On a problem of G Fejes Toth

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Dedicated to the memory of Professor K G Ramanathan

Abstract. A solution is given for the following Problem of G Fejes Toth: In 3-space find the thinnest lattice of balls such that every straight line meets one of the balls.

Keywords. Spheres (balls); lattices; thinnest arrangements.

1. Introduction

1.1 The object of this note is to give a solution of the following problem of G Fejes Toth [2]:

In 3-space find the thinnest lattice arrangement of closed balls such that every straight line meets these balls.

As pointed out by G Fejes Toth himself this is in some sense the first unsolved case of the more general problem:

In n-space find the thinnest lattice arrangement of closed balls such that every k-dimensional (0 ≤ k ≤ n − 1) flat meets one of these balls.

For k = 0, this is the problem of thinnest lattice coverings by spheres, while for k = n − 1, Makai [4] has shown that the problem can be reduced to that of the closest lattice packings of spheres. Thus the solution is known for k = 0, n ≤ 5 and for 0 ≤ k = n − 1 ≤ 7. (See any book dealing with packings and coverings, e.g. Lekkerkerker and Gruber [3]). The problem above can be generalised to one for other "bodies" also. In the case of convex bodies, Makai [4] has shown that a theorem analogous to the one for spheres holds if k = n − 1. Our solution to the Fejes Toth problem stated in the beginning is contained in the following Theorems I and II and the remark after Theorem II.

(We shall throughout be working in the three-dimensional Euclidan space R^3).

Theorem I. Let K be the sphere |x| ≤ 1. Let \Lambda be a lattice with determinant d(\Lambda). If every straight line meets a ball K + A, A∈\Lambda, then d(\Lambda) ≤ 2(4/3)^3.

Theorem II. Let K be the sphere |x| ≤ 1 and \Lambda be the lattice generated by 4/3(1, 1, 0), 4/3(0, 1, 1) and 4/3(1, 0, 1). Then every straight line meets a sphere K + A, A∈\Lambda.
Remark Our proof of Theorem I (see §4.4) shows that “up to” orthogonal transformations the lattice \( \Lambda \) of Theorem II is the only “critical” lattice.

For convenience we replace Theorems I and II by the equivalent Theorems I', II':

**Theorem I'.** Let \( K \) be the sphere \( |x| \leq 3/4 \) and \( \Lambda \) a lattice with determinant \( d(\Lambda) \). If every straight line meets a ball \( K + \Lambda, \Lambda \in \Lambda \), then \( d(\Lambda) \leq 2 \).

**Theorem II'.** Let \( K \) be the sphere \( |x| \leq 3/4 \) and \( \Lambda \) the lattice generated by \((1,1,0), (0,1,1)\) and \((1,0,1)\). Then every straight line meets a \( K + \Lambda, \Lambda \in \Lambda \).

2. Proof of Theorem I'

2.1. Let \( K \) be the sphere \( |x| \leq 3/4 \) and \( \Lambda \) a lattice. Let \( \Lambda_1 \in \Lambda \). Let \( \Pi \) be the plane through \( O \) perpendicular to \( OA_1 \). Let \( \Lambda^* \) be the (orthogonal) projection of \( \Lambda \) on \( \Pi \). Let \( C \) be the circle \( K \cap \Pi \). All lines parallel to \( OA_1 \) meet a \( K + \Lambda, \Lambda \in \Lambda \) is equivalent to the statement: the circles \( C + \Lambda^*, \Lambda^* \in \Lambda^* \) cover \( \Pi \), i.e. the “covering radius” \( \rho(\Lambda^*) \) of \( \Lambda^* \) is \( \leq 3/4 \).

2.2. Let \( A_1, A_2, A_3 \) be a basis of \( \Lambda \). Let \( L \) be the matrix \((A_1, A_2, A_3)\) with \( A_1, A_2, A_3 \) written as column vectors. The positive definite quadratic form \( f(x) = f(x_1, x_2, x_3) = X' L' L X \), where \( X' = (x_1, x_2, x_3) \) is called the quadratic form of \( \Lambda \) w.r.t. the basis \( A_1, A_2, A_3 \). Its determinant \( d(f) = \det(L' L) = d^2(\Lambda) \). If \((B_1, B_2, B_3) = (A_1, A_2, A_3) \) is any other basis of \( \Lambda \), the \( U \in \text{GL}(3, \mathbb{Z}) \) and the corresponding quadratic form \( X' U' L' L U X \) is equivalent to \( f(X) \). In fact the quadratic forms corresponding to different bases of \( \Lambda \) consist of the class of quadratic forms equivalent to \( f \).

Again if \( f(x) = X' L' L X = X' M' M X \), then \( M = T L \), where \( T \) is orthogonal and the lattice \( T \Lambda \) with basis \( T A_1, T A_2, T A_3 \) is an orthogonal transform of \( \Lambda \). We may note that \( T K = K \), and \( \Lambda \) has the property of Theorem I' if and only if \( T \Lambda \) has.

2.3. Let \( f(x) = \sum a_{ij} x_i x_j \), \( a_{ij} = a_{ji} \) be the real positive definite quadratic form corresponding to a basis \( A_1, A_2, A_3 \) of \( \Lambda \). Write

\[
 f = a_{11} \left( x_1 + \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3 \right)^2 + g(x_2, x_3) = \left( a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \right)^2 + \left( a_{22} x_2 + a_{23} x_3 \right)^2 + \left( a_{32} x_2 + a_{33} x_3 \right)^2,
\]

and \( f \) is the quadratic form of a lattice \( \Lambda_1 = T \Lambda, T \) orthogonal, with respect to the basis \( B_1 = T A_1, B_2 = T A_2, B_3 = T A_3 \), and \( B_1 = (a_{11}, 0, 0), B_2 = (a_{12}, a_{22}, a_{32}), B_3 = (a_{13}, a_{23}, a_{33}) \). Every line parallel to \( OA_1 \) meets a \( K + \Lambda, \Lambda \in \Lambda \) if and only if every line parallel to \( OB_1 \) meets a \( K + B, B \in \Lambda_1 \). Since \( B_1 \) is the point \((a_{11}, 0, 0)\), the plane \( \Pi \) of 2.1 is \( x_1 = 0 \) and the projection \( \Lambda^* \) of \( \Lambda_1 \) on \( \Pi \) is the lattice generated by \((0, a_{22}, a_{32})\) and \((0, a_{23}, a_{33})\), while

\[
 g(x_2, x_3) = (a_{22} x_2 + a_{23} x_3)^2 + (a_{32} x_2 + a_{33} x_3)^2.
\]

Let \( \rho = \rho(\Lambda^*) \) be the covering radius of \( \Lambda^* \) and \( R(g) = \rho^2 \). \( R(g) \) depends only on \( g \),
because if $g$ is a quadratic form of another lattice $\Lambda^*_1$, then $\Lambda^*_1 = T\Lambda^*$, where $T$ is orthogonal and the covering radius of $\Lambda^*_1$ is the same as that of $\Lambda^*$.)

By §2.1 all lines parallel to $OA$, meet a $K + \Lambda$, $\Lambda \in \Lambda$ if and only if $\rho(\Lambda)^* \leq 3/4$, if and only if $R(\Lambda) \leq 9/16$. Since every primitive lattice point can be extended to a basis of $\Lambda$, all lines parallel to lines $OA$, $\Lambda \in \Lambda$ meet the balls $K + \Lambda$, $\Lambda \in \Lambda$ if and only if for all forms $f' \sim f$, the corresponding "sections" $g'(x_2, x_3)$ have $R(g') \leq 9/16$. More precisely, the hypothesis of Theorem I' implies the following:

Let $\Lambda$ be a lattice. Let $f(x) = \sum a_{ij}x_ix_j$, $a_{ij} = a_{ji}$ be any quadratic form of $\Lambda$. Let

$$f(x) = a_{11} \left( x_1 + \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3 \right)^2 + g(x_2, x_3).$$

Then

$$R(g) \leq 9/16.$$

To prove Theorem I' it is enough to prove

**Theorem 1A.** Let $f(x) = \sum a_{ij}x_ix_j$, $a_{ij} = a_{ji}$ be a real positive definite quadratic form with determinant $d(f)$. Let $f' \sim f$; write

$$f'(x) = a'_{11} \left( x_1 + \frac{a'_{12}}{a'_{11}} x_2 + \frac{a'_{13}}{a'_{11}} x_3 \right)^2 + g'(x_2, x_3).$$

If $R(g') \leq 9/16$ for each $f' \sim f$, then $d(f) \leq 4$.

2.4. Let $f(x) = \sum a_{ij}x_ix_j$, $a_{ij} = a_{ji}$ be a positive definite quadratic form. Let

$$f(x) = a_{11} \left( x_1 + \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3 \right)^2 + g(x_2, x_3).$$

Then

$$a_{11}g = (a_{11}a_{22} - a_{12}^2)x_2^2 + 2(a_{11}a_{23} - a_{12}a_{13})x_2x_3 + (a_{11}a_{33} - a_{13}^2)x_3^2$$

$$= A_{33}x_2^2 - 2A_{23}x_2x_3 + A_{22}x_3^2$$

$$= G', \text{ say,}$$

where $A_{ij}$ are the entries of the matrix adjoint to $(a_{ij})$. Since $g = a_{11}^{-1}G'$, $R(g) = a_{11}^{-1}R(G')$. If

$$G = A_{22}x_2^2 + 2A_{23}x_2x_3 + A_{33}x_3^2,$$

then $G \sim G'$ and

$$R(G) = R(G'), \text{ and}$$

$$R(g) = a_{11}^{-1}R(G).$$

(a)

Let $A = (a_{ij})$, $\text{adj } A = (A_{ij})$. Then $A \text{ adj } A = \det(A)I$, and $\det(\text{adj } A) = (\det A)^2$. Write

$$F(x) = \text{adj } f(x) = \sum A_{ij}x_ix_j$$

Then

$$d(F) = \det(A_{ij}) = (\det A)^2 = d^2(f).$$

(b)

Since

$$A(\text{adj } A) = (\det A)I = d(f)I, \text{ and } (\text{adj } A) \text{ adj } A = d(F)I = d^2(f)I,$$
we have
\[ \frac{1}{d(f)}A = \frac{1}{d^2(f)} \text{adj}(\text{adj} A) \]
i.e.
\[ \frac{1}{d(f)}(a_{ij}) = \frac{1}{d^2(f)} \text{adj}(A_{ij}) \]
Equating elements in the leading position, we get
\[ \frac{1}{d(f)}a_{11} = \frac{1}{d^2(f)}(A_{22}A_{33} - A_{23}^2) \]
\[ = \frac{1}{d^2(f)}d(G), \]
and \( a_{11}^{-1} = d(f)/d(G) = \sqrt{d(F)/d(G)}, \) and, by (a),
\[ R(g) = R(G)\sqrt{d(F)/d(G)}. \]
Therefore,
\[ R(g) \leq 9/16 \text{ iff } R(G) \leq 9/16 \text{ d}(G)/d(F)^{1/2} \quad (c) \]
and
\[ d(F) = d^2(f). \quad (d) \]
It is well known that if \( f \sim f' \), then \( \text{adj} f \sim \text{adj} f' \) and vice versa, i.e., the class of forms equivalent to \( \text{adj} f \) is the class of adjoints of forms \( \sim f \).

Let \( F(x_1, x_2, x_3) = \sum A_{ij}x_i x_j \) be a definite quadratic form and \( F_1 \sim F \). Let \( G(x_2, x_3) = F_1(0, x_2, x_3) \) be called a partial sum of \( F \) and let \( S \) be the set of partial sums of \( F \). Since \( F(x_1, x_2, x_3) \sim F(x_3, x_1, x_2) \) the set of partial sums of \( F \) consists of the forms \( G(x_2, x_3) = F'(x_1, x_2, 0) \) for all forms \( F' \sim F(x) \).

We can replace Theorem IA by (see (c) and (d) above).

**Theorem IB.** Let \( F(x_1, x_2, x_3) = \sum A_{ij}x_i x_j, A_{ij} = A_{ji} \) be a positive definite quadratic form. Suppose for every partial sum \( G \) of \( F \) we have \( R(G) \leq 9/16 \text{ d}(G)/\sqrt{d(F)}. \) Then \( d(F) \leq 16. \)

It is clear that we can replace \( F \) by any equivalent form without affecting the hypothesis or conclusion of the theorem. For convenience we alter the notation a little bit and state Theorem IB as:

**Theorem IC.** Let \( f(x_1, x_2, x_3) = \sum a_{ij}x_i x_j, a_{ij} = a_{ji} \) be a positive definite quadratic form. Suppose for every partial sum \( g(x_1, x_2) = f'(x_1, x_2, 0) \), where \( f' \sim f \), we have \( R(g) \leq 9/16 \text{ d}(g)/\sqrt{d(f)}, \) then \( d(f) \leq 16. \)

### 3. Proof of Theorem IC

#### 3.1 A basis \( A, B \) of a two-dimensional lattice \( \Lambda \) is said to be reduced if the angle \( \Theta \) of the \( \Delta OAB \) is largest and lies between \( 60^\circ \) and \( 90^\circ \), equivalently \( \Delta OAB \) is acute
angled with largest angle at $O$. In this case the covering radius of $\Lambda$ is the circumradius of $\Delta OAB$. (see e.g. Dickson [1], pp. 160).

Now suppose $A, B$ generate a two-dimensional lattice and $\Delta OAB$ is acute angled. Then $(A_1, B_1) = (A, B)$ or $(-A, B - A)$ or $(-B, A - B)$ is a reduced basis of $\Lambda$ and its covering radius is the circumradius of $\Delta OA_1 B_1 = \text{the circumradius of } \Delta OAB$. Thus if $A, B$ generate $\Lambda$ and $\Delta OAB$ is acute angled, then the covering radius $\rho(\Lambda)$ of $\Lambda$ is the circumradius of $\Delta OAB$.

Let $g(x, y) = ax^2 + 2bxy + cy^2$ be positive definite. Let $g(x, y) = (ax + by)^2 + (yx + \delta y)^2$.

Let $A = (x, \gamma)$, $B = (\beta, \delta)$. Then $A, B$ generate a lattice $\Lambda$ and $\rho(\Gamma) = \rho^2(\Lambda)$. The triangle $OAB$ is acute angled if the square of each side $\leq$ sum of squares of the other two sides, i.e., if

\[
\begin{align*}
\frac{a}{c} &\leq \frac{a + c - 2b}{a + c} , \\
\frac{b}{a} &\leq \frac{a + c - 2b}{a + c} , \\
\frac{c}{a} &\leq \frac{a + c - 2b}{a + c} ,
\end{align*}
\]

i.e.

\[
\begin{align*}
b &\leq c , \\
a &\leq a , \\
b &\geq 0 , \text{ i.e.} \\
0 &\leq b \leq \min(a, c).
\end{align*}
\]

Therefore, if $0 \leq b \leq \min(a, c)$, then $R(g) = \text{(circumradius of triangle OAB)} = ac(a + c - 2b)/4 \, d(g)$. (If $ABC$ is an acute angle triangle with sides $a, b, c$ circumradius $\rho$ and area $\Delta$, then

\[
\frac{\rho}{\rho^3} = \frac{a b c}{8 \sin A \sin B \sin C} = \frac{a^3 b^3 c^3}{64(1/2 \sin A)(1/2 \sin B)(1/2 \sin C)}
\]

so that

\[
\rho^2 = \frac{a^2 b^2 c^2}{4(2\Delta)^2}.
\]

3.2 Let $f(x_1) = \Sigma a_{ij} x_j x_j$, $a_{ij} = a_{ji}$ be a positive definite form, all of whose partial sums $g(x_1, x_2)$ have $R(g) \leq 9/16 \, d(g)/\sqrt{d(f)}$. We have to show $d(f) \leq 16$.

By replacing $f$, by an equivalent form reduced in the sense of Gauss and Sieber (see, e.g. Dickson [1], Th 103, pp. 171), we can suppose

\[
\begin{align*}
0 &\leq a_{11} \leq a_{22} \leq a_{33} , \\
2|a_{12}| &\leq a_{11} , \\
2|a_{13}| &\leq a_{11} , \\
2|a_{23}| &\leq a_{22} , \\
\text{and} \\
a_{ij} , i \neq j , \text{ all have the same sign} , \\
a_{11} + a_{22} + 2(a_{12} + a_{13} + a_{23}) &\geq 0 .
\end{align*}
\]
We divide the proof into two cases:

**case I:** all \( a_{ij}, i \neq j, \) are negative (or 0),

**case II:** all \( a_{ij}, i \neq j, \) are positive (or 0).

4. Proof of Theorem IC Case I

4.1 Clearly \( g_1 = f(0, x_2, x_3), \ g_2 = f(x_1, 0, x_3) \) and \( g_3 = f(x_1, x_2, 0) \) are all partial sums of \( f. \) If \( \Sigma A_{ij} x_i x_j \) is adjoint to \( f, \) then

\[
d(g_1) = A_{11}, \ d(g_2) = A_{22}, \ d(g_3) = A_{33}.
\]

Also each \( g \) is equivalent to one with the cross term of opposite sign. Therefore, applying the formula of §3.1,

\[
R(g_1) = a_{22} a_{33} (a_{22} + a_{33} + 2a_{23}) / 4 A_{11},
R(g_2) = a_{33} a_{11} (a_{33} + a_{11} + 2a_{31}) / 4 A_{22}, \text{ and}
R(g_3) = a_{11} a_{22} (a_{11} + a_{22} + 2a_{12}) / 4 A_{33}
\]

By the hypothesis \( R(g_i) \leq 9/16 \ d(g_i) / \sqrt{d(f)}, \) and we have

\[
a_{22} a_{33} (a_{22} + a_{33} + 2a_{23}) / 4 A_{11} \leq 9/16 A_{11} / \sqrt{d(f)}
\]

or

\[
4 a_{22} a_{33} (a_{22} + a_{33} + 2a_{23}) \sqrt{d(f)} \leq 9 A_{11}^2.
\]  \quad (1)

Similarly,

\[
4 a_{33} a_{11} (a_{33} + a_{11} + 2a_{31}) \sqrt{d(f)} \leq 9 A_{22}^2,
\]  \quad (2)

and

\[
4 a_{11} a_{22} (a_{11} + a_{22} + 2a_{12}) \sqrt{d(f)} \leq 9 A_{33}^2.
\]  \quad (3)

4.2 Define \( \beta_{12}, \beta_{23}, \beta_{13} \) by

\[
a_{12} = -\beta_{12} \sqrt{a_{11} a_{22}}, \ a_{13} = -\beta_{13} \sqrt{a_{11} a_{33}},
\]

\[
a_{23} = -\beta_{23} \sqrt{a_{22} a_{33}},
\]  \quad (4)

and put

\[
t_1 = (a_{11}/a_{22})^{1/2}, \ t_2 = (a_{22}/a_{33})^{1/2}.
\]  \quad (5)

The reduction conditions (A) of §3.2 give

\[
0 \leq t_1, \ t_2 \leq 1 \quad (6)
\]

\[
0 \leq \beta_{12} \leq \frac{1}{2} t_1, \ 0 \leq \beta_{13} \leq \frac{1}{2} t_1 t_2, \ 0 \leq \beta_{23} \leq \frac{1}{2} t_2,
\]  \quad (7)

and

\[
a_{11} + a_{22} + 2(a_{12} + a_{13} + a_{23}) \geq 0 \text{ becomes}
\]

\[
a_{11} + a_{22} \geq 2(\beta_{12} \sqrt{a_{11} a_{22}} + \beta_{13} \sqrt{a_{11} a_{33}} + \beta_{23} \sqrt{a_{22} a_{33}}),
\]
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so that, dividing by \( \sqrt{a_{22}a_{33}} \), we get

\[
t_1^2 + t_2 \geq 2(\beta_{12}t_1t_2 + \beta_{13}t_1 + \beta_{23}).
\] (8)

Now, if we write

\[
g(t_1, t_2) = t_1^2t_2 + t_2 - 2(\beta_{12}t_1t_2 + \beta_{13}t_1 + \beta_{23}),
\]

then

\[
\frac{\partial g}{\partial t_1} = 2t_1t_2 - 2\beta_{12}t_2 - 2\beta_{13} \geq 2t_1t_2 - t_1t_2 - t_1t_2 \quad \text{(By (7))}
\]

\[
\geq 0,
\]

\[
\frac{\partial g}{\partial t_2} = t_1^2 + 1 - 2\beta_{12}t_1
\]

\[
= 1 + t_1(t_1 - 2\beta_{12}) \geq 1 > 0. \quad \text{(By (7))}
\]

Therefore, (8) remains true if we replace \( t_1, t_2 \) by 1, i.e.

\[
\beta_{12} + \beta_{13} + \beta_{23} \leq 1.
\] (B)

Also,

\[
d(f) = a_{11}a_{22}a_{33} + 2a_{12}a_{13}a_{23} - a_{33}a_{12}^2 - a_{11}a_{23}^2 - a_{12}a_{13}^2
\]

\[
= a_{11}a_{22}a_{33}(1 - 2\beta_{12}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2)
\]

\[
= a_{11}a_{22}a_{33}\Delta, \text{ say.} \quad \text{(C)}
\]

4.3 Using inequality 1 of §4.1, together with the arithmetic geometric mean inequality, we get

\[
9A_{11}^2 \geq 4a_{22}a_{33}(a_{22} + a_{33} + 2a_{23})\sqrt{d(f)}
\]

\[
\geq 8a_{22}a_{33}(\sqrt{a_{22}a_{33} + a_{23}})\sqrt{d(f)}
\]

\[
= 8a_{22}a_{33}\sqrt{a_{11}a_{22}a_{33}\Delta}\sqrt{\sqrt{a_{22}a_{33} + a_{23}}}
\]

\[
= 8\sqrt{a_{11}\Delta(a_{22}a_{33})^{3/2}\sqrt{a_{22}a_{33} + a_{23}}},
\]

so that

\[
8\sqrt{a_{11}\Delta} \leq 9(a_{22}a_{33} - a_{23}^2)/(a_{22}a_{33})^{3/2}\sqrt{a_{22}a_{33} + a_{23}}
\]

\[
= 9\left\{1 - \frac{a_{23}^2}{a_{22}a_{33}}\right\}^{2/3}\left\{1 + \frac{a_{23}}{\sqrt{a_{22}a_{33}}}\right\}
\]

\[
= 9(1 - \beta_{23}^2)^2/(1 - \beta_{23})
\]

\[
= 9(1 - \beta_{23})(1 + \beta_{23})^2, \text{ and}
\]

\[
\sqrt{a_{11}\Delta} \leq \frac{9}{8}(1 - \beta_{23})(1 + \beta_{23})^2.
\] (9)

Similarly, (2), (3) give

\[
\sqrt{a_{22}\Delta} \leq \frac{9}{8}(1 - \beta_{23})(1 + \beta_{23})^2,
\] (10)
and

$$\sqrt{a_{33}^2} \leq \frac{9}{8} (1 - \beta_{12})(1 + \beta_{12})^2 \quad (11)$$

Multiplying (9), (10), and (11), we get

$$\sqrt{d(f)} = \sqrt{a_{11}a_{22}a_{33}^2} \leq (9/8)^3 (1 - \beta_{12})(1 - \beta_{23})(1 - \beta_{13})
\quad (1 + \beta_{12})^2(1 + \beta_{23})^2(1 + \beta_{13})^2/\Delta
\quad = h(\beta_{12}, \beta_{23}, \beta_{13}), \text{ say (D)}$$

4.4 Our object now is to use (D) above to show that the condition (B) of §4.2 (i.e. 
$$\beta_{12} + \beta_{23} + \beta_{13} \leq 1$$) implies \(\sqrt{d(f)} \leq 4\). (This will, of course, prove theorem IC in case I).

We note that if \(\beta_{12} + \beta_{23} + \beta_{13} \leq 1\), one of the \(\beta\)'s must be \(\leq 1/3\). Increasing the \(\beta\) increases the numerator of \(h\) and decreases its denominator

$$\Delta = (1 - 2\beta_{12}\beta_{23}\beta_{13} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2),$$

because

$$\frac{d}{dx} (1 - x)(1 + x)^2 = -(1 + x)^2 + 2(1 - x^2)$$

$$= (1 + x)(1 - 3x) \geq 0 \text{ if } x \leq 1/3.$$  

Increasing the \(\beta\)'s appropriately, we can assume

$$\beta_{12} + \beta_{23} + \beta_{13} = 1. \quad (E)$$

Putting \(\beta_{23} = 1 - \beta_{12} - \beta_{13}\), we have

$$\Delta = 1 - 2\beta_{12}\beta_{13}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2$$
$$= 1 - 2\beta_{12}\beta_{13} - (1 - \beta_{12} - \beta_{13}) - \beta_{12}^2 - \beta_{13}^2 - (1 - \beta_{12} - \beta_{13})^2$$
$$= 1 - 2\beta_{12}\beta_{13} + 2\beta_{12}\beta_{13}(\beta_{12} + \beta_{13}) - (1 - \beta_{12} - \beta_{13})^2$$
$$= 2(\beta_{12} + \beta_{13})(1 + \beta_{12}\beta_{13} - \beta_{12} - \beta_{13})$$
$$= 2(\beta_{12} + \beta_{13})(1 - \beta_{12})(1 - \beta_{13}), \quad (12)$$

while

$$\quad (1 - \beta_{12})(1 - \beta_{13})(1 - \beta_{23})(1 + \beta_{12})^2(1 + \beta_{13})^2(1 + \beta_{23})^2$$
$$= (1 - \beta_{12})(1 - \beta_{13})(\beta_{12} + \beta_{13})(1 + \beta_{12})^2(1 + \beta_{13})^2(2 - \beta_{12} - \beta_{13})^2,$$

so that (D) gives

$$\sqrt{d(f)} \leq (9/8)^3 (1 - \beta_{12})(1 - \beta_{13})(\beta_{12} + \beta_{13})(1 + \beta_{12})^2(1 + \beta_{13})^2$$
$$\quad (2 - \beta_{12} - \beta_{13})^2/2(\beta_{12} + \beta_{13})(1 - \beta_{12})(1 - \beta_{13})$$
$$= (9^3/2^{10})(1 + \beta_{12}^2)(1 + \beta_{13}^2)(2 - \beta_{12} - \beta_{13})^2. \quad (F)$$
Also (7) gives $0 \leq \beta_{12} \leq 1/2, 0 \leq \beta_{13} \leq 1/2$. We now observe

Lemma. The maximum of $f(x, y) = (1 + x)(1 + y)(2 - x - y)$, subject to $0 \leq x, y \leq 1$ is attained only when $x = y = 1/3$ and has the value $(4/3)^3$.

Proof. By the inequality of arithmetic geometric mean

$$f(x, y) = (1 + x)(1 + y)(2 - x - y) \leq \left(\frac{1 + x + 1 + y + 2 - x - y}{3}\right)^3 = (4/3)^3,$$

and the equality occurs if $1 + x = 1 + y = 2 - x - y = 4/3$, i.e. $x = y = 1/3$.

Using the Lemma in (F), we get

$$\sqrt{d(f)} \leq \frac{9^3}{2^{10}(4/3)^6} = 2^2 = 4,$$

which proves Theorem I(C) in this case.

We also note that $d(f)$ can be 16 only if

$$\beta_{12} = 1/3, \beta_{13} = 1/3, \beta_{23} = 1/3,$$

$$\Delta = \frac{222}{333} = 2(2/3)^3,$$

and by (9), (10), (11)

$$\sqrt{a_\|\Delta} = \frac{9}{8} \frac{2}{3} (4/3)^2$$

i.e.

$$a_\| = (4/3)^2 \frac{3^3}{16} = 3,$$

i.e.,

$$f(x_1, x_2, x_3) = 3 \sum_{1 \leq i \leq 3} x_i^2 - 2 \sum_{1 \leq i < j \leq 3} x_i x_j$$

5. Proof of Theorem I(C), Case II

5.1 In this case $f = \Sigma a_{ij} x_i x_j, a_{ij} = a_{ji}$; and

$$0 < a_{11} \leq a_{22} \leq a_{33},$$

$$0 \leq 2a_{12}, 2a_{13} \leq a_{11}, 0 \leq 2a_{23} \leq a_{22}.$$

Writing

$$a_{ij} = \beta_{ij} \sqrt{a_\| a_{jj}}, i \neq j$$

We have

$$0 \leq \beta_{ij} \leq \frac{1}{2}.$$
We divide this case into two subcases:

(a) at least one $\beta_{ij} \leq 0.459$, $i \neq j$,
(b) $0.459 < \beta_{ij} \leq 1/2$ for $i,j, i \neq j$.

6. Proof of Theorem IC Case II (a)

6.1 As in § 4.1, considering the partial sums $f(0, x_2, x_3)$, $f(x_1, 0, x_3)$, $f(x_1, x_2, 0)$, and noting $a_{ij} \geq 0$, we get

$$4a_{22}a_{33}(a_{22} + a_{33} - 2a_{23})\sqrt{d(f)} \leq 9A_{11}^2,$$  (1')

and

$$4a_{11}a_{22}(a_{11} + a_{22} - 2a_{12})\sqrt{d(f)} \leq 9A_{33}^2$$  (3')

Also

$$d(f) = a_{11}a_{22}a_{33} + 2a_{12}a_{13}a_{23} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}$$

$$= a_{11}a_{22}a_{23}(1 + 2\beta_{12}\beta_{22}\beta_{33} - \beta_{12}^2 - \beta_{23}^2 - \beta_{31}^2)$$

$$= a_{11}a_{22}a_{33}A', \text{ say}$$  (C')

from (1') and (C') we get, applying A–G mean inequality,

$$9A_{11}^2 \geq 8a_{22}a_{23}(\sqrt{a_{22}a_{33} - a_{23}^2})\sqrt{d(f)}$$

$$= 8\sqrt{a_{11}A'(a_{22}a_{33})^2(1 - \beta_{23})},$$

so that

$$8\sqrt{a_{11}A'} \leq 9(a_{22}a_{33} - a_{23}^2)/a_{22}a_{33})^2(1 - \beta_{23})$$

$$= 9(1 - \beta_{23})^2/(1 - \beta_{23})$$

$$= 9(1 - \beta_{23})(1 + \beta_{23})^2$$  (4')

Similarly, (2'), (3') and (C') give

$$8\sqrt{a_{22}A'} \leq 9(1 - \beta_{13})(1 + \beta_{13})^2$$  (5')

$$8\sqrt{a_{33}A'} \leq 9(1 - \beta_{12})(1 + \beta_{12})^2.$$  (6')

Multiplying (4'), (5'), (6'), we get

$$8^3\sqrt{d(f)}\Delta' \leq 9^3(1 - \beta_{12})(1 - \beta_{13})(1 - \beta_{23})$$

$$(1 + \beta_{12})^2(1 + \beta_{13})^2(1 + \beta_{23})^2,$$

and

$$\sqrt{d(f)} \leq (9/8)^3(1 - \beta_{12})(1 - \beta_{13})(1 - \beta_{23})$$

$$(1 + \beta_{12})^2(1 + \beta_{13})^2(1 + \beta_{23})^2/1 + 2\beta_{12}\beta_{13}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2$$

$$= F, \text{ say}.$$  (F')
Make the substitution
\[ x_1 = 1 + \beta_{12}, \quad x_2 = 1 + \beta_{13}, \quad x_3 = 1 + \beta_{23}. \]
Then
\[ 1 \leq x_i \leq 3/2, \text{ and at least one } x_i \leq 1.459. \]
Noting
\[ 2x_1 x_2 x_3 - (x_1 + x_2 + x_3 - 2)^2 \]
\[ = 2(1 + \beta_{12})(1 + \beta_{13})(1 + \beta_{23}) - (1 + \beta_{12} + \beta_{13} + \beta_{23})^2 \]
\[ = 1 + 2\beta_{12} \beta_{13} \beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2 = \Delta', \]
We get, from \((F')\),
\[ \sqrt{d(F)} \leq (9/8)^3 (2 - x_1)(2 - x_2)(2 - x_3)x_1^2 x_2^2 x_3^2 / \]
\[ = F(x_1, x_2, x_3), \text{ say.} \]
It is, therefore, enough to prove that if \(1 \leq x_1 \leq 3/2\) and at least one \(x_i \leq 1.459\), then \(F(x_1, x_2, x_3) \leq 4\).
Now \(\partial F / \partial x_1\) has the same sign as
\[ (4x_1 - 3x_1^2)(2x_1 x_2 x_3 - (x_1 + x_2 + x_3 - 2)^2) \]
\[ - (2x_2 x_3 - 2(x_1 + x_2 + x_3 - 2))x_1^2 (2 - x_1), \]
which has the same sign as
\[ (4 - 3x_1)(2x_1 x_2 x_3 - (x_1 + x_2 + x_3 - 2)^2) \]
\[ - 2x_1(2 - x_1)(x_2 x_3 - x_1 - x_2 - x_3 + 2) \]
\[ = 4x_1 x_2 x_3(1 - x_1) + (x_1 + x_2 + x_3 - 2) \]
\[ \{4x_1 - 2x_1^2 - (4 - 3x_1)(x_1 + x_2 + x_3 - 2)\} \]
\[ = 4x_1 x_2 x_3(1 - x_1) + (x_1 + x_2 + x_3 - 2) \]
\[ \{x_1^2 - (4 - 3x_1)(x_1 + x_3 - 2)\} \]
\[ = G(x_1, x_2, x_3), \text{ say.} \]
Writing \(x = ((x_2 + x_3)/2)\), and noting,
\[ x_2 x_3 \leq ((x_2 + x_3)/2)^2 = x^2, \quad 1 - x_1 \leq 0, \]
\[ G(x_1, x_2, x_3) \geq 4x_1 x^2(1 - x_1) + (x_1 + 2x - 2) \]
\[ \{x_1^2 - (4 - 3x_1)(2x - 2)\} \]
\[ = (x_1 - 2)^2 \{x_1 - 4(x - 1)^2\} \]
\[ = (x_1 - 2)^2 \{x_1 - 1 + 1 - 4(x - 1)^2\} \]
\[ \geq (x_1 - 2)^2(x_1 - 1), \quad \text{because } 0 \leq x - 1 \leq \frac{1}{2} \]
\[ \geq 0. \]
Therefore, \( (\partial F/\partial x_1) \geq 0 \). Similarly \( (\partial F/\partial x_2) \geq 0 \), \( (\partial F/\partial x_3) \geq 0 \), and the maximum of \( F \) will occur at \( x_1 = 1.459, x_2 = 1.5, x_3 = 1.5 \), so that \( F \leq F(1.459, 1.5, 1.5) = 3.99... < 4 \), and the Theorem is proved in this case.

7. Proof of Theorem IC Case II (b)

7.1 In this case \( 0.459 \leq \beta_{ij} \leq 0.5 \) for all \( i, j, i \neq j \). We first note that the inequality (1'), (2'), (3') of §6.1 is valid in this case also.

Since

\[
f(x_1, x_2, x_3) \sim f(x_1 - x_2, x_2, x_3),
\]

The form

\[
g(x_2, x_3) = \frac{f(x_1 - x_2, x_2, x_3)}{f(x_1, x_2, x_3)} = \frac{(a_{11} + a_{22} - 2a_{12})x_2^2 + 2(a_{23} - a_{13})x_2x_3 + a_{33}x_3^2}{f(x_1, x_2, x_3)}
\]

is a partial sum of \( f \).

Since

\[
g(x_2, x_3) \sim g(x_2, -x_3),
\]

\[
g(x_2, x_3) \sim (a_{11} + a_{22} - 2a_{12})x_2^2 - 2|a_{23} - a_{13}|x_2x_3 + a_{33}x_3^2 = g'(x_2, x_3), \text{ say.}
\]

Then \( R(g) = R(g') \).

Since

\[
0 \leq 2|a_{23} - a_{13}| \leq \max(2a_{23}, 2a_{13}) \leq a_{22} \leq a_{11} - 2a_{12},
\]

and

\[
|2(a_{23} - a_{13})| \leq a_{22} \leq a_{33},
\]

\[
R(g) = R(g') = a_{33}(a_{11} + a_{22} - 2a_{12})
\]

\[
(a_{11} + a_{22} - 2a_{12} + a_{33} - 2|a_{23} - a_{13}|)/4d(g),
\]

where

\[
d(g) = (a_{11} + a_{22} - 2a_{12})a_{33} - (a_{23} - a_{13})^2
\]

\[
= A_{11} + A_{22} + 2(a_{23}a_{13} - a_{12}a_{33})
\]

\[
= A_{11} + A_{22} + 2A_{12}.
\]

Since

\[
R(g) \leq \frac{9}{16}d(g)/\sqrt{d(f)},
\]

we have

\[
a_{33}(a_{11} + a_{22} - 2a_{12})(a_{11} + a_{22} + a_{33} - 2a_{12} - 2|a_{23} - a_{13}|)
\]

\[
\sqrt{d(f)} \leq \frac{9}{4}(A_{11} + A_{22} + 2A_{12})^2.
\]  

(13)

Permuting \( x_1, x_2, x_3 \), we get two similar inequalities.

Using

\[
\beta_{ij}(a_{ii}a_{jj})^{1/2} = a_{ij}, t_1 = \sqrt{a_{11}/a_{22}}, t_2 = \sqrt{a_{22}/a_{33}}, \text{ we have}
\]

\[
(a_{11} + a_{22} - 2a_{12}) = (a_{11}a_{22})^{1/2}(t_1 + t_1^{-1} - 2\beta_{12}),
\]
On a problem of G Fejes Toth

\[
(a_{11} + a_{22} + a_{33} - 2a_1a_2 - 2|a_{23} - a_{13}|)
= (a_{11} a_{22})^{1/2} \left\{ t_1 + t_1^{-1} + \frac{1}{t_1 t_2^2} - 2\beta_{12} - \frac{2}{t_1 t_2} |\beta_{23} - \beta_{13} t_1| \right\},
\]

\[A_{11} + A_{22} + 2A_{12} = a_{22} a_{33} - a_{23}^2 + a_{11} a_{33} - a_{22}^2 + 2(a_{23} a_{13} - a_{12} a_{33})
= a_{22} a_{33}(1 - \beta_{23}^2) + a_{11} a_{33}(1 - \beta_{13}^2) + 2(a_{11} a_{22})^{1/2}(\beta_{23} \beta_{13} a_{33} - \beta_{12} a_{33})
= a_{33}(a_{11} a_{22})^{1/2} \left\{ t_1^{-1} (1 - \beta_{23}^2) + t_1 (1 - \beta_{13}^2) + 2(\beta_{13} \beta_{23} - \beta_{12}) \right\},
\]

and (13) becomes

\[
\sqrt{d(f)} \leq \frac{9}{4} a_{33} a_{11} a_{22} [t_1 (1 - \beta_{13}^2) + t_1^{-1} (1 - \beta_{23}^2)]
+ 2(\beta_{13} \beta_{23} - \beta_{12})] \sqrt{a_{33} a_{11} a_{22} (t_1 + t_1^{-1} - 2\beta_{12})} \left( t_1 + t_1^{-1} + \frac{1}{t_1 t_2^2} - 2\beta_{12} - \frac{2}{t_1 t_2} |\beta_{23} - \beta_{13} t_1| \right)
\]

or

\[
\sqrt{d(f)} \leq \frac{9}{4} a_{33} [t_1 (1 - \beta_{13}^2) + t_1^{-1} (1 - \beta_{23}^2)]
+ 2(\beta_{13} \beta_{23} - \beta_{12})] \sqrt{(t_1 + t_1^{-1} - 2\beta_{12})}
\left( t_1 + t_1^{-1} + \frac{1}{t_1 t_2^2} - 2\beta_{12} - \frac{2}{t_1 t_2} |\beta_{23} - \beta_{13} t_1| \right).
\]

Now (3') can be written as

\[
\sqrt{d(f)} \leq \frac{9}{4} a_{33} [a_{11} a_{22} (a_{11} + a_{22} - 2a_{12})]
= \frac{9}{4} (a_{11} a_{22} - a_{12})^2 / a_{11} a_{22} (a_{11} + a_{22} - 2a_{12})
= \frac{9}{4} (a_{11} a_{22})^{1/2} (1 - \beta_{12}^2)^2 / (t_1 + t_1^{-1} - 2\beta_{12}),
\]

using (C'), we have

\[
\sqrt{a_{11} a_{22} a_{33}} \leq \frac{9}{4} (a_{11} a_{22})^{1/2} (1 - \beta_{12}^2)^2 / (t_1 + t_1^{-1} - 2\beta_{12})
\]

so that

\[
a_{33} \leq (9/4)^2 (1 - \beta_{12}^2)^4 / (t_1 + t_1^{-1} - 2\beta_{12})^2.\]
Substituting in (14), we get

\[
\sqrt{d(f)} \leq (9/4)^3(1 - \beta_{13}^2)^4(1 - \beta_{13}^2) + \beta_{13}^2 \beta_{23} - \beta_{12})]^{2/3} \\
\Delta'(t_1 + t_1^{-1} - 2\beta_{12})^3(t_1 + t_1^{-1} + 1/t_1 t_2^2 - 2\beta_{12} - 2/t_1 t_2) \\
|\beta_{23} - \beta_{13} t_1| \cdot (15)
\]

Since

\[
t_1 \leq 1; |\beta_{23} - \beta_{13} t_1| \leq 1, 1/t_2 \geq 1,
\]

\[
2x - 2|\beta_{23} - \beta_{13} t_1| \geq 0 \text{ if } x = \frac{1}{t_2} \geq 1,
\]

\[
\sqrt{d(f)} \leq (9/4)^3(1 - \beta_{13}^2)^4(1 - \beta_{13}^2) + \beta_{13}^2 \beta_{23} - \beta_{12})]^{2/3} \\
\Delta'(t_1 + t_1^{-1} - 2\beta_{12})^3(t_1 + t_1^{-1} - 2\beta_{12} - 2t_1^{-1} \\
|\beta_{23} - \beta_{13} t_1| \cdot (16)
\]

Writing \(t\) for \(t_1\) for convenience, we have

\[
0 \leq t \leq 1,
\]

and

\[
\sqrt{d(f)} \leq (9/4)^3(1 - \beta_{13}^2)^4(1 - \beta_{13}^2) + \beta_{13}^2 \beta_{23} - \beta_{12})]^{2/3} \\
\Delta'(t + t^{-1} - 2\beta_{12})^3(t + t^{-1} - 2\beta_{12} - 2t^{-1}|\beta_{23} - \beta_{13} t|). \quad (17)
\]

Since

\[
0 \leq \beta_{13}, \beta_{23}, \beta_{12} \leq \frac{1}{2},
\]

\[
t(1 - \beta_{13}) + t^{-1}(1 - \beta_{23}) + 2(\beta_{13} \beta_{23} - \beta_{12}) \\
\geq 3/4(t + t^{-1}) - 2\beta_{12} \\
\geq 3/2 - 1 > 0,
\]

\[
2\beta_{13} \beta_{23} \leq t\beta_{13}^2 + t^{-1} \beta_{23}^2,
\]

we have, from (17),

\[
\sqrt{d(f)} \leq (9/4)^3(1 - \beta_{13}^2)^4(t + t^{-1} - 2\beta_{12})^2/ \\
\Delta'(t + t^{-1} - 2\beta_{12})^3(t + t^{-1} - 2\beta_{12} - 2t^{-1}|\beta_{23} - \beta_{13} t|) \\
= (9/4)^3(1 - \beta_{13}^2)^4/ \\
\Delta'(t + t^{-1} - 2\beta_{12})(t + t^{-1} - 2\beta_{12} - 2t^{-1}|\beta_{23} - \beta_{13} t|). \quad (18)
\]

Now, let

\[
F(t) = t + \frac{2}{t} - 2\beta_{12} - \frac{2}{t}|\beta_{23} - \beta_{13} t|.
\]
If $\beta_{23} \geq \beta_{13} t$,

$$F'(t) = 1 - \frac{2}{t^2} + \frac{2\beta_{23}}{t^2} \leq 1 - \frac{2}{t^2} + \frac{1}{t^2}$$

$$= 1 - \frac{1}{t^2} \leq 0, \text{ because } t \leq 1,$$

while, if $\beta_{23} < \beta_{13} t$,

$$F'(t) = 1 - \frac{2}{t^2} - \frac{2\beta_{23}}{t^2} \leq 1 - \frac{2}{t^2} < 0.$$

Therefore, in all cases,

$$F(t) \geq F(1)$$

$$= 3 - 2\beta_{12} - 2|\beta_{23} - \beta_{13}|$$

$$\geq 3 - 2\beta_{12} - 2 \times 0.041$$

$$= 2.918 - 2\beta_{12},$$

because $|\beta_{23} - \beta_{13}| \leq 0.5 - 0.459 = 0.041$.

Also

$$t + \frac{1}{t} - 2\beta_{12} \geq 2 - 2\beta_{12}. $$

Therefore, (18) implies

$$\sqrt{d(f)} \leq (9/4)^3(1 - \beta_{12}^2)^4/(2.918 - 2\beta_{12})(2 - 2\beta_{12})\Delta'.$$

Now

$$\Delta' = 1 + 2\beta_{12}\beta_{13}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2,$$

$$\frac{\partial \Delta'}{\partial \beta_{13}} = 2\beta_{12}\beta_{23} - 2\beta_{13}$$

$$\leq \frac{11}{22} - 2(0.459)$$

$$< 0.$$

Similarly

$$\frac{\partial \Delta'}{\partial \beta_{23}} < 0,$$

therefore,

$$\Delta' \geq 1 + 2\beta_{12}\frac{11}{22} - \beta_{12}^2 - \frac{1}{4} - \frac{1}{4}$$

$$= \frac{1}{2}(1 + \beta_{12} - 2\beta_{12}^2)$$

$$= \frac{1}{2}(1 - \beta_{12})(1 + 2\beta_{12}).$$
Writing $\beta$ for $\beta_{1,2}$, for convenience, (19) gives
\[
\sqrt{d(f)} \leq (9/4)^3 (1 - \beta^2)^4/(2.918 - 2\beta)(1 + 2\beta)(1 - \beta)^2
\]
\[
= \frac{1}{2} (9/4)^3 \frac{(1 - \beta)^2(1 + \beta)^4}{(1.459 - \beta)(1 + 2\beta)}
\]
\[
= \frac{1}{2} (9/4)^3 \frac{1 - \beta}{1.459 - \beta} \frac{(1 - \beta)(1 + \beta)^4}{1 + 2\beta}
\]
\[
= \frac{1}{2} (9/4)^3 g(\beta) h(\beta), \text{ say.}
\]

Now
\[
g(\beta) = \frac{1 - \beta}{1.459 - \beta} = 1 - \frac{0.459}{1.459 - \beta}
\]
is a decreasing function of $\beta$. Therefore,
\[
g(\beta) \leq g(0.459).
\]

Again
\[
h(\beta) = (1 - \beta)(1 + \beta)^4/(1 + 2\beta),
\]
\[
\frac{h'(\beta)}{h(\beta)} = -\frac{1}{1 - \beta} + \frac{4}{1 + \beta} - \frac{2}{1 + 2\beta}
\]
\[
= \frac{4 + 4\beta - 8\beta^2 - 1 - 3\beta - 2\beta^2 - 2 + 2\beta^2}{(1 - \beta)^2(1 + 2\beta)}
\]
\[
= \frac{(8\beta^2 - \beta - 1)}{(1 - \beta)^2(1 + 2\beta)} < 0,
\]
because
\[
8\beta^2 - \beta - 1 \geq 8(0.459)^2 - (0.459) - 1
\]
\[
> 8(0.459)^2 - (0.459) - 1
\]
\[
= 1.62 - 1.459 > 0.
\]
Therefore
\[
h(\beta) \leq h(0.459), \text{ and}
\]
\[
\sqrt{d(f)} \leq \frac{1}{2} (9/4)^3 g(0.459) h(0.459)
\]
\[
= \frac{729}{128} \frac{(1 - 0.459)^2(1 + 0.459)^4}{1(1 + 0.918)} = 3.93 \ldots < 4.
\]

Thus $d(f) < 16$ in this case also and the proof of Theorem IIC is complete.

8. Proof of Theorem II'

8.1 Let $K$ be the sphere $|x| \leq 3/4$ and $\Lambda$ the lattice generated by $(1, 1, 0), (0, 1, 1), (1, 0, 1)$. We have to show that every straight line $l$ meets a $k + A, A \in \Lambda$. 

We divide the proof into two parts:

(a) The lines \( l \) are parallel to “lattice lines” \( OA, A \in \Lambda \),

(b) \( l \) is not parallel to any lattice line.

9. Proof of Theorem II' Case (a)

9.1 The quadratic form

\[
f(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_3 + x_1)^2
= 2 \sum_{i < j \leq 3} x_i x_j
\]

is the quadratic form of \( \Lambda \) corresponding to the given basis. The adjoint of \( f \) is

\[
F(x_1, x_2, x_3) = 3 \sum_{i < j \leq 3} x_i x_j.
\]

As explained in §2.3, Theorem II' in case (a) will follow if we can show that for every partial sum \( G \) of \( F \), \( R(G) < \frac{9}{16} d(G) / d(F) \). We note that \( F(x_1, x_2, x_3) = (x_1 + x_2 - x_3)^2 + (x_2 + x_3 - x_1)^2 + (x_3 + x_1 - x_2)^2 \). For integers \( x_i \), \( x_1 + x_2 - x_3 \), \( x_2 + x_3 - x_1 \), \( x_3 + x_1 - x_2 \) are all even or all odd. Therefore, the possible non-zero values of \( F \) for integers \( x_i \) are 3, 4, 8, 11, ... in ascending order, i.e. the values can be 3, 4 or \( \geq 8 \).

Let \( G'(x_1, x_2) \) be a partial sum of \( F \) and \( G(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2, \) \( 0 \leq b \leq a \leq c, a > 0, \) be the reduced form equivalent to \( G' \). Then

\[
R(G') = R(G) = ac(a + c - 2b)/4(ac - b^2)
\]

and we have to prove

\[
a c(a + c - 2b) \leq \frac{9}{16}(ac - b^2)^2,
\]

because \( d(F) = 16 \).

We shall prove this by contradiction, i.e. we shall show that

\[
a c(a + c - 2b) > \frac{9}{16}(ac - b^2)^2
\]

is not possible.

Since the values of \( G \) for integers \( x_i \) are a subset of the values of \( F \) for integers \( x_i \), we have the following possibilities:

(i) \( a = 3 \), (ii) \( a = 4 \), (iii) \( a \geq 8 \).

(i) \( a = 3 \), so that \( b = 0 \) or \( 1, \ c \geq 3 \).

If \( b = 0 \), \( ac(a + c - 2b) > \frac{9}{16}(ac - b^2)^2 \), then

\[
3ac(3 + c) > \frac{9}{16}(3c)^2
\]

i.e.

\[
11c^2 - 48c < 0
\]
c(11c - 48) < 0,

and

c = 3 or c = 4, and

\[ G(x_1, x_2) = 3x_1^2 + 3x_2^2 \text{ or } 3x_1^2 + 4x_2^2 \]

takes the value 6 or 7 for integers \( x_i \). Since 6, 7 are not possible values of \( F \), this case is not possible. If

\[ b = 1, \quad ac(a + c - 2b) > 9/16(ac - b^2)^2, \]

then

\[ 16c(1 + c) > 3(3c - 1)^2 \]

i.e.

\[ 11c^2 - 34c + 3 < 0 \]

i.e.

\[ (c - 3)(11c - 1) < 0, \]

which is impossible, because \( c \geq 3 \).

(ii) Let \( a = 4 \), so that \( b = 0, 1 \) or 2 and \( c \geq 4 \).

Then \( ac(a + c - 2b) > 9/16(ac - b^2)^2 \) implies

\[ 64c(4 + c - 2b) > 9(4c - b^2)^2 \]

or

\[ 80c^2 - c(72b^2 - 128b + 256) + 9b^4 < 0. \]

\( b = 0 \) gives

\[ 80c^2 < 256c \]

and \( c < 4 \), which is impossible,

\( b = 1 \) gives

\[ 80c^2 - 200c + 9 = 80c(c - 4) + 120c + 9 < 0, \]

which is not possible, because \( c \geq 4 \),

and \( b = 2 \) gives

\[ 80c^2 - 288c + 144 = 80c(c - 4) + 32c + 144 < 0, \]

which is again not possible.

(iii) \( a \geq 8 \).

By the Theorem of Lagrange, since \( G \) is reduced,

\[ ac \leq 4/3 d(G) = 4/3(ac - b^2), \]

so that

\[ (ac - b^2) \geq 3/4ac, \]

and

\[ ac(a + c - 2b) > 9/16(ac - b^2)^2 \]

implies

\[ ac(a + c - 2b) > 9/16 \cdot 9/16 a^2 c^2 \]
(a + c) > \frac{81}{256} \text{ac},

so that

\[ \frac{1}{a} + \frac{1}{c} \geq \frac{81}{256}. \]

But

\[ a \geq 8, \; c \geq 8, \text{ and } \frac{1}{a} + \frac{1}{c} \leq \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{8}{256}, \]

which shows that this case is also impossible.

We have thus completed the proof of Theorem II' in case (a).

10. Proof of Theorem II' Case (b)

10.1 Let \( l \) be a straight line not parallel to a lattice line. Let \( \Pi \) be the plane through \( O \) perpendicular to \( l \). Let \( \Lambda_1 \) be the projection of \( \Lambda \) on \( \Pi \). Then the lines parallel to \( l \) meet the spheres \( K + A, A \in \Lambda \) if and only if the circles \( C + A, A \in \Lambda_1 \) cover \( \Pi \), where \( C \) is the circle \( K \cap \Pi \), i.e. \( C \) is the circle of radius 3/4. We have then to show that every point of \( \Pi \) is within the distance 3/4 from some point of \( \Lambda_1 \).

If \( \text{Proj} A = \) projection of the point \( A \) of \( R^3 \) on \( \Pi \), then \( \text{Proj} (A - B) = \text{Proj} A - \text{Proj} B \), and it follows that \( \Lambda_1 \) is an additive subgroup of the group \( \Pi \) under addition. Also, since \( \Lambda \) is "three-dimensional", \( \Lambda_1 \) is "two-dimensional". One can easily see that for \( \Lambda_1 \), we have the following possibilities:

(i) If \( O \) is not a limit point of \( \Lambda_1 \), then \( \Lambda_1 \) is a two-dimensional lattice, and since \( \text{Proj} (mA + nB) = m \text{Proj} A + n \text{Proj} B \), one can easily see that \( l \) is parallel to a lattice line \( OA \) of \( \Lambda_1 \), and this case does not arise,

(ii) If \( O \) is a limit point of \( \Lambda_1 \), and all points of \( \Lambda_1 \) near enough to \( O \) lie on a straight line \( \alpha \) through \( O \), then \( \Lambda_1 \) is dense on \( \alpha \), and consists of points lying dense on lines parallel to \( \alpha \) at the same distance \( \delta \) say, between consecutive ones, and

(iii) \( \Lambda_1 \) is dense everywhere in \( \Pi \), in which case there is nothing to prove.

We have, therefore, to consider case (ii) only. In this case \( \Lambda \) is distributed in the planes orthogonal to \( \Pi \) through the lines parallel to \( \alpha \) of \( \Lambda_1 \). These planes are at a distance \( \delta \) apart (i.e. consecutive planes are at a distance \( \delta \) from each other). The part of \( \Lambda \) in the plane through \( \alpha \) is a two dimensional lattice \( \Lambda_2 \) and the parts in other planes are its translates. The determinant \( d(\Lambda) = \delta \cdot d(\Lambda_2) \), where \( d(\Lambda_2) \) is the determinant of \( \Lambda_2 \).

We notice that the squares of the distances between lattice points of \( \Lambda \) are the values of \( f = 2\Sigma x_i^2 - 2\Sigma x_i x_j \), so that these squared distances are at least 2, and \( \Lambda \) provides a packing for spheres of radius \((1/2)\sqrt{2}\). Therefore, \( \Lambda_2 \) provides a packing for circles of radius \( 1/\sqrt{2} \). Since the density of the closest lattice packings of circles is \( \pi/2\sqrt{3} \), we get

\[ \pi/2d(\Lambda_2) \leq \pi/2\sqrt{3} \]
and 

\[ d(\Lambda_2) \geq \sqrt{3}. \]

Since 

\[ d(\Lambda) = 2, \quad \delta \leq 2/\sqrt{3} < 3/2. \]

Thus the distance \( \delta \) between consecutive lines parallel to \( l \) on which \( \Lambda_1 \) is dense is <3/2. Let \( P \in \Pi \), then \( P \) is at a distance \( \leq \delta/2 < 3/4 \) from one of these lines and at a distance <3/4 from some point of \( \Lambda_1 \), which completes the proof.

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References