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Local zeta functions of general quadratic polynomials*

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Dedicated to the memory of Professor K G Ramanathan

Abstract. This paper is concerned with the kind of local zeta functions now often called Igusa's local zeta functions: A simple closed form of such a zeta function for an arbitrary quadratic form, its variants, and an application are given.

Keywords. Local zeta function; quadratic polynomials.

1. Introduction

The local zeta functions which we shall consider in this paper are of the form

$$Z(s) = \int_L |f(x)|_K^s \mathrm{d}x, \quad \mathrm{Re}(s) > 0,$$

where L is a lattice in a vector space V over a p-adic field K, $|\cdot|_K$ is an absolute value on K, f is a polynomial function on V, s is a complex variable, and $\mathrm{d}x$ is the Haar measure on V normalized as $Z(0) = 1$. If q is the cardinality of the residue class field of K and if char $(K) = 0$, our general theorem states that $Z(s)$ is a rational function of q^{-s}; cf. [2]. Furthermore $Z(s)$ has been computed in a large number of cases. We might add that in some cases computations are very difficult and that no algorithm to compute $Z(s)$ for a general f is known.

Now, in the difficult cases where $Z(s)$ has been computed, one has often encountered auxiliary integrals of the above form where f is a quadratic polynomial with coefficients depending on some parameters. In this paper we shall give a simple closed form to $Z(s)$ in the case where f is an arbitrary nondegenerate quadratic form and also in some inhomogeneous cases; for the sake of simplicity we have assumed that q is odd. The formula has turned out to be useful for the computation of $Z(s)$ in some new cases and for the simplification in some known cases. We have included just one application at the end of the paper.

2. Preliminaries

As in the Introduction, we denote by V a (finite dimensional) vector space over a p-adic field K and by L a lattice in V. If S is any subset of V, we shall denote by KS the K-span of S in V. If O_K is the maximal compact subring of K, then L is a free

O_K-submodule of V such that $KL = V$. We shall denote by $n(L)$ the rank of L, i.e., $n(L) = \dim(KL)$. We choose π from O_K such that πO_K becomes the maximal ideal of O_K and denote by q the cardinality of the residue class field $O_K/\pi O_K$, i.e., $O_K/\pi O_K = \mathbf{F}_q$. In general if R is any associative ring with 1, we shall denote by $R^\times$ the group of units of R. We shall assume, once and for all, that q is odd; then

$$(O_K^\times)/(O_K^\times)^2 \simeq \mathbf{F}_q^\times/(\mathbf{F}_q^\times)^2 \simeq \{\pm 1\}.$$

We shall denote the corresponding homomorphism of $O_K^\times$ to $\{\pm 1\}$ by χ.

We take a quadratic form Q on V, i.e., a K-valued function on V satisfying $Q(ax) = a^2 Q(x)$ for every a in K and x in V such that $Q(x, y) = Q(x + y) - Q(x) - Q(y)$ is K-bilinear in x, y. If $\{w_1, \ldots, w_n\}$ is a K-basis for V so that $n = \dim(V)$, then

$$Q\left(\sum_{1 \leqslant i \leqslant n} x_i w_i\right) = \sum_{1 \leqslant i \leqslant n} Q(w_i) x_i^2 + \sum_{i<j} Q(w_i, w_j) x_i x_j$$

for every $x_1, \ldots, x_n$ in K. Therefore if $\{w_1, \ldots, w_n\}$ is an O_K-basis for L, then Q is O_K-valued on L if and only if the coefficients of the above homogeneous polynomial of degree 2 in $x_1, \ldots, x_n$ are all in O_K. We shall assume that Q is nondegenerate, i.e., that $Q(x, y) = 0$ for all y in V implies $x = 0$. If $\{w_1, \ldots, w_n\}$ is an O_K-basis for L and if h is the square matrix of degree n with $Q(w_i, w_j)$ as its (i,j)-entry for $1 \leqslant i, j \leqslant n$, then $\det(h) \neq 0$. We define the discriminant $D(L, Q)$ of (L, Q) as

$$D(L, Q) = (-1)^{n(n-1)/2} \cdot \det(h).$$

We observe that $D(L, Q)(O_K^\times)^2$ is independent of the choice of the O_K-basis for L. We call (L, Q) *unimodular* if Q is O_K-valued on L and if $D(L, Q)$ is in $O_K^\times$. In the general case (L, Q) has the following *Jordan decomposition*:

There exist O_K-submodules $L_1, \ldots, L_t$ of L with L as their sum and a sequence of integers $e_1 < \cdots < e_t$ such that $KL_1, \ldots, KL_t$ are mutually orthogonal with respect to $Q(x, y)$ and (L_i, Q_i), where $Q_i = \pi^{-e_i} Q | KL_i$, is unimodular for $1 \leqslant i \leqslant t$. Furthermore if we put $n_i = n(L_i)$, then

$$\operatorname{inv}(L, Q) = \{n_i, e_i, \chi(D(L_i, Q_i)); 1 \leqslant i \leqslant t\}$$

characterizes the isomorphism class of (L, Q).

We refer to O'Meara [5], Chapter IX for the theory of Jordan decompositions. We keep in mind that $M = L_{j_1} + \cdots + L_{j_r}$, where $1 \leqslant j_1 < \cdots < j_r \leqslant t$, gives the Jordan decomposition of $(M, Q|KM)$.

3. $Z(s)$ for a quadratic form

We start from the Jordan decomposition of (L, Q) recalled in the previous section. If we define Q^0 as $Q_1 + \cdots + Q_t$, i.e., as

$$Q^0(x_1 + \cdots + x_t) = Q_1(x_1) + \cdots + Q_t(x_t)$$

for all $x_1, \ldots, x_t$ respectively in $KL_1, \ldots, KL_t$, then (L, Q^0) is unimodular. We put

$\chi(L, Q) = \chi(D(L, Q^0))$ if

$$n \equiv n_1 e_1 + \cdots + n_t e_t \equiv 0 \bmod 2$$

and $\chi(L, Q) = 0$ otherwise. In view of the formal identity

$$\left(\sum_i n_i\right)\left(\sum_i n_i - 1\right) = \sum_i n_i(n_i - 1) + 2 \cdot \sum_{i<j} n_i n_j,$$

if $n \equiv \Sigma n_i e_i \equiv 0 \bmod 2$, then

$$\chi(L, Q) = \chi(-1)^{\Sigma_{i<j} n_i n_j} \cdot \prod_{1 \leqslant i \leqslant t} \chi(D(L_i, Q_i)).$$

We observe that $\chi(L, Q)$ is not only independent of the choice of the O_K-basis for L, but also it remains invariant under $Q \mapsto u_1 Q$ and $\pi \mapsto u_2 \pi$ for any u_1, u_2 in $O_K^\times$. This follows from the fact that $D(L, Q^0)$ is multiplied, for a fixed O_K-basis for L, by u^n under $Q \mapsto uQ$ and by u^{-e}, where $e = \Sigma n_i e_i$, under $\pi \mapsto u\pi$ for every u in $O_K^\times$. In the special case where $e_1 \equiv \cdots \equiv e_t \bmod 2$ we shall write $\chi(L)$ instead of $\chi(L, Q)$. In other words if $e_1 \equiv \cdots \equiv e_t \bmod 2$, then we define $\chi(L)$ as follows:

$$\chi(L) = \begin{cases} \chi(D(L, Q^0)) & n(L) \text{ even} \\ 0 & n(L) \text{ odd.} \end{cases}$$

The notation $\chi(M)$ will be used for

$$M = L_{j1} + \cdots + L_{jr},$$

where $1 \leqslant j_1 < \cdots < j_r \leqslant t$ and $e_{j1} \equiv \cdots \equiv e_{jr} \bmod 2$, relative to $Q|KM$ with the understanding that $\chi(M) = 1$ for $r = 0$.

In the Introduction we have defined $Z(s)$ as

$$Z(s) = \int_L |f(x)|_K^s \mathrm{d}x, \quad \mathrm{Re}(s) > 0.$$

We shall normalize the absolute value $|\cdot|_K$ on K as $|\pi|_K = q^{-1}$ so that $|a|_K$ for every a in $K^\times$ becomes the module of the multiplication by a in K. Furthermore, for the sake of simplicity, we put

$$[a, b] = 1 - q^{-(a+bs)}, \quad [a, b]_+ = 1 + q^{-(a+bs)}, \quad [a] = [a, 0], \quad [a]_+ = [a, 0]_+,$$

in which a, b are mostly nonnegative integers.

Theorem 1. *The local zeta function $Z(s)$ defined as*

$$Z(s) = \int_L |Q(x)|_K^s \mathrm{d}x, \quad \mathrm{Re}(s) > 0$$

has the following closed form in terms of inv(L, Q): *For each i, $1 \leqslant i \leqslant t$ put*

$$Z_i = [1][0,1]/[1,1][m_1, 2][m_2, 2] \cdot q^{-1/2 \cdot \Sigma_{j<i} n_j(e_i - e_j) - e_i s}$$
$$\cdot \{\chi(M_1)[n_i] q^{-m_1 - s} + \chi(M_2)[m_2, 2] q^{-m_1/2} - \chi(M_3)[m_1, 2] q^{-m_2/2}\},$$

where

$$M_1 = \sum_{j<i, e_j \not\equiv e_i} L_j, \quad M_2 = \sum_{j<i, e_j \equiv e_i} L_j, \quad M_3 = M_2 + L_i,$$

$$m_1 = \sum_{j<i} n_j, m_2 = m_1 + n_i.$$

Then

$$Z(s) = Z_1 + \cdots + Z_t.$$

In the above definition of M_1, M_2 the congruences are mod 2. We observe that the condition "$e_{j1} \equiv \cdots \equiv e_{jr} \bmod 2$" is satisfied by each M_k, hence $\chi(M_k)$ is defined in terms of $\mathrm{inv}(M_k, Q|KM_k)$, hence of $\mathrm{inv}(L, Q)$. In fact, if L_j occurs in M_k, then $e_j \equiv e_i + 1$, e_i, $e_i \bmod 2$ respectively for $k = 1, 2, 3$.

4. Some lemmas

We first state a "p-adic stationary phase formula," abbreviated as SPF, as a lemma:

Lemma 1. *Let $f(x)$ denote an element of the polynomial ring $O_K[x_1, \ldots, x_n]$ and $\bar{f}(x)$ its image in $\mathbf{F}_q[x_1, \ldots, x_n]$ under the canonical homomorphism $O_K \to \mathbf{F}_q$; let $\bar{f}^{-1}(0)$ denote the subset of $\mathbf{F}_q^n$ defined by $\bar{f}(\xi) = 0$, $\bar{S}$ its subset defined by*

$$(\partial\bar{f}/\partial x_1)(\xi) = \cdots = (\partial\bar{f}/\partial x_n)(\xi) = 0,$$

and S the preimage of $\bar{S}$ under $O_K^n \to \mathbf{F}_q^n$; put $N = card(\bar{f}^{-1}(0))$. Then

$$\int_{O_K^n} |f(x)|_K^s dx = (1 - q^{-n}N) + (N - \mathrm{card}(\bar{S}))[1]q^{-n-s}/[1,1] + \int_S |f(x)|_K^s dx.$$

We refer to [4] for the proof of a more general statement. If $f(x)$ is homogeneous of degree $d \geqslant 1$ and if $\bar{S} = \{0\}$, then SPF gives

$$\int_{O_K^n} |f(x)|_K^s dx = \{(1 - q^{-n}N)[1,1] + (N-1)[1]q^{-n-s}\}/[1,1][n,d].$$

If further $d = 2$, i.e., if (O_K^n, Q) is unimodular for $Q = f$, then N has the following well-known expression:

$$N = \begin{cases} q^{n-1} + \chi(D)[1]q^{n/2} & n \text{ even} \\ q^{n-1} & n \text{ odd,} \end{cases}$$

in which $D = D(O_K^n, Q)$; cf., e.g., Bourbaki [1]. If we use our notation that $\chi(L)$ for $L = O_K^n$ represents $\chi(D)$ or 0 according as n is even or odd, then we get

$$\int_L |Q(x)|_K^s dx = [1]\{1 - \chi(L)[0,1]q^{-n/2} - q^{-n-s}\}/[1,1][n,2],$$

which is also well known (in the usual notation).

For the sake of clarity we have separated the following lemmas from the proof of Theorem 1:

Lemma 2. *If we define* A_i *as*

$$A_i = [1]\{1 - \chi_i[0,1]q^{-n_i/2} - q^{-n_i-s}\}/[1,1][n_i,2]$$

for $i = 1, 2$ *and* A_{12} *similarly as*

$$A_{12} = [1]\{1 - \chi_{12}[0,1]q^{-(n_1+n_2)/2} - q^{-n_1-n_2-s}\}/[1,1][n_1+n_2,2],$$

where $\chi_1, \chi_2, \chi_{12}$ *are variables, then*

$$[n_1+n_2,2](A_{12} - A_1) = [1][0,1]q^{-n_1/2}/[1,1][n_1,2]$$
$$\cdot\{[n_2]q^{-n_1/2-s} + \chi_1[n_1+n_2,2] - \chi_{12}[n_1,2]q^{-n_2/2}\},$$
$$([n_1,2] - [n_1+n_2,2])A_1 + [n_2,2]q^{-n_1-s}A_2 = [1][0,1]q^{-n_1-s}/[1,1][n_1,2]$$
$$\cdot\{\chi_1[n_2]q^{-n_1/2-s} + [n_1+n_2,2] - \chi_2[n_1,2]q^{-n_2/2}\}.$$

Lemma 3. *If* a, b, c *are integers and* x, y *are variables, then*

$$(1-x^b)(1-x^{a+b+c}y^2) - (1-x^{b+c})(1-x^{a+b}y^2) + x^b(1-x^c)(1-x^a y^2) = 0,$$

hence $[b][a+b+c,2] - [b+c][a+b,2] + q^{-b}[c][a,2] = 0.$

The verifications are all straightforward. In Lemma 2 we shall take $\chi(L_i)$ as χ_i for $i = 1, 2$ and $\chi(L_1 + L_2)$ as χ_{12}, this in the case where $e_1 \equiv e_2 \bmod 2$.

5. Proof of Theorem 1. If $t = 1$, i.e., if $(L, \pi^{-e_1}Q)$ is unimodular, then we have seen that.

$$Z(s) = q^{-e_1 s}A_1.$$

On the other hand, by definition

$$Z_1 = [1][0,1]/[1,1][0,2][n_1,2]\cdot q^{-e_1 s}\{[n_1]q^{-s} + [n_1,2]$$
$$-\chi(L_1)[0,2]q^{-n_1/2}\};$$

and clearly they are equal. Therefore we shall assume that $t \geqslant 2$ and apply an induction on t.

If for every integer j we put

$$\varphi(j) = \int_L |Q_1(x_1) + \sum_{i\geqslant 2} \pi^{e_i - e_1 - 2j}Q_i(x_i)|_K^s dx,$$

then we can write

$$Z(s) = \int_L \left| \sum_{1\leqslant i\leqslant t} \pi^{e_i}Q_i(x_i)\right|_K^s dx = q^{-e_1 s}\varphi(0).$$

We define an integer k as $e_2 - e_1 = 2k$ or $e_2 - e_1 = 2k+1$ according as $e_2 - e_1$ is even or odd, and for $j < k$ we apply SPF to $\varphi(j)$. Then after a small computation we get

$$\varphi(j) = [n_1,2]A_1 + q^{-n_1-2s}\varphi(j+1).$$

If we take $j = 0, 1, \ldots, k-1$ and eliminate $\varphi(1), \ldots, \varphi(k-1)$ from the resulting relations, we get

$$\varphi(0)=[n_1k,2k]A_1+q^{-k(n_1+2s)}\varphi(k),$$

hence

$$(*)\qquad Z(s)=[n_1k,2k]q^{-e_1s}A_1+q^{-n_1k-(e_1+2k)s}\varphi(k).$$

We shall separate cases according as e_2-e_1 is even or odd.

Case 1: $e_2-e_1=2k$.

If we put

$$Q'(x)=Q_1(x_1)+Q_2(x_2)+\sum_{i\geqslant 3}\pi^{e_i-e_2}Q_i(x_i),$$

then we get

$$\varphi(k)=\int_L|Q'(x)|_K^s\,dx.$$

Since $(L_1+L_2)+\cdots+L_t$ is the Jordan decomposition of (L,Q'), we can apply an induction on t to $\varphi(k)$. In that way we get

$$\varphi(k)=A_{12}+q^{n_1(e_2-e_1)/2+e_2s}\sum_{i\geqslant 3}Z_i.$$

By $(*)$ we have only to verify, therefore, that

$$Z_1+Z_2=[n_1k,2k]q^{-e_1s}A_1+q^{-n_1k-(e_1+2k)s}A_{12},\ \text{i.e.,}$$

$$Z_1=q^{-e_1s}A_1,\quad Z_2=q^{-n_1(e_2-e_1)/2-e_2s}(A_{12}-A_1).$$

We already know the first identity and we can verify the second identity by Lemma 2.

Case 2: $e_2-e_1=2k+1$

In this case we have

$$\varphi(k)=\int_L|Q_1(x_1)+\sum_{i\geqslant 2}\pi^{e_i-e_2+1}Q_i(x_i)|_K^s\,dx.$$

If for every integer j we put

$$\begin{cases}\psi(2j)=\displaystyle\int_L|Q_1(x_1)+\pi Q_2(x_2)+\sum_{i\geqslant 3}\pi^{e_i-e_2-2j+1}Q_i(x_i)|_K^s\,dx\\ \psi(2j+1)=\displaystyle\int_L|Q_2(x_2)+\pi Q_1(x_1)+\sum_{i\geqslant 3}\pi^{e_i-e_2-2j}Q_i(x_i)|_K^s\,dx,\end{cases}$$

then $\varphi(k)=\psi(0)$. In the special case where $t=2$ we only have $\psi(0)$ and $\psi(1)$. If we apply SPF to them, we get

$$\psi(0)=[n_1,2]A_1+q^{-n_1-s}\psi(1),\quad \psi(1)=[n_2,2]A_2+q^{-n_2-s}\psi(0).$$

By eliminating $\psi(1)$ from these, we get

$$\varphi(k)=\{[n_1,2]A_1+[n_2,2]q^{-n_1-s}A_2\}/[n_1+n_2,2].$$

By $(*)$ we have only to verify that

$$Z_1+Z_2=[n_1k,2k]q^{-e_1s}A_1+q^{-n_1k-(e_1+2k)s}\varphi(k), \text{ i.e.,}$$
$$Z_1=q^{-e_1s}A_1, \quad Z_2=q^{-n_1(e_2-e_1-1)/2-(e_2-1)s}(\varphi(k)-A_1).$$

Again the second identity can be verified by Lemma 2.

In the general case where $t\geqslant 3$ we apply SPF to $\psi(2j)$ and $\psi(2j+1)$ respectively for $2j\leqslant e_3-e_2$ and $2j<e_3-e_2$; then we get

$$\psi(2j)=[n_1,2]A_1+q^{-n_1-s}\psi(2j+1)$$
$$\psi(2j+1)=[n_2,2]A_2+q^{-n_2-s}\psi(2j+2).$$

By using these relations alternatively, we can express $\varphi(k)=\psi(0)$ by $\psi(2j)$ and also by $\psi(2j+1)$ both for $2j\leqslant e_3-e_2$. We can take $(e_3-e_2)/2$ or $(e_3-e_2-1)/2$ as j according as e_3-e_2 is even or odd. In that way we get

$$\varphi(k)=[(n_1+n_2)(e_3-e_2)/2,e_3-e_2]\{[n_1,2]A_1 + [n_2,2]q^{-n_1-s}A_2\}/[n_1+n_2,2]+q^{-(n_1+n_2)(e_3-e_2)/2-(e_3-e_2)s}\psi(e_3-e_2)$$

if e_3-e_2 is even and

$$\varphi(k)=\{[(n_1+n_2)(e_3-e_2+1)/2,e_3-e_2+1][n_1,2]A_1 + [(n_1+n_2)(e_3-e_2-1)/2,e_3-e_2-1][n_2,2]q^{-n_1-s}A_2\}/[n_1+n_2,2] + q^{-(n_1+n_2)(e_3-e_2-1)/2-n_1-(e_3-e_2)s}\psi(e_3-e_2)$$

if e_3-e_2 is odd. Again we shall separate cases.

Case 2.1 e_3-e_2 even.

If we put

$$Q''(x)=Q_1(x_1)+\pi(Q_2(x_2)+Q_3(x_3))+\sum_{i\geqslant 4}\pi^{e_i-e_3+1}Q_i(x_i),$$

then we get

$$\psi(e_3-e_2)=\int_L|Q''(x)|_K^s\,dx.$$

Since $L_1+(L_2+L_3)+\cdots+L_t$ is the Jordan decomposition of (L,Q''), by induction we get

$$\psi(e_3-e_2)=B+q^e\sum_{i\geqslant 4}Z_i,$$

in which

$$B=A_1+[1][0,1]/[1,1][n_1,2][n_1+n_2+n_3,2]\cdot q^{-n_1/2-s}$$

$$\{\chi(L_1)[n_2+n_3]q^{-n_1-s}+[n_1+n_2+n_3,2]q^{-n_1/2}$$
$$-\chi(L_2+L_3)[n_1,2]q^{-(n_1+n_2+n_3)/2}\},$$
$$e=n_1(e_3-e_1-1)/2+n_2(e_3-e_2)/2+(e_3-1)s.$$

Therefore, as before, we have only to verify that

$$Z_2=q^{-n_1(e_2-e_1-1)/2-(e_2-1)s}/[n_1+n_2,2]$$
$$\cdot\{([n_1,2]-[n_1+n_2,2])A_1+[n_2,2]q^{-n_1-s}A_2\},$$
$$Z_3=q^{-e}\{B-([n_1,2]A_1+[n_2,2]q^{-n_1-s}A_2)/[n_1+n_2,2]\}.$$

The first identity follows from Lemma 2. As for the second identity, we eliminate A_2 by Lemma 2; then we see that the coefficients of $\chi(L_2)$, $\chi(L_2+L_3)$ on both sides are trivially equal and the coefficients of $\chi(L_1)$ are equal by Lemma 3.

Case 2.2: e_3-e_2 odd

If we put

$$Q''(x)=Q_2(x_2)+\pi(Q_1(x_1)+Q_3(x_3))+\sum_{i\geq 4}\pi^{e_i-e_3+1}Q_i(x_i),$$

then $L_2+(L_1+L_3)+\cdots+L_t$ becomes the Jordan decomposition of (L,Q'') and

$$\psi(e_3-e_2)=\int_L|Q''(x)|_K^s\mathrm{d}x.$$

Therefore if we apply the permutation of the subscripts 1 and 2 to the above expressions for B, e, we get

$$\psi(e_3-e_2)=B+q^e\sum_{i\geq 4}Z_i$$

with the new B, e. Furthermore the identity for Z_2 to be verified is the same as in the previous case and the one for Z_3 becomes

$$Z_3=q^{-e}\{B-([n_2,2]A_2+[n_1,2]q^{-n_2-s}A_1)/[n_1+n_2,2]\}.$$

We need no new verification because the old Z_3 and the new Z_3 differ only by the permutation of the subscripts 1 and 2. The induction is thus complete and Theorem 1 is proved.

6. Remarks

We have shown in Theorem 1 that

$$\int_L\left|\sum_{1\leq i\leq t}\pi^{e_i}Q_i(x_i)\right|_K^s\mathrm{d}x=\sum_{1\leq i\leq t}Z_i,$$

in which L is the direct sum of free O_K-submodules $L_1,\ldots,L_t$ of V, (L_i,Q_i) is unimodular for $1 \leqslant i \leqslant t$, $e_1 < \cdots < e_t$. We observe that if $dx_1 \cdots dx_i$ denotes the Haar measure on $K(L_1 + \cdots + L_i)$ such that $L_1 + \cdots + L_i$ is of measure 1, then

$$\int_{L_1 + \quad + L_i} \left| \sum_{j \leqslant i} \pi^{e_j} Q_j(x_j) \right|_K^s dx_1 \cdots dx_i = \sum_{j \leqslant i} Z_j$$

for $1 \leqslant i \leqslant t$. This is the first remark.

The second remark is that in Theorem 1 the condition $e_1 < \cdots < e_t$ can be replaced by $e_1 \leqslant \cdots \leqslant e_t$. This can be proved as follows: if $e_i = e_{i+1}$, we put

$$L_i^* = L_i + L_{i+1}, \quad n_i^* = n_i + n_{i+1}, \quad e_i^* = e_i$$

and define $M_1^*, M_2^*, M_3^*, m_1^*, m_2^*$ relative to $L_1 + \cdots + L_i^*$; then $M_1^* = M_1$, $M_2^* = M_2$, $m_1^* = m_1$. Therefore on the RHS we get

$$\begin{aligned} Z_i + Z_{i+1} = {} & [1][0,1]/[1,1][m_1^*,2][m_2^*,2] \cdot q^{-1/2 \cdot \Sigma_{j<i} n_j(e_i^* - e_j) - e_i^* s} \\ & \cdot \{\{\chi(M_1^*)[n_1^*]q^{-m_1^* - s} + \chi(M_2^*)[m_2^*,2]q^{-n_1^*/2} \\ & - \chi(M_3^*)[m_1^*,2]q^{-m_2^*/2}\}. \end{aligned}$$

In fact, this is an identity with $\chi(M_1)$, $\chi(M_2)$, $\chi(M_3)$, $\chi(M_3^*)$ as variables: The coefficients of $\chi(M_2)$, $\chi(M_3)$, $\chi(M_3^*)$ on both sides are trivially equal and the coefficients of $\chi(M_1)$ are equal by Lemma 3. Therefore we can shorten $Z_1 + \cdots + Z_t$ as many times as we have equalities in $e_1 \leqslant \cdots \leqslant e_t$ and the shortened expression is equal to $Z(s)$ on the LHS by Theorem 1.

7. $Z(s)$ for quadratic polynomials

If $f(x)$ is any quadratic polynomial on V with nondegenerate second degree part, we can eliminate the first degree part by translation in V. We shall discuss two inhomogeneous cases which often appear in applications.

Theorem 2. *Let Q denote any nondegenerate quadratic form on V, L a lattice in V, and $L = L_1 + \cdots + L_t$ the Jordan decomposition of (L, Q) as in Theorem 1; take any integer $e \geqslant e_t$, u from $O_K^\times$, and define a quadratic form Q^* on $V + K$ as $Q^*(x+y) = Q(x) + \pi^e u y^2$; and put*

$$M_1 = \sum_{e_i \not\equiv e} L_i, \quad M_2 = \sum_{e_i \equiv e} L_i, \quad M_3 = M_2 + O_K.$$

Then

$$\begin{aligned} \int_L |Q(x) + \pi^e u|_K^s dx = {} & \int_L |Q(x)|_K^s dx \\ & + [0,1]|D(L,Q)|_K^{-1/2} q^{-(n/2+s)e}/[1,1][n,2] \\ & \cdot \{\chi(M_1)[1]q^{-n-s} + \chi(M_2)[n+1,2]q^{-n/2} \\ & - \chi(M_3)[n,2]q^{-(n+1)/2}\}, \end{aligned}$$

in which $n = \dim(V)$ and $\chi(M_3)$ is defined relative to $(L + O_K, Q^*)$.

Proof. If we denote by $\mathrm{d}y$ the Haar measure on K such that O_K is of measure 1, then by splitting O_K into $O_K^\times$ and πO_K we get

$$\int_{L+O_K} |Q(x)+\pi^e uy^2|_K^s \mathrm{d}x\mathrm{d}y = [1]\cdot\int_L |Q(x)+\pi^e u|_K^s \mathrm{d}x + q^{-1}$$
$$\int_{L+O_K} |Q(x)+\pi^{e+2}uy^2|_K^s \mathrm{d}x\mathrm{d}y.$$

We apply Theorem 1 to the two integrals over $L+O_K$ by using the remarks in the previous section. Then by a small elementary computation we get the formula in the theorem.

COROLLARY

In the same notation as in Theorem 2,

$$\int_{L+O_K} |Q(x)+\pi^e y|_K^s \mathrm{d}x\mathrm{d}y = \int_L |Q(x)|_K^s \mathrm{d}x$$
$$+[1][0,1]|D(L,Q)|_K^{-1/2} q^{-(n/2+s)e}/[1,1][n,2]$$
$$\cdot\{\chi(M_1)q^{-n-s}+\chi(M_2)q^{-n/2}\}.$$

Proof. By splitting O_K into $\pi^{2k}O_K^\times$, $\pi^{2k+1}O_K^\times$ and $\{0\}$ for all $k\geqslant 0$ we get

$$\int_{L+O_K} |Q(x)+\pi^e y|_K^s \mathrm{d}x\mathrm{d}y = \sum_{k\geqslant 0} q^{-2k}\int_{O_K^\times}\left\{\int_L |Q(x)+\pi^{e+2k}u|_K^s \mathrm{d}x\right\}\mathrm{d}u$$
$$+\sum_{k\geqslant 0} q^{-2k-1}\int_{O_K^\times}\left\{\int_L |Q(x)+\pi^{e+2k+1}u|_K^s \mathrm{d}x\right\}\mathrm{d}u.$$

We apply Theorem 2 to the two integrals over L and then use the fact that the integral of $\chi(M_3)$ over $O_K^\times$ is 0 in view of

$$\int_{O_K^\times} \chi(u)\mathrm{d}u = [1]/2\cdot(1-1)=0.$$

The rest is a small elementary computation.

8. An application

A typical form of difficult auxiliary integrals we have encountered is

$$\int_{O_K^m\times O_K^n} |{}^t yA(x)y+b(x)|_K^s \mathrm{d}x\mathrm{d}y,$$

in which $A(x)$ is a nonsingular symmetric matrix of degree n with entries in $O_K[x_1,\ldots,x_m]$ and, up to a factor in O_K, certain powers of $\det(A(x))$ and $b(x)$ are equal; cf., e.g., [3], p. 218. In this last section we shall give a closed form to the simplest integral of that form, i.e., to

$$Z(s) = \int_{\mathrm{Sym}_n(O_K) \times O_K^n} |{}^t yxy \pm \pi^e \det(x)|_K^s \, dx\,dy,$$

in which Sym_n denotes the space of symmetric matrices of degree n and $e \geqslant 0$.

We first state the Jordan decomposition theorem in a matrix form: We choose representatives $1, \varepsilon$ of $O_K^\times/(O_K^\times)^2$ so that $\Sigma\chi(u)$ for $u = 1, \varepsilon$ is 0; we let $GL_n(O_K)$ act on $\mathrm{Sym}_n(O_K) \cap GL_n(K)$ as $(g, x) \mapsto g \cdot x = gx\,{}^t g$. Then every orbit has a unique representative of the form

$$h = \begin{pmatrix} \pi^{e_1} h_1 & & \\ & \ddots & \\ & & \pi^{e_t} h_t \end{pmatrix}, \quad h_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & u_i \end{pmatrix},$$

in which $u_i = 1, \varepsilon$ for every i and $0 \leqslant e_1 < \cdots < e_t$. Furthermore if

$$\deg(h_i) = n_i, \quad d_i = (-1)^{n_i(n_i-1)/2} u_i$$

for $1 \leqslant i \leqslant t$ and if μ denotes the Haar measure on $\mathrm{Sym}_n(K)$ such that $\mathrm{Sym}_n(O_K)$ is of measure 1, then

$$\mu(GL_n(O_K) \cdot h) = \mu(GL_n) / \prod_{1 \leqslant i \leqslant t} \mu(O(h_i)) \cdot q^{-1/2 \cdot \Sigma_i n_i e_i (n_i + 2 \cdot \Sigma_{j > i} n_j + 1)},$$

in which $\mu(GL_n)$, $\mu(O(h_i))$ are the canonical volumes of $GL_n(O_K)$, $O(h_i)(O_K)$:

$$\mu(GL_n) = \prod_{1 \leqslant j \leqslant n} [j],$$

$$\mu(O(h_i)) = 2 \cdot \prod_{1 \leqslant j < n_i/2} [2j] \cdot \begin{cases} 1 - \chi(d_i) q^{-n_i/2} & n_i \text{ even} \\ 1 & n_i \text{ odd} \end{cases}$$

for $1 \leqslant i \leqslant t$. The above formula must be well known; it can be proved by an induction on t. The following lemma can also be proved by an induction on t:

Lemma 4. *Let s_i denote any complex number with* $\mathrm{Re}(s_i) > 0$ *and* $r_i = 0, 1$ *for* $1 \leqslant i \leqslant t$; *put*

$$\Phi(r, s) = \sum_{k_i} q^{-(k_1 s_1 + \cdots + k_t s_t)},$$

in which $k_1, \ldots, k_t$ are integers satisfying $0 \leqslant r_1 + 2k_1 < \cdots < r_t + 2k_t$. Then the series is absolutely convergent and it represents

$$q^{-\Sigma_{1 < i \leqslant t}(i - 1 + \Sigma_{1 < j \leqslant i} r_j (r_{j-1} - 1)) s_i} \Big/ \prod_{1 \leqslant i \leqslant t} (1 - q^{-\Sigma_{i \leqslant j \leqslant t} s_j}).$$

We have separated a special case of Theorem 3 in the following lemma:

Lemma 5. *We have*

$$\int_{\mathrm{Sym}_n(O_K)\times O_K^n} |{}^t yxy|_K^s \mathrm{d}x\mathrm{d}y = [1][n]/[1,1][n,2].$$

Proof. We put $W_n = O_K^n - \pi O_K^n$ and split O_K^n into $\pi^k W_n$ and $\{0\}$ for all $k \geqslant 0$; then we get

$$\int_{\mathrm{Sym}_n(O_K)\times O_K^n} |{}^t yxy|_K^s \mathrm{d}x\mathrm{d}y = \sum_{k\geqslant 0} q^{-(n+2s)k} \int_{W_n} \left\{ \int_{\mathrm{Sym}_n(O_K)} |{}^t yxy|_K^s \mathrm{d}x \right\} \mathrm{d}y.$$

If $\eta = {}^t(10\cdots 0)$, then every y in W_n can be written as $g\eta$ for some g in $GL_n(O_K)$ and the action of $GL_n(O_K)$ on $\mathrm{Sym}_n(O_K)$ is measure preserving. Therefore the above integral over $\mathrm{Sym}_n(O_K)$ is $[1]/[1,1]$, hence the RHS becomes $[1]/[1,1]\cdot[n]/[n,2]$.

Theorem 3. *We take a partition* $n = n_1 + \cdots + n_t$ *of* n, *choose* $r_i = 0, 1$, *put*

$$s_i = n_i\left(n + 2s + n_i + 2\sum_{j>i} n_j\right)$$

for $1 \leqslant i \leqslant t$, *and for a given* $e \geqslant 0$ *we split* $\{1, 2, \ldots, t\}$ *into two subsets* I, J *as*

$$I(resp. J) = \left\{ i; r_i \equiv (resp. \not\equiv) e + \sum_{1\leqslant j\leqslant t} n_j r_j \bmod 2 \right\};$$

and finally we define A_S *for* $S = I, J$ *as*

$$A_S = \prod_{i\in S} \delta_0(n_i) q^{-n_i/2} \Big/ \prod_{1\leqslant i\leqslant t}\ \prod_{1\leqslant j\leqslant n_i/2} [2j],$$

in which $\delta_r(m)$ *for any integers* r, m *represents* 1 *or* 0 *according as* $m - r$ *is even or odd. Then*

$$\int_{\mathrm{Sym}_n(O_K)\times O_K^n} |{}^t yxy + (-1)^{n(n+1)/2}\pi^e \det(x)|_K^s \mathrm{d}x\mathrm{d}y = [1][n]/[1,1][n,2]$$

$$+ \prod_{1\leqslant i\leqslant n} [i]\cdot[0,1] q^{-(n/2+s)e}/[1,1][n,2]$$

$$\cdot \sum_{n_i}\sum_{r_i} \{[n+1,2]A_I + ([1]q^{-n/2-s} - \delta_1(n)[n,2]q^{-1/2})A_J\}$$

$$\cdot q^{-1/2\cdot(n+\Sigma_j r_j s_j)}\Phi(r,s)$$

with $\Phi(r,s)$ *as in Lemma* 4.

Proof. We decompose the LHS according to the above-recalled splitting of $\mathrm{Sym}_n(O_K)\cap GL_n(K)$ and apply Theorem 2 to each integral. Then by using the expression for $\mu(GL_n(O_K)\cdot h)$ and Lemma 5 we get

$$\mathrm{LHS} = [1][n]/[1,1][n,2] + \mu(GL_n)[0,1]q^{-(n/2+s)e}/[1,1][n,2]$$

$$\sum_{n_i, e_i, u_i} q^{-1/2\cdot\Sigma_j s_j e_j} \Big/ \prod_{1\leq j\leq t} \mu(O(h_j))$$
$$\cdot\{\chi(M_1)[1]q^{-n-s} + \chi(M_2)[n+1,2]q^{-n/2} - \chi(M_3)[n,2]q^{-(n+1)/2}\}.$$

Furthermore if $e_i \equiv r_i \bmod 2$ with $r_i = 0, 1$ for $1 \leq i \leq t$, then

$$\sum_{u_1,\ldots,u_t = 1,\varepsilon} 1 \Big/ \prod_{1\leq j\leq t} \mu(O(h_j))\cdot\chi(M_k) = A_J, A_I, \delta_1(n)A_J$$

respectively for $k = 1, 2, 3$. Therefore the summation in e_i, u_i above becomes

$$\sum_{r_i}\{[n+1,2]A_I + ([1]q^{-n/2-s} - \delta_1(n)[n,2]q^{-1/2})A_J\}\cdot q^{-1/2\cdot(n+\Sigma_j r_j s_j)}\Phi(r,s).$$

This completes the proof.

We might mention that in Theorem 3 if we replace $(-1)^{n(n+1)/2}$ by ± 1, then we have only to insert the factor $\chi(\pm(-1)^{n(n+1)/2})$ after $\delta_1(n)$. The special case of Theorem 3 for $n = 4$, $e = 1$ appears as the lowest codimensional new partial integral in the computation of $Z(s)$ for the degree 8 invariant of Spin_{14}. In that case the theorem gives

$$\int_{\mathrm{Sym}_4(O_K)\times O_K^4} |{}^t yxy + \pi\det(x)|_K^s \mathrm{d}x\mathrm{d}y = [1]\{[4] + [3]q^{-4-s}$$
$$-[7]q^{-5-2s}\}/[1,1][5,2][7,2].$$

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