be a reductive group over a field k, acting linearly in a vector space V, everything being defined over k. Let $\xi \in V(k)$ be a non-zero instable vector. Then there is a cocharacter of G which is defined over k such that ξ is instable for the corresponding $\mathbf{G}_m$-action.

One knows that this result holds if k is perfect. But there are counter examples for non-perfect fields (see [7, 5, 6]). This might explain that application of the method of Mostow and Tamagawa in the case of arbitrary function fields has not been successful.

1. Preliminaries

1.1. In the sequel k denotes a global field, i.e. either a finite extension of $\mathbf{Q}$ or a function field of dimension one with a finite field of constants. In the latter case k is a finite separable extension of a purely transcendental extension k_0 of a finite field, of transcendence degree one. In the first case we put $k_0 = \mathbf{Q}$.

Let G be a k-group, i.e. a linear algebraic group which is defined over k. We refer to [2] for the theory of linear algebraic groups. Denote by $G(k)$ the group of k-rational points of G and by $G(A, k)$ or $G(A)$ the corresponding adele group. It is a locally compact group, containing $G(k)$ as a discrete subgroup (see [10, Ch. 1] for the basic results on adeles).

We denote by $G(A, k)^0$ or $G(A)^0$ the closed subgroup of $G(A, k)$ consisting of the adeles g such that for each rational character χ of G which is defined over k the idele norm $|\chi(g)|$ equals one. Then $G(k)$ is a subgroup of $G(A, k)^0$. If G is the multiplicative group then $G(A, k)$ is the group $I(k)$ of ideles and $G(A, k)^0$ is the group $I(k)^0$ of ideles of norm one. We denote by $C(G, k)$ or $C(G)$ the quotient space $G(A, k)^0/G(k)$. It is well-known that $I(k)^0/k^*$ is compact.

If $\phi: G \to H$ is a homomorphism of k-groups we denote by $\tilde{\phi}$ the induced homomorphism of topological spaces $C(G, k) \to C(H, k)$.

1.2. We now review some auxiliary results. Let H be a closed k-subgroup of G and let i be the injection $H \to G$. We say that a triple (V, ρ, ξ) is a *k-representation of G adapted to H* if V is a vector space over k (in the sense of algebraic geometry), $\rho: G \to GL(V)$ is a k-representation of G and $\xi \in V(k)$ is a non-zero vector such that H is the stabilizer of the line L through ξ for the G-action in V defined by ρ. It is known that such a triple exists. We may assume that ρ induces a k-isomorphism of the quotient space G/H onto the G-orbit of L in the projective space $\mathbf{P}(V)$ (see [2, 5.1, 6.8]).

1.3. *Lemma. $\tilde{\imath}$ is a homeomorphism of $C(H, k)$ onto a closed subspace of $C(G, k)$.*

It is clear that $\tilde{\imath}$ is an injective continuous map. Let (V, ρ, ξ) be as above. Then $H(A)^0.G(k)$ is the inverse image of $I(k)^0\rho(G(k)).\xi$ under the continuous map $g \mapsto \rho(g)^{-1}v$ of $G(A)$ to $V(A)$. It suffices to prove that the induced map $i': X/H(A)^0 \cdot (G(k)/G(k)) \to G(A)/G(k)$ is a homeomorphism onto a closed subspace. Since $I(k)^0$ is the product of k^* and a compact set and since $k^*\rho(G(k))v$ is discrete in $V(A)$ we have that $I(k)^0\rho(G(k)).\xi$ is closed. Hence $H(A)^0.G(k)$ is closed in $G(A)$, and the image of i' is closed. Since i' is open (see for example [10, p. 28–29]) it is a homeomorphism.

1.4. *Restriction of the ground field*

Let l be a finite separable extension of k. If G is an l-group denote by $H = \Pi_{l/k} G$ the k-group obtained from G by restricting the ground field to k ([10, 1.3]). Denote by

$\phi_G: H \to G$ the canonical l-morphism. If G is a k-group we also have a k-homomorphism $\psi_G: G \to H$ with $\phi_G \circ \psi_G = id$. We have an isomorphism of topological groups

$$H(A,k) \to G(A,l), \tag{1}$$

inducing an isomorphism $H(k) \simeq G(l)$.

If G is a k-group, the composite of ψ_G and the isomorphism (1) is the canonical injection $G(A,k) \to G(A,l)$.

1.5. *Lemma. Let l be a finite separable extension of k. The canonical injection $C(G,k) \to C(G,l)$ is a homeomorphism onto a closed subspace.*

This follows from the preceding observations and 1.3.

Let H be a normal k-subgroup of the k-group G and let $\pi: G \to G/H$ be the canonical homomorphism. A *section over k* for π is a k-morphism $\sigma: G/H \to G$ such that $\pi \circ \sigma = id$. Then $G(l) \to (G/H)(l)$ is surjective for any k-algebra l.

1.6. *Lemma. Assume that σ is a section over k.*

(i) *The canonical homomorphism $G(A) \to (G/H)(A)$ is surjective;*

(ii) *Assume that the group of k-characters of H is trivial and that both $C(H)$ and $C(G/H)$ are compact. Then $C(G)$ is compact.*

We skip the easy proof of (i). In the situation of (ii) let K and K' be compact sets in $(G/H)(A)$ and $H(A)$ such that $(G/H)(A) = K.(G/H)(k)$ and $H(A) = K'.H(k)$, respectively. Then $G(A)^0$ is a closed subset of $\sigma(K).K'.G(k)$, and the assertion follows.

A special case where sections exist, is when G is the semi-direct product over k of H and a k-subgroup L. In that case $G(A,k)$ is the semi-direct product of $H(A,k)$ and $L(A,k)$.

1.7. PROPOSITION

Let G be a connected solvable k-group which is split over k. Then $C(G,k)$ is compact.

For split groups see [2, § 15]. The proposition is well-known if $G = \mathbf{G}_a$ or $\mathbf{G}_m$. In the general case G is the semi-direct product of a k-split maximal torus and its unipotent radical, which is also k-split. By the lemma the proof is reduced to the case that G is either a torus or a unipotent group. The first case reduces to the case of $\mathbf{G}_m$. In the second case we have a normal sequence of connected split k-subgroups such that the successive quotients are all k-isomorphic to $\mathbf{G}_a$. Since for any connected normal k-subgroup of our group G sections exist (by a result of Rosenlicht, see [9, Th. 1]), the lemma reduces this case to $\mathbf{G}_a$.

1.8. *Heights*

Let V be a finite dimensional vector space over k, in the sense of algebraic geometry. Denote by $V(A)$ the corresponding adele space and by $GL(V,A)$ the group of invertible automorphisms of the A-module $V(A)$. This group is isomorphic to the adele group $GL(V)(A)$. We transport the structure of topological group of the latter group to $GL(V,A)$.

We say that $x \in V(A)$ is *primitive* if there is $g \in GL(V,A)$ such that $g.x$ is a non-zero element of $V(k)$. For all places v of k we choose on $V_v = V(k_v)$ a norm $\| \; \|_v$ compatible with the absolute value on k_v and such that for almost v we have that for $x \in V(k_v)$ the norm $\|x\|_v$ equals the maximum of the absolute values of x with respect to a fixed basis of $V(k)$ (the same then holds for any other basis, for almost all v). For $x \in V(A)$ primitive we put

$$\|x\| = \prod_v \|x_v\|_v. \tag{2}$$

We call such a function on the set of primitive elements of $V(A)$ a *height*.

We list some properties of a height $\| \ \|$.

(a) *for all ideles t and all primitive* $x \in V(A)$ *we have* $\|t.x\| = |t| \, \|x\|$;

(b) *if* $\| \ \|'$ *is another height, the ratio* $\|x\|^{-1} \|x\|'$ *lies in a fixed compact subset of* $\mathbf{R}_+^*$, *where* $x \in V(A)$ *is primitive*;

(c) *if* (x_n) *is a sequence of primitive vectors which converges to* 0 *in* $V(A)$ *then* $\|x_n\|$ *tends to* 0 *in* $\mathbf{R}$;

(d) *If* (x_n) *is a sequence of primitive vectors such that* $\|x_n\|$ *tends to* 0 *in* $\mathbf{R}$ *then there exist* λ_n *in k such that the sequence* $(\lambda_n x_n)$ *converges to* 0 *in* $V(A)$.

This is well-known (see [5, 1.1]). (a) and (b) are easy. To prove (c) it suffices to consider the case that in (2) we have for *all* places v that $\|x_v\|$ is the maximum of the absolute values of coordinates with respect to a given basis of $V(k)$. Let

$$K = \{x \in V(A) | \, \|x_v\| = 1 \text{ for all } v\}.$$

This is a compact set and for any primitive x there is an idele t such that $t.x \in K$. Using this fact, the proof of (c) is straightforward and the proof of (d) reduces to the case that V has dimension one, which is well-known.

1.9. *Reduction theory for GL(2)*

We now take $k = k_0$. Let V be the standard 2-dimensional vector space. So $V(k) = k^2$, $V(A) = A^2$, $GL(V, A) = GL(2, A)$, where $A = A(k_0)$. We use a particular height. If $k_0 = \mathbf{Q}$ we define for a finite place $\|x_v\|$ to be the maximum of the absolute values of the coordinates with respect to the canonical basis (e_i). For the infinite place v of $\mathbf{Q}$ it is the Euclidean length of x_v. If k_0 is a function field we take $\|x_v\|$ to be the maximum of the absolute values of these coordinates, for all places.

For all places v the subgroup M_v of $GL(2, k_v)$ preserving $\| \ \|$ is compact. So $M = \Pi_v M_v$ is a compact subgroup of $GL(2, A)$. It is well-known that any $g \in GL(2,A)$ can be written in the form $g = m.t$, with $m \in M$ and t upper triangular. Denote the first and last diagonal ideles of such a t by t_1, t_2. For any $c > 0$ let $T(c)$ be the set of upper triangular elements t in $GL(2, A)$ with $|t_1/t_2| \leqslant c$. The next result goes back to Gauss.

1.10 PROPOSITION

There exists a constant $c > 0$ *such that* $GL(2, A) = M.T(c).GL(2, k)$.

The proof will show that we may take $c = 2/\sqrt{3}$. Let $g \in GL(2, A)$. We have to find $\gamma \in GL(2, k)$ such that $g\gamma \in M.T(c)$. Since $V(k)$ is discrete in $V(A)$ it follows from property (d) of heights that the set of numbers $\|g.\xi\|$ where ξ runs through the non-zero vectors of $V(k)$, is bounded away from zero. We may therefore assume that $\|g.e_1\| \leqslant \|g.\xi\|$ for all such ξ. Put $g = m.t$, as before. Then $g.e_1 = t_1$ and for all $\lambda, \mu \in k$, not both zero, and $u \in A$ we have

$$|t_1| \leqslant \|(\lambda + \mu u)t_1 e_1 + \mu t_2 e_2\|.$$

Put $x = t_1/t_2$. Then

$$|x| \leqslant \|(u+v)xe_1 + e_2\|, \tag{3}$$

for all $v \in k$.

If $k = \mathbf{Q}$ we multiply x by an element of k^* (modifying u and v) such as to obtain an idele whose components at the finite places are all 1. Then take $v \in k$ such that $\|u_v + v\| \leqslant 1$ for all finite v and $\leqslant 1/2$ at the infinite place. Now (3) gives

$$|x| \leqslant \sqrt{(1 + |x|^2/4)},$$

whence $|x| \leqslant 2/\sqrt{3}$.

If k is a field of rational functions in one indeterminate over a finite field with q elements we take as infinite place the obvious one. Proceeding as before we see that we can now find v such that even $\|u_v + \mu\| \leqslant q^{-1}$ at the infinite place. The inequality (3) gives $|x| \leqslant 1$. We have a bound as required for $t_1 t_2^{-1}$. The same argument works for $SL(2)$ and gives the following.

1.11. COROLLARY.

There exists a compact subgroup M' of $SL(2, A)$ and a constant $c > 0$) such that $SL(2, A) = M'.(T(c) \cap SL(2, A)).SL(2, k)$.

2. Reduction theory

2.1. *Statement of the main results*

We now assume that G is a connected reductive k-group. We fix a minimal parabolic k-subgroup P of G. Let U be its unipotent radical. It is a k-split unipotent group. We also fix a maximal k-split torus S of G which lies in P. The centralizer L of S is a k-Levi group of P and P is the semi-direct product over k of L and U. We denote by R the root system of (G, S), by R^+ the set of positive roots defined by P and by Δ the basis of R defined by R^+. For the basic facts on reductive k-groups we refer to [4].

The homogeneous space G/P is a projective k-variety. It follows from [loc.cit., 4.13] that the canonical map $G(A)/P(A) \to (G/P)(A)$ is a homeomorphism. It follows that $G(A)/P(A)$ is compact.

Let $X(P)$ be the group of k-characters of P. We have a homomorphism $P(A) \to \mathrm{Hom}(X(P), \mathbf{R}_+^*)$ whose kernel if $P(A)^0$, and there is a similar homomorphism for S. One knows that restriction of characters identifies $X(P)$ with a subgroup of finite index of $X(S)$. It then readily follows that there is a finite subset F of $P(A)$ such that $P(A) = F.S(A).P(A)^0$.

We conclude that there is a compact subset K of $G(A)$ such that

$$G(A)^0 = K.(S(A) \cap G(A)^0).P(A)^0.$$

If c is a strictly positive constant we define $S(c)$ to be the set of $s \in (S(A) \cap G(A)^0)$ such that $|\alpha(s)| \leqslant c$ all $\alpha \in \Delta$ and we put $\mathscr{S}(c) = K.S(c).P(A)^0$. The following statements are the main results of reduction theory.

(C) *(Compactness theorem) $C(G)$ is compact if and only if G is anisotropic (i.e. $P = G$).*
(F) *(Fundamental set theorem) If G is isotropic there is c with $G(A)^0 = \mathscr{S}(c).G(k)$.*

2.2. *Remarks.* (a) Since the Levi group L of P is anisotropic and the unipotent radical U is k-split, it follows from the compactness theorem that there is a compact set K_1 in $L(A)^0$ such that $L(A)^0 = K_1.L(k)$. The fundamental set theorem then implies that $G(A)^0 = K.K_1.S.U(A)G(k)$. Using 1.7 we conclude that there are compact sets $K' \subset G(A)^0$ and $K'' \subset U(A)$ with $G(A)^0 = K'.T(c).K''.G(k)$.
(b) Let V be a vector space over k such that G is a closed k-subgroup of $GL(V)$. Assume that $C(G)$ is compact and let K_0 be a compact subset of $G(A)^0$ such that $G(A)^0 = K_0.G(k)$. Choose an open neigbourhood U of 0 in $V(A)$ such that $U \cap V(k) = \{0\}$ and that $K_0^{-1}.U \subset U$. Then $G(A)^0.V(k) \cap U = \{0\}$. It follows that 0 is an isolated point of $G(A)^0.V(k)$. On the other hand, if G is isotropic over k there exists a non-trivial k-split subtorus S of the commutator subgroup of G. Then $S(A)$ is a subgroup of $G(A)^0$. Let $\xi \in V(k)$ of S be a weight vector of S whose weight χ cannot be extended to a character of G and choose a sequence (s_n) in $S(A)$ such that $\chi(s_n)$ converges to zero in A. Then $(s_n.\xi)$ converges to 0 in $V(A)$. We conclude that G is anisotropic if $C(G)$ is compact. Then proof of the converse statement is the crucial part of the proof of the compactness theorem.

2.3. *Auxiliary results*

The parabolic k-subgroups of G containing P are parametrized by the subsets of Δ. If $\Pi \subset \Delta$ we denote by P_Π the parabolic k-subgroup containing P such that the root system of the Levi group of P_Π which contains S has basis Π (so $P_\varnothing = P$).

We number the elements of Δ, say $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. For $i \in [0, r]$ put

$$P_i = P_{\{\alpha_{i+1}, \ldots, \alpha_r\}}.$$

So $P_0 = G$, $P_r = P$. For $i \in [1, r]$ let (ρ_i, V_i, ξ_i) be a representation adapted to P_i. Notice that for $j \geqslant i$ we have that the restriction of ρ_j to L_i is adapted to the parabolic k-subgroup $L_i \cap P_j$ of L_i.

We fix a height $\| \, \|_i$ on $V_i(A)$. For $c > 0$ we put

$$\mathscr{R}(c) = \{g \in G(A)^0 \mid \|\rho_i(g)\xi_i\|_i \leqslant c \|\rho_i(g\gamma)\xi_i\|_i \text{ for } i \in [1, r], \gamma \in P_i(k)\}.$$

It follows from property (d) of heights that $G(A)^0 = \mathscr{R}(1).G(k)$.

The next lemma gives a reduction to rank one, following [5, 9.3].

2.4. *Lemma. Assume that (F) holds for the k-groups $L_i (1 \leqslant i \leqslant r-1)$.*

(i) *There is c' such that $\mathscr{R}(c) \subset \mathscr{S}(c')$;*
(ii) *(F) holds for G.*

(ii) follows from (i), by the remark we just made. We prove (i) by induction on the rank r. If $r = 1$ then ξ_1 is a weight vector for S whose weight is a rational multiple of the only simple root α. It follows from (F) that there is a constant c_0 such that for $g \in G(A)^0$ there is $\gamma \in G(k)$ with $g\gamma = xsy$, where $x \in K$, $s \in S(A) \cap G(A)^0$, $y \in P(A)^0$ and

$|\alpha(s)| \leqslant c_0$. Hence $\inf_{\gamma\in G(k)} \|\rho_1(g).\xi_1\|_1 \leqslant c_0$. It follows that if $g = xsy \in \mathscr{R}(c)$ we have $\|g.\xi_1\|_1 \leqslant c_0$, from which one concludes that $|\alpha(s)|$ must be bounded by a constant.

Now let $r > 1$ and take $g = xsy \in \mathscr{R}(c)$. We can write $y = y_1 u_1$, where $y_1 \in L_1(A)^0$, $u_1 \in U_1(A)$ (U_1 denoting the unipotent radical of P_1). It is immediate that sy_1 lies in a set like $\mathscr{R}(c)$ for the group L_1. By induction it follows that $|\alpha_i(s)|$ is bounded by a constant for $i > 1$. A similar argument, using the rank one group L_{r-1} gives a bound for $|\alpha_1(s)|$ and (i) follows.

2.5. For $\alpha \in \Delta$ we denote by $(\rho_\alpha, V_\alpha, \xi_\alpha)$ a representation adapted to the maximal parabolic subgroup $P(\alpha) = P_{\Delta-\{\alpha\}}$. Then ξ_α is a weight vector for S whose weight χ_α is a strictly positive multiple of α.

We denote by W the Weyl group $N_G(S)/Z_G(S)$. It is the Weyl group of the root system R. If $\Pi \subset \Delta$ denote by W_Π the Weyl group of the Levi group of P_Π containing S, relative to S. Then W_Π is a parabolic subgroup of W. For $\alpha \in \Pi$ we put $W(\alpha) = W_{\Delta-\{\alpha\}}$.

For $w \in W$ denote by $\dot{w}$ a representative lying in the group of rational points $N_G(S)(k)$. By Bruhat's lemma we have $G(k) = \cup_{w\in W} U(k)\dot{w}P(k)$. Fix $\gamma \in G(k)$ and write $\gamma = \mu\dot{w}v$ where $\mu \in U(k)$, $w \in W$, $v \in P(k)$. Put $\mathscr{S}'(c') = K'.S(c').P(A)^0$, where K' is another compact set and $c' > 0$. Let $g = xsy = x's'y'\gamma$ with $x \in K$, $x' \in K'$, $s, s' \in S(A) \cap G(A)^0$, $y, y' \in P(A)^0$, be an element of $\mathscr{S}(c) \cap \mathscr{S}'(c')\gamma$.

2.6. PROPOSITION

Assume that $C(L)$ is compact. Let $\alpha \in \Delta$.

If the set of positive numbers $|\alpha(s)|$, where $g = xsy = x's'y'\gamma$ runs through $\mathscr{S}(c) \cap \mathscr{S}'(c')\gamma$, is not bounded away from zero then $\gamma \in P(\alpha)$.

This is a variant of [5, lemme 3]. Since $C(L)$ is compact we may assume (changing K and K') that y and y' lie in $U(A)$. Then $g' = sy(\gamma)^{-1}(s'y')^{-1}$ lies in the compact set $K^{-1}.K'$. Let $\beta \in \Delta$ and fix a height $\| \ \|$ on $V_\beta(A)$. We have

$$\|\rho_\beta(g').\xi_\beta\| = |\chi_\beta(s')^{-1}| \cdot |(\chi_\beta(w.s)| \, \|syv^{-1}s^{-1}\dot{w}^{-1}.\xi_\beta\|.$$

Since $C(U, k)$ is compact it follows from properties (c) and (d) of heights, using that ξ_β is a positive multiple of β, that $|\beta((s')^{-1}(w.s))|$ lies in a compact set of $\mathbf{R}_+^*$. We conclude that there is a constant d such that for all $\beta \in \Delta$ we have

$$|(w^{-1}.\beta)(s)| \leqslant d|\beta(s')| \leqslant c'd. \tag{4}$$

Now assume that $w \notin W(\alpha)$. Then there is a positive root δ such that $w^{-1}.\delta = -\Sigma_{\beta\in\Delta} n_\beta \beta$, with $n_\alpha > 0$. It follows from (4) that there is a constant e such that

$$|(w^{-1}.\delta)(s)| \leqslant e.$$

On the other hand we have

$$|w^{-1}.\delta(s)| = \prod_{\beta\in\Delta} |\beta(s)^{-n_\beta}| \geqslant |\alpha(s)|^{-n_\alpha} \prod_{\beta\neq\alpha} c^{-n_\beta}.$$

Since the numbers n_β are bounded the last two inequalities imply that $|\alpha(s)|$ is bounded below by a strictly positive constant. So if $|\alpha(s)|$ is not bounded away from zero, we must have $w \in W(\alpha)$ and $\gamma \in P(\alpha)$, proving the proposition.

3. Proof of the main results

3.1. *A reduction*

The field k is a finite separable extension of a subfield k_0 which is either $\mathbf{Q}$ or a field of rational functions in one variable over a finite field. Let $H = \Pi_{k/k_0} G$ be the k_0-group obtained by restriction of the ground field. We use the notations of 1.4. We have the following facts.

(a) *ϕ_G induces isomorphisms $H(k_0) \simeq G(k)$, $H(A, k_0) \simeq G(A, k)$.*
See 1.4. We denote these isomorphisms also by ϕ_G.
(b) *ϕ_G induces an isomorphism $H(A, k_0)^0 \simeq G(A, k)^0$.*
For the (easy) proof see [1, p. 14].
The maximal k-split torus S is a k_0-group. The composite of ψ_S and the canonical morphism $\Pi_{k/k_0} S \to H$ is a k_0-homomorphism $\mu: S \to H$.
(c) *$S' = \mu S$ is a maximal k_0-split torus of H and there is a bijection of R onto the root system of (H, S').*
This is straightforward.
(d) *$P' = \Pi_{k/k_0} P$ is a minimal parabolic subgroup over k_0 in H and*

$$\phi_G(P'(A, k_0)^0) = P(A, k)^0.$$

(e) *For $c > 0$ there exists c' such that $\phi_G(S'(c)) \subset S(c')$.*

See [4, no. 6] and [1, p. 14] for (d) and (e). The facts just stated imply that it suffices to prove the statments (*C*) and (*F*) in the case that $k = k_0$, which we assume from now on.

3.2. *Split groups*

We first prove (*F*) in the case that G is split over $k (= k_0)$, i.e. that S is a maximal torus of G. In that case P is a Borel group, which is a k-split connected solvable group. By 1.7 we know that $C(L)$ is compact. Also, the groups L_i of 2.3 are split over k. It then follows from 2.3 by an easy induction that property (*F*) for G is a consequence of the following lemma.

3.3. *Lemma. If G has semi-simple rank one and is split over k then (F) holds.*

A k-group G with these properties is k-isomorphic to a product $H \times T$, where T is a k-split torus and H is one of the groups $GL(2)$, $SL(2)$, $PGL(2)$. (This must be well-known, but as I do not know a reference a proof will be sketched below.) It suffices to prove the lemma for H. If $H = PGL(2)$ there is the obvious map $C(GL(2), k) \to C(H, k)$, which is surjective. (Notice that for any field l the canonical map $GL(2, l) \to PGL(2, l)$ is surjective.) A set $\mathscr{S}(c)$ for $GL(2)$ is mapped onto a similar set for H. So we may assume that $H = GL(2)$ or $SL(2)$ and then 1.10 and 1.11 establish the lemma.

The proof of the result on the structure of G uses the root datum of (G, S), say $(X, X^\vee, R, R^\vee)$. Here X is the character group of S, $R = \{\pm\alpha\}$ the root system, $X^\vee$ the dual of X and $R^\vee = \{\pm\alpha^\vee\}$ the dual root system. If $\alpha = 2\chi \in 2X$ then X is the direct sum of $\mathbf{Z}\chi$ and $(\alpha^\vee)^\perp$ and G is k-isomorphic to the direct product of SL_2 and a torus. Similarly, if $\alpha^\vee \in 2X^\vee$ then G is isomorphic to PGL_2 times a torus. If $\alpha \notin 2X$, $\alpha^\vee \notin 2X^\vee$ choose $\lambda \in X^\vee$ with $\langle\alpha, \lambda\rangle = 1$. Put $X_0^\vee = \mathbf{Z}\alpha^\vee + \mathbf{Z}\lambda$. Then X is the direct sum of $(X_0^\vee)^\perp$ and a rank two sublattice X_0 containing α and $X^\vee$ is the direct sum of $X_0^\vee$ and $X_0^\perp$. Then $X_0^\vee$ can be identified with the dual of X_0 and these two lattices are ingredients of a root datum for GL_2. It follows that G is k-isomorphic to the product of GL_2 and a torus.

3.4. Now assume that G is arbitrary. We fix a maximal torus T of G which is defined over k and contains S. We also fix a finite separable Galois extension l of k which splits T. Denote by Γ the Galois group of l over k. We denote by $\tilde{R}$ the root system of (G,T) and by $\tilde{W}=N_G(T)/T$ its Weyl group. If $w\in\tilde{W}$ we denote by $\dot{w}\in\tilde{W}$ a representative in $G(l)$. The roots of the relative root system R are the non-trivial restrictions to S of the roots of $\tilde{R}$. We fix a system of positive roots $\tilde{R}^+$ such that the roots in R^+ are restrictions of roots in $\tilde{R}^+$. Let $\tilde{\Delta}$ be the basis defined by $\tilde{R}^+$. If $\Pi\subset\tilde{\Delta}$ we denote by $\tilde{P}_\Pi$ the corresponding parabolic l-subgroup.

Since T is split over l, all characters of T are defined over l. Hence the Galois group Γ acts on $\tilde{R}$. Let $B\supset P$ be the Borel subgroup defined by $\tilde{R}^+$ and let $\tilde{U}$ be its unipotent radical. For $s\in\Gamma$ there is $w_s\in\tilde{W}$ such that $s.B=\mathrm{Int}(\dot{w}_s)B$. There is an action ι of Γ on $\tilde{\Delta}$ such that for $s\in\Gamma$, $\alpha\in\tilde{\Delta}$ we have $s.\alpha=w_s(L(s).\alpha)$. We then have $s.P_\Pi=\mathrm{Int}(\dot{w}_s)P_{\iota(s).\Pi}$ if $\Pi\subset\tilde{\Delta}$.

3.5. *Proof of* (*C*). As we remarked in 2.2 the burden of the proof of (*C*) is to show that $C(G,k)$ is compact if G is anisotropic. This we now assume. We identify $G(A,k)^0$ with a closed subgroup of $G(A,l)^0$. The Galois group Γ acts on the latter group, and $G(A,k)^0$ is the set of elements fixed by all of Γ. Since G is l-split, we know that (*C*) and (*F*) hold over l. Put $\mathscr{S}'(c)=K.T(c).\tilde{U}(A,l)$, where K is a compact set. We assume that $G(A,l)^0=\mathscr{S}'(c).G(l)$. Take $g\in G(A,k)^0$. There are $x\in K$, $t\in T(c)$, $u\in U(A,l)$, $\gamma\in\tilde{G}(l)$ with $g=xtu\gamma$. For all $s\in\Gamma$ we have $s(xtu\gamma)=xtu\gamma$. This can be rewritten as

$$((s.x)\dot{w}_s)(w_s^{-1}(s.t)).u'=xtu(\gamma(s.\gamma)^{-1}\dot{w}_s),$$

where $u'=\mathrm{Int}(\dot{w}_s^{-1})(s.u)\in U$. Let $\alpha\in\tilde{\Delta}$. It follows from 2.6 that there is a constant d such that if $|\alpha(t)|\leqslant d$ we have $\gamma(s.\gamma)^{-1}\dot{w}_s\in\tilde{P}(\alpha)$, for all $s\in\Gamma$. Then $s.\mathrm{Int}(\gamma^{-1})\tilde{P}(\iota(s)^{-1}.\alpha)=\mathrm{Int}((s.\gamma)^{-1}\dot{w}_s)\tilde{P}(\alpha)=\mathrm{Int}(\gamma^{-1})\tilde{P}(\alpha)$ and the proper parabolic subgroup

$$Q=\bigcap_{s\in\Gamma}\mathrm{Int}(\gamma^{-1})\tilde{P}(\iota(s).\alpha)$$

is Γ-stable, hence is defined over k. But this is impossible if G is anisotropic. It follows that for all $\alpha\in\tilde{\Delta}$ we have $|\alpha(t)|\geqslant d$. Since $U(A,l)/U(l)$ is compact we conclude that the image of $G(A,k)^0$ in $C(G,l)$ is relatively compact. By 1.2 we have that $C(G,k)$ is compact and (*C*) follows.

3.6. *Proof of* (*F*). Let G be arbitrary. By (*C*) we know now that $C(G,l)$ is compact. Application of 2.4 shows that it suffices to establish (*F*) in the case the G has semi-simple rank one. This we now assume. We proceed as in 3.5 and use the notations introduced there. In the present case, the parabolic k-subgroup Q must be k-conjugate to P. This means that, after multiplying g on the right by an element of $G(k)$, we may assume that $Q=P$. But it is known (see [4, no. 6]) that there is a unique Γ-orbit $\mathscr{O}$ in $\tilde{\Delta}$ such that P is the intersection of the $\tilde{P}(\alpha)$ with $\alpha\in\mathscr{O}$. The roots in $\mathscr{O}$ restrict to the simple root of Δ. We conclude that the $\alpha \in \tilde{\Delta}$ such that $|\alpha(t)|$ is not bounded away from zero lie in $\mathcal{O}$ and also that if $Q = P$ the element γ must lie in $P(l)$.

Let (ρ, V, ξ) be a k-representation of G adapted to P. Then ξ is a weight vector for the maximal k-split torus S, whose weight χ is a positive multiple of the only simple root α of the root system R. Fix a height $\| \; \|$ on $V(A,k)$. Let $g \in G(A,k)^0$ and write $g = ysz$, where $s \in S(A,k) \cap G(A,k)^0$, $z \in P(A,k)^0$ and y lies in a fixed compact set (see 2.1). Then

$$\|\rho(g).\xi\| = |\chi(s)| \, \|\rho(y).\xi\|.$$

By property (b) of heights we conclude that $|\chi(s)|^{-1} \|\rho(g).\xi\|$ lies in a compact subset of $\mathbf{R}_+^*$.

On the other hand we can view g as an element of $G(A,l)$. As in 3.5 we write $g = xtu\gamma$. We may assume that $\gamma \in P(l)$. Now (ρ, V, ξ) obviously is an l-representation of G adapted to P, and ξ is a highest weight vector for that representation (relative to the Borel group B). Its weight ψ is a linear combination of the roots in $\tilde{\Delta}$, with non-negative rational coefficients. It follows that for $t \in T(c)$ the numbers $|\psi(t)|$ lie in a compact set of $\mathbf{R}$. Take a height on $V(A,l)$. We denote it by $\| \; \|$ and we assume (as we may) that its restriction to $V(A,k)$ is the previous height. Then

$$\|\rho(g).\xi\| = |\psi(t)| \, \|\rho(x).\xi\|,$$

and we see that $\|\rho(g).\xi\|$ lies in a bounded set. Then the same holds for $|\chi(s)|$ and $|\alpha(s)|$, which means that g lies in a set $\mathcal{S}(c')$. It follows that $G(A,k) = \mathcal{S}(c').G(k)$, which we had to prove.

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