

Vector bundles as direct images of line bundles

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Let X be a smooth irreducible projective variety over an algebraically closed field K and E a vector bundle on X . We prove that, if $\dim X \geq 1$, there exist a smooth irreducible projective variety Z over K , a surjective separable morphism $f: Z \rightarrow X$ which is finite outside an algebraic subset of codimension ≥ 3 in X and a line bundle L on X such that the direct image of L by f is isomorphic to E . When X is a curve, we show that Z, f, L can be so chosen that f is finite and the canonical map

$$H^1(Z, \mathcal{O}) \rightarrow H^1(X, \text{End } E)$$

is surjective.

Keywords. Projective variety; algebraic vector bundle; line bundle; direct image; finite morphism.

1. Introduction

Let X be a smooth irreducible projective variety over an algebraically closed field and E a vector bundle on X . We prove in this paper first that, if $\dim X \geq 1$, E is the direct image of a line bundle L on a smooth irreducible projective variety Z by a morphism $f: Z \rightarrow X$ which is finite outside an algebraic subset of codimension ≥ 3 in X . Moreover one can choose the morphism f to be separable and to have the property that all higher direct images of L by f are zero [Theorem 4.2].

In particular if $\dim X \leq 2$ the morphism f may be chosen to be finite. In the case of surfaces this result has been proved by R.L.E. Schwarzenberger for rank two vector bundles [5, Theorem 3]. We also give an example of a vector bundle on \mathbb{P}_3 which cannot be obtained as the direct image of a line bundle on a smooth variety by a finite morphism.

In the second part of the paper we consider the case when X is a curve. We prove in this case that Z, L and f can be so chosen that the canonical homomorphism (see 5.1)

$$H^1(Z, \mathcal{O}_Z) \rightarrow H^1(X, \text{End } E)$$

is surjective (Theorem 6.4). This result was proved for a “very stable” vector bundle E by Beauville-Narasimhan-Ramanan in the case of a curve over \mathbb{C} by using the Hitchin map [3]. (For the significance of this result see Remark 6.5).

Let $\pi: \mathbf{P}(E) \rightarrow X$ be the projective bundle associated to E . The variety Z is constructed as the subscheme (of $\mathbf{P}(E)$) of zeros of a generic section of the tangent bundle along the fibres of π twisted by a suitable ample line bundle on X pulled up to $\mathbf{P}(E)$; the line bundle L is simply taken to be the restriction of $\mathcal{O}_{\mathbf{P}(E)}(1)$ to Z .

The scheme Z is essentially the scheme of ‘eigenstates’ of a generic twisted endomorphism of E . However, in general, Z is not the spectral variety of the twisted endomorphism; the canonical map from the spectral variety into X is always a finite morphism.

2. Sections of the tangent bundle of a projective space

Let V be a finite dimensional vector space of dimension ≥ 2 over an algebraically closed field K and $\mathbf{P} = \mathbf{P}(V)$ the projective space of hyperplanes in V . We have the exact sequences of vector bundles on $\mathbf{P}(E)$:

$$0 \rightarrow \Omega^1(1) \rightarrow V_{\mathbf{P}} \rightarrow \mathcal{O}(1) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_{\mathbf{P}} \rightarrow V_{\mathbf{P}}^* \otimes \mathcal{O}(1) \rightarrow \Theta \rightarrow 0,$$

where Θ denotes the tangent bundle of \mathbf{P} and V^* the dual of V . We obtain an exact sequence of vector spaces

$$0 \rightarrow K \rightarrow V^* \otimes V \rightarrow H^0(\mathbf{P}, \Theta) \rightarrow 0.$$

If $\text{End}^0(V) := \text{End}(V)/(\text{Scalar endomorphisms})$ we have $\text{End}^0(V) = H^0(\mathbf{P}, \Theta)$.

If an endomorphism T of V leaves a hyperplane ξ invariant, T induces an endomorphism of the one dimensional space V/ξ . The subspace $(V/\xi)^*$ of V^* is an eigenspace of the transpose of T . If s_T is the section of Θ defined by T , we have $s_T(\xi) = 0$ if and only if $T(\xi) \subset \xi$. Thus we can view the subscheme $Z = Z_T$ of zeros of s_T as the scheme of “eigenstates” of T . Moreover the “eigenvalue” of T is a section of \mathcal{O}_Z ; in fact it is the section of \mathcal{O}_Z corresponding to the morphism $\mathcal{O}_Z(1) \rightarrow \mathcal{O}_Z(1)$ induced by T from:

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega^1(1) & \rightarrow & V_{\mathbf{P}} & \rightarrow & \mathcal{O}(1) \rightarrow 0 \\ & & & & T \downarrow & & \\ & & & & V_{\mathbf{P}} & \rightarrow & \mathcal{O}(1) \rightarrow 0. \end{array}$$

Observe that the scheme Z_T has dimension ≥ 1 if and only if the transpose of T has an eigenspace of dimension ≥ 2 corresponding to some eigenvalue.

PROPOSITION 2.1

Consider the exact sequence of vector bundles

$$0 \rightarrow F \rightarrow \text{End}^0(V)_{\mathbf{P}} \rightarrow \Theta \rightarrow 0$$

(F being defined as the kernel of homomorphism $\text{End}^0(V)_{\mathbf{P}} \rightarrow \Theta$). Let $p: F \rightarrow \text{End}^0(V)$ be the restriction to F of the projection $\text{End}^0(V) \times \mathbf{P} \rightarrow \text{End}^0(V)$. Then there exists an open subset Ω in $\text{End}^0(V)$, whose complement is of codimension ≥ 3 , such that the morphism $p: p^{-1}(\Omega) \rightarrow \Omega$ is finite.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & K_{\mathbf{P}} & & K_{\mathbf{P}} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & F^1 & \rightarrow & \text{End}(V)_{\mathbf{P}} & \rightarrow & \Theta \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & F & \rightarrow & \text{End}^0(V)_{\mathbf{P}} & \rightarrow & \Theta \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Let $q: F^1 \rightarrow \text{End}(V)$ be the projection. We shall show that there exists an open set U of $\text{End}(V)$ which is saturated for the map $\text{End}(V) \rightarrow \text{End}^0(V)$ and whose complement is of codimension ≥ 3 such that the morphism $q: q^{-1}(U) \rightarrow U$ is finite. This will prove the proposition.

For each subspace W of dimension $k \geq 2$ of V^* , consider the subspace of $\text{End}(V^*)$ consisting of those endomorphisms whose restriction to W is a scalar endomorphism of W . The dimension of this space is $1 + (r - k)^2 + k(r - k)$. Varying W over the Grassmannian $G(r, k)$ we get a vector bundle $W(r, k)$ over $G(r, k)$ and the dimension of the total space of this bundle is

$$1 + (r - k)^2 + k(r - k) + k(r - k) = 1 + r^2 - k^2.$$

We have a natural morphism $\pi_k: W(r, k) \rightarrow \text{End}(V)$ which maps an endomorphism to its transpose. If $S_k := \pi_k(W(r, k))$, we have $\dim S_k \leq (1 + r^2 - k^2)$ and $\text{codim } S_k \geq k^2 - 1 \geq 3$. Let $S = \bigcup_{2 \leq k \leq r} S_k$ and $U = \text{End}(V) - S$. We have $\text{codim } S \geq 3$ and S is saturated for the map $\text{End}(V) \rightarrow \text{End}^0(V)$. The fibres of $q: q^{-1}(U) \rightarrow U$ are finite and q is proper. Hence q is a finite morphism.

3. Sections of the (twisted) relative tangent bundle of a projective bundle

Let E be a vector bundle of rank $r \geq 2$ on a smooth irreducible projective variety X of dimension ≥ 1 over K . Let $\pi: \mathbf{P}(E) \rightarrow X$ be the associated projective bundle. We have the exact sequences on $\mathbf{P}(E)$:

$$0 \rightarrow \Omega_{\mathbf{P}(E)}^1 \rightarrow \pi^*(E) \rightarrow \mathcal{O}_{\mathbf{P}(E)}(1) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_{\mathbf{P}(E)} \rightarrow \pi^*(E^*) \otimes \mathcal{O}(1) \rightarrow \Theta_{\mathbf{P}(E)} \rightarrow 0$$

where $\Theta_{\mathbf{P}(E)}$ (resp. $\Omega_{\mathbf{P}(E)}^1$) denotes the relative tangent (resp. cotangent) bundle along the fibres of π . Let $\text{End}^0(E)$ denote the vector bundle $\text{End}(E)/\mathcal{O}_X$. We have an exact sequence of vector bundles on $\mathbf{P}(E)$:

$$0 \rightarrow F \rightarrow \pi^*(\text{End}^0(E)) \rightarrow \Theta_{\mathbf{P}(E)} \rightarrow 0.$$

Let M be a line bundle on X . We obtain the exact sequence:

$$0 \rightarrow F \otimes \pi^*(M) \rightarrow \pi^*(\text{End}^0(E) \otimes M) \rightarrow \Theta_\pi \otimes \pi^*(M) \rightarrow 0$$

PROPOSITION 3.1

Let $p: F \otimes \pi^*(M) \rightarrow \text{End}^0(E) \otimes M$ be the canonical morphism (of total spaces of geometric vector bundles over $\mathbf{P}(E)$ and X respectively). Then there exists an open subset Ω of $\text{End}^0(E) \otimes M$ whose complement is of codimension ≥ 3 such that the morphism $p: p^{-1}(\Omega) \rightarrow \Omega$ is finite.

Proof. This follows from Proposition 2.1.

PROPOSITION 3.2

There exists an ample line bundle M on X such that a generic section s of $\Theta_\pi \otimes \pi^*(M)$ (i.e. for s in a non-empty open subset of $H^0(\mathbf{P}(E), \Theta_\pi \otimes \pi^*(M))$) satisfies the following conditions:

- (a) The scheme Z of zeros of s is smooth and irreducible.
- (b) The morphism $\pi|_Z: Z \rightarrow X$ is surjective and separable.
- (c) There exists a closed subset S of X of codimension ≥ 3 such that the morphism

$$\pi: Z \setminus \pi^{-1}(S) \rightarrow X \setminus S$$

is finite.

Proof. Let ξ be an ample line bundle on X . Then the line bundle $\pi^*(\xi^k) \otimes \mathcal{O}(1)$ is very ample on $\mathbf{P}(E)$ for $k \geq k_0$. [4, II, Prop. 7.10, p. 161]. We may also assume that for $k \geq k_0$, the bundle $\xi^k \otimes E^*$ is generated by its sections. Let $M = \xi^{2k}$. Since the sections of the very ample line bundle $\pi^*(\xi^k) \otimes \mathcal{O}(1)$ generate its first order jet bundle and $\pi^*(\xi^k \otimes E^*)$ is generated by its sections, we see that the sections of $\pi^*(M \otimes E^*) \otimes \mathcal{O}(1)$ generate its first order jet bundle [7, Lemma 5]. Since $\Theta_\pi \otimes \pi^*(M)$ is a quotient bundle of $\pi^*(M \otimes E^*) \otimes \mathcal{O}(1)$, the sections of $\Theta_\pi \otimes \pi^*(M)$ generate its first order jet bundle. Now by [7, Theorem 1] the zero scheme Z of a generic section s of $\Theta_\pi \otimes \pi^*(M)$ is smooth. We will prove in the next proposition (Proposition 4.1) that Z is irreducible.

Let $x_0 \in X$ and $S := \pi^{-1}(x_0)$ the fibre over x_0 . Let W be the image of the homomorphism

$$H^0(\mathbf{P}(E), \Theta_\pi \otimes \pi^*(M)) \rightarrow H^0(S, \Theta_\pi \otimes \pi^*(M)|_S).$$

(We may even assume that $W = H^0(S, \Theta_\pi \otimes \pi^*(M)|_S)$, by choosing k large enough). Then the first order jets of elements of W generate the first order jet bundle of $\Theta_\pi \otimes \pi^*(M)|_S$; hence, again by [7, Theorem 1], for a generic element σ of W , the zero subscheme (of S) defined by σ is smooth. Thus we see that for a generic section of $\Theta_\pi \otimes \pi^*(M)$ the zero scheme Z is smooth and irreducible and Z intersects S transversally. It follows that there exists a point $z_0 \in Z \cap S$ such that the differential of $\pi|_Z$ at z_0 is an isomorphism. This proves that $\pi|_Z: Z \rightarrow X$ is surjective and (assuming Z to be irreducible) separable. (Observe that Z intersects every fibre $\pi^{-1}(x), x \in X$,

for otherwise the tangent bundle of the projective space $\pi^{-1}(x)$ would contain a trivial line bundle.)

$$\begin{aligned} \text{Let } \Sigma &:= H^0(\mathbf{P}(E), \pi^*(M) \otimes \Theta_\pi) \\ &= H^0(X, M \otimes \pi_*(\Theta_\pi)) \\ &= H^0(X, M \otimes \text{End}^0(E)). \end{aligned}$$

Consider the morphisms

$$\begin{array}{c} \varphi: \Sigma \times X \xrightarrow{\varphi} M \otimes \text{End}^0(E) \\ \downarrow p_X \\ X. \end{array}$$

where the evaluation map φ is a smooth morphism, being a surjection of vector bundles. Let Ω be the open subset of $M \otimes \text{End}^0(E)$ defined in Proposition 3.1 and N its complement. Then

$$\dim \varphi^{-1}(N) \leq \dim \Sigma + \dim X - 3.$$

By considering $p_X: \varphi^{-1}(N) \rightarrow X$ we see that for a generic section s of $M \otimes \text{End}^0(E)$ we have

$$\dim(\varphi^{-1}(N) \cap \{s \times X\}) \leq (\dim X) - 3.$$

Let $S \subset X$ be defined to be $p_X(\varphi^{-1}(N) \cap \{s \times X\})$. Then $\dim S \leq (\dim X) - 3$ and

$$\pi|_Z: Z \setminus \pi^{-1}(S) \rightarrow X \setminus S$$

is finite.

Thus for a generic section s of $\Theta_\pi \otimes \pi^*(M)$ all the conditions a), b) and c) are satisfied.

4. Koszul resolution of the zero scheme Z

Proof of Theorem 4.2

PROPOSITION 4.1

Let s be a section of $\Theta_\pi \otimes \pi^*(M)$ over $\mathbf{P}(E)$ such that the zero scheme Z of s is smooth. We then have

- a) $\pi_*(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbf{P}(E)}(1)) \simeq E$ and $R^i \pi_*(\mathcal{O}_Z(1)) = 0$ for $i \geq 1$.
- b) $\pi_*(\mathcal{O}_Z)$ has a filtration $0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_i \subset \dots \subset F_r = \pi_*(\mathcal{O}_Z)$ such that $F_i/F_{i-1} \simeq M^{-(i-1)} (= (M^*)^{\otimes (i-1)})$ for $1 \leq i \leq r$ (In particular $F_1 \simeq \mathcal{O}_Z$).
- c) Z is irreducible (if $\dim X \geq 1$ and M is ample).

Proof. Using our assumption on Z , we have a Koszul resolution for \mathcal{O}_Z on $\mathbf{P}(E)$: [1, Ch I, Lemma 4.2 and Ch. III, Propositions 4.10 and 4.11]

$$(A) \quad 0 \rightarrow \bigwedge^{r-1} (\Omega_\pi^1 \otimes \pi^*(M^*)) \rightarrow \dots \rightarrow \bigwedge^2 (\Omega_\pi^1 \otimes \pi^*(M^*)) \rightarrow \Omega_\pi^1 \otimes \pi^*(M^*) \rightarrow \mathcal{O}_{\mathbf{P}(E)} \rightarrow \mathcal{O}_Z \rightarrow 0$$

and also a resolution of $\mathcal{O}_Z(1)$:

$$(B) \ 0 \rightarrow \Omega_\pi^{r-1} \otimes (\pi^*(M^*))^{r-1} \otimes \mathcal{O}(1) \rightarrow \cdots \rightarrow \Omega_\pi^1 \otimes \pi^*(M^*) \otimes \mathcal{O}(1) \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}(1)|_Z \rightarrow 0$$

Now we have, for the projective space \mathbf{P} , the Bott vanishing theorem:

$$H^i(\mathbf{P}, \Omega^p(1)) = 0 \text{ for } p \geq 1 \text{ and all } i \text{ [6, Théorème 1].}$$

Hence we have

$$R^i \pi_* (\Omega_\pi^p \otimes \pi^*((M^*)^{\otimes p}) \otimes \mathcal{O}(1)) = 0$$

for $p \geq 1$ and all i . Splitting B into short exact sequences we deduce that

$$\pi_*(\mathcal{O}_Z(1)) = \pi_*(\mathcal{O}_{\mathbf{P}(E)}(1)) = E$$

and $R^i \pi_*(\mathcal{O}_Z(1)) = 0$ for $i > 0$.

For proving (b) we observe that $R^q \pi_*(\Omega_\pi^p) = 0$ for $p \neq q$ and

$$R^p \pi_* (\Omega_\pi^p) = \mathcal{O}_X \text{ for } 0 \leq p \leq (r-1) \text{ [6].}$$

Splitting (A) into short exact sequences we obtain b).

To prove c), since $\dim X \geq 1$ and M is ample we have $H^0(X, M^{-k}) = 0$ for $k > 0$. Using the filtration of $\pi_*(\mathcal{O}_Z)$ given in b), we see that

$$H^0(Z, \mathcal{O}_Z) = H^0(X, \pi_*(\mathcal{O}_Z)) = H^0(X, \mathcal{O}_X) = K.$$

Since Z is smooth it follows that Z is irreducible.

Theorem 4.2 *Let X be a smooth irreducible projective variety of dimension ≥ 1 over an algebraically closed field K . Let E be a vector bundle on X . Then there exist a smooth irreducible projective variety Z over K , a line bundle L on Z and a surjective separable morphism $f: Z \rightarrow X$ having in addition the following properties:*

1) *there exists a closed subset S in X of codimension ≥ 3 such that the morphism*

$$f: Z \setminus f^{-1}(S) \rightarrow X \setminus S$$

is finite.

2) *we have $f_*(L) \simeq E$ and $R^i f_*(L) = 0$ for $i > 0$.*

Proof. We may assume that E is of rank ≥ 2 . Choose an ample line bundle M on X satisfying the conditions of Proposition 3.2. Let L be the restriction of $\mathcal{O}_{\mathbf{P}(E)}(1)$ to Z and f be the restriction of $\pi: \mathbf{P}(E) \rightarrow X$ to Z . Then by Proposition 4.1(a) we have

$$f_*(L) \simeq E \text{ and } R^i f_*(L) = 0 \text{ for } i > 0.$$

5. The map D

Let $f: Z \rightarrow X$ be a morphism and L a line bundle on Z such that $f_*(L) = E$ is a vector bundle of rank r on X . The morphism $f^*(f_*(L)) \rightarrow L$ gives rise to a morphism

$$f_*(\mathcal{O}_Z) \otimes E \rightarrow f_*(L) = E$$

which may be viewed as a morphism

$$D: f_*(\mathcal{O}_Z) \rightarrow E^* \otimes E.$$

(D gives the canonical $f_*(\mathcal{O}_Z)$ -module structure on $f_*(L)$).

Suppose that $f: Z \rightarrow X$ is a finite surjective morphism of smooth varieties. Then f is flat [1, Ch. V, Cor. 3.6]. We have

$$H^1(Z, \mathcal{O}_Z) = H^1(X, f_*(\mathcal{O}_Z)).$$

The homomorphism

$$D_*: H^1(Z, \mathcal{O}_Z) \rightarrow H^1(X, \text{End } E) \tag{5.1}$$

induced by D is the infinitesimal deformation map (at L) for the variation of the direct image bundles as L deforms as a line bundle on X [2, Lemma 1.3.1].

Since f is finite, the map $f^*(E) \rightarrow L$ is surjective and we have an exact sequence

$$0 \rightarrow N \rightarrow f^*(E) \rightarrow L \rightarrow 0$$

of vector bundles on Z . From this we get an exact sequence of vector bundles.

$$0 \rightarrow \mathcal{O}_Z \rightarrow f^*(E^*) \otimes L \rightarrow N^* \otimes L \rightarrow 0.$$

Since f is flat and finite, $f_*(\mathcal{O}_Z)$ is a vector bundle on X of rank r and $f_*(N^* \otimes L)$ is a vector bundle. So we have an exact sequence of vector bundles on X :

$$0 \rightarrow f_*(\mathcal{O}_Z) \rightarrow E^* \otimes E \rightarrow f_*(N^* \otimes L) \rightarrow 0.$$

Observe that $f_*(\mathcal{O}_Z)/\mathcal{O}_X$ is a vector subbundle of rank $(r - 1)$ of the vector bundle $\text{End}(E)/\mathcal{O}_X = \text{End}^0(E)$. Thus we have

Lemma 5.2 Let $f: Z \rightarrow X$ be a finite morphism of smooth varieties and L a line bundle on Z . If $E := f_*(L)$ is a vector bundle of rank r , then the vector bundle $\text{End}^0(E)$ contains a vector subbundle of rank $(r - 1)$.

Let us get back to the situation in § 3 and § 4.

PROPOSITION 5.3

Let s be a section of $\Theta_\pi \otimes \pi^*(M)$ and Z the zero subscheme of s with the property that the canonical map $E = \pi_*(\mathcal{O}_{\mathbf{P}(E)}(1)) \rightarrow f_*(\mathcal{O}_Z(1))$ is an isomorphism, where $f = \pi|_Z$. Suppose that T is a section of $\text{End}(E) \otimes M$ such that the image of T in $H^0(\mathbf{P}(E), \Theta_\pi \otimes \pi^*(M))$ is s . Then there is a homomorphism $\mu: M^{-1} \rightarrow f_*(\mathcal{O}_Z)$ and a commutative diagram

$$\begin{array}{ccc} f_*(\mathcal{O}_Z) & \xrightarrow{D} & \text{End } E \\ \mu \swarrow & & \nearrow T \\ & M^{-1} & \end{array}$$

where D is defined at the beginning of this section (§ 5).

Proof. Consider the diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & \Omega_x^1 \otimes \pi^*(M)^{-1} & \xrightarrow{i} & \pi^*(E \otimes M^{-1}) & \rightarrow & \mathcal{O}(1) \otimes \pi^*(M^{-1}) \rightarrow 0 \\
 & & & & \downarrow \tilde{T} & & \\
 0 & \longrightarrow & \Omega_x^1 & \longrightarrow & \pi^*(E) & \xrightarrow{g} & \mathcal{O}(1) \longrightarrow 0
 \end{array}$$

where \tilde{T} is induced by T . The homomorphism $g \circ \tilde{T} \circ i: \Omega_x^1 \otimes M^{-1} \rightarrow \mathcal{O}(1)$ gives the section s of $\Theta_x \otimes \pi^*(M)$. So \tilde{T} induces on Z a homomorphism $\lambda: \mathcal{O}_Z(1) \otimes f^*(M^{-1}) \rightarrow \mathcal{O}_Z(1)$ and we have a commutative diagram

$$\begin{array}{ccc}
 \pi^*(E) \otimes \pi^*(M^{-1}) & \xrightarrow{g \otimes 1} & \mathcal{O}_Z(1) \otimes \pi^*(M^{-1}) \\
 \downarrow \tilde{T} & & \downarrow \lambda \\
 \pi^*(E) & \xrightarrow{g} & \mathcal{O}_Z(1)
 \end{array}$$

Considering λ as a section of $\mathcal{O}_Z \otimes \pi^*(M)$ we obtain the section $\pi_*(\lambda)$ of $M \otimes \pi_*(\mathcal{O}_Z)$ which we view as a homomorphism $\mu: M^{-1} \rightarrow f_*(\mathcal{O}_Z)$. Since by assumption $E \rightarrow \pi_*(\mathcal{O}_Z(1))$ is an isomorphism, it follows, from the above diagram, that T corresponds to $\pi_*(\lambda): M^{-1} \otimes E \rightarrow E$. But by the definitions of D and μ we see that $\pi_*(\lambda)$ corresponds to $D \circ \mu$. This means that $T = D \circ \mu$.

6. Vector bundles on curves

Lemma 6.1. Let X be a smooth, projective irreducible curve over K and F a vector bundle of rank ≥ 2 on X . Let ξ be an ample line bundle on X . Then there exists an integer l_0 such that for $l \geq l_0$, ξ^{-l} is a subbundle of F and the induced homomorphism $H^1(X, \xi^{-l}) \rightarrow H^1(X, F)$ is surjective.

Proof. Let first F be of rank 2. Since for all large l , $\xi^l \otimes F^*$ contains a trivial line subbundle, we get an exact sequence

$$0 \rightarrow \xi^{-l} \rightarrow F \rightarrow \xi^l \otimes \det F \rightarrow 0.$$

Choose l large enough so that $H^1(X, \xi^l \otimes \det F) = 0$.

Now suppose that F is a vector bundle of rank ≥ 3 . Then we can find a filtration of F by subbundles

$$0 \subset F_1 \subset F_2 \dots \subset F_i \subset \dots \subset F_{r-1} = F$$

such that $\text{rank}(F_i) = i + 1$ (in particular $\text{rank } F_1 = 2$) and such that $H^1(X, F_i/F_{i-1}) = 0$, for $i \geq 2$. Now choose a line subbundle ξ^{-l} of F_1 with $H^1(F_1/\xi^{-l}) = 0$. We see easily, by induction on i , that $H^1(X, F/\xi^{-l}) = 0$.

Remark 6.2 Note that $H^1(X, \xi^{-l}) \rightarrow H^1(X, F)$ is surjective if and only if $H^1(F/\xi^{-l}) = 0$, as $H^2(X, \xi^{-l}) = 0$.

COROLLARY 6.3

Let F be a vector bundle on X . Then there exists an integer l_0 such that, for $l \geq l_0$, for a generic section σ of $\xi^l \otimes F$ the map $H^1(X, \xi^{-l}) \rightarrow H^1(X, F)$ induced by σ is surjective.

Theorem 6.4 Let X be a smooth projective irreducible curve over an algebraically closed field K and let E be a vector bundle on X . Then there exist a smooth projective irreducible curve Z over K , a line bundle L on Z and a finite surjective separable morphism $f: Z \rightarrow X$ such that

- 1) $f_*(L) \simeq E$
- 2) and the homomorphism (defined in 5.1)

$$H^1(Z, \mathcal{O}_Z) \rightarrow H^1(X, \text{End } E)$$

is surjective.

Proof. Choose an ample line bundle M as in the proof of Theorem 4.2. We may also choose M to have the further properties:

- a) $H^1(X, M) = 0$
- and
- b) a generic section σ of $H^0(X, \text{End } E \otimes M)$ verifies the condition that the homomorphism

$$H^1(X, M^{-1}) \rightarrow H^1(X, \text{End } E)$$

is surjective (use Corollary 6.3).

By condition a) the map

$$H^0(X, \text{End } E \otimes M) \rightarrow H^0(X, \text{End}^0 E \otimes M)$$

is surjective.

Now a generic section s of $H^0(\mathbf{P}(E), \Theta_x \otimes \pi^*(M)) = H^0(X, \text{End}^0(E) \otimes M)$ is the image of a section T of $\text{End}(E) \otimes M$ with the property that the homomorphism

$$H^1(X, M^{-1}) \rightarrow H^1(X, \text{End } E)$$

induced by T is surjective and satisfies conditions a), b) and c) of Proposition 3.2.

Choose Z, L, f as in the proof of Theorem 4.2. Then $f_*(L) = E$.

To prove 2), observe that the factorisation, given in Proposition 5.3,

$$\begin{array}{ccc} f_*(\mathcal{O}_Z) & \xrightarrow{D} & \text{End } E \\ \mu \swarrow & & \nearrow T \\ & M^{-1} & \end{array}$$

induces a commutative diagram:

$$\begin{array}{ccc} H^1(Z, \mathcal{O}_Z) = H^1(X, f_*(\mathcal{O}_Z)) & \xrightarrow{D_*} & H^1(X, \text{End } E) \\ \mu_* \swarrow & & \nearrow T_* \\ & H^1(X, M^{-1}) & \end{array}$$

Since, by choice, T_* is surjective, the homomorphism D_* is forced to be surjective.

Remark 6.5 If E is a stable bundle on X and if Z, f and L are chosen as in Theorem 6.4, we see that f_* gives a *dominant* separable rational morphism from an appropriate component of $\text{Pic}(Z)$ into the moduli space of vector bundles on X of rank $r = rkE$ and degree $d = \text{degree } E$ (compare [3]). We thus obtain ‘most’ stable bundles of a given rank and degree as direct images of line bundles on a *fixed* covering Z of X .

7. The example

We now give an example of a rank 2 vector bundle on the projective space $\mathbf{P}_3(\mathbf{C})$ which cannot be obtained as the direct image of a line bundle by a *finite* morphism $f: Z \rightarrow \mathbf{P}_3(\mathbf{C})$, with Z smooth.

Let E be a stable vector bundle of rank 2 on $\mathbf{P}_3(\mathbf{C})$ with $c_1(E) = 0$ and $c_2(E) > 0$. If E were the direct image of a line bundle by $f: Z \rightarrow X$, with Z smooth and f finite, the bundle $\text{End}^0(E)$ would contain a line *subbundle* L by Lemma 5.2. If $\xi = L^{-1}$, we would have

$$c_3(\xi \otimes \text{End}^0 E) = 0. \text{ We have}$$

$$c_3(\xi \otimes \text{End}^0 E) = 4c_1(\xi)c_2(E) + c_1(\xi)^3.$$

But the bundle L and hence ξ , is non-trivial, since $h^0(\mathbf{P}_3, \text{End}^0 E) = 0$, E being stable. So $c_1(\xi) \neq 0$ and we would have

$$c_1(\xi)(4c_2(E) + c_1(\xi)^2) = 0,$$

a contradiction.

Acknowledgements

The first author (AH) carried out this work in the framework of the Vector Bundle group of Europroj. The second author (MSN) would like to thank CNRS and Université de Nice-Sophia Antipolis for hospitality when part of this work was done.

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