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On Fourier coefficients of Maass cusp forms in 3-dimensional hyperbolic space

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Dedicated to the memory of Professor K G Ramanathan

Abstract. In this article we establish the analogue of a theorem of Kuznetsov (theorem 6 of [3]) in the case of 3-dimensional hyperbolic space. We also consider a generalization of this result for higher dimensional hyperbolic spaces and discuss the relevant ingredients of a proof.

Keywords. Fourier coefficients, Maass cusp forms; 3-dimensional hyperbolic space; Kuznetsov theorem.

Let $\mathbb{H}_3 = \{w = z + jy | z = x_1 + ix_2 \in \mathbb{C}, x_1, x_2, y \in \mathbb{R}, y > 0, ij = -ji, j^2 = -1 = i^2\}$ be the 3-dimensional hyperbolic space. The group $PSL(2; \mathbb{C})$ acts on $\mathbb{H}_3$ via the mappings $w \to \gamma\langle w\rangle := (aw + b)(cw + d)^{-1}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2; \mathbb{C})$. If R denotes the ring of integers in an imaginary quadratic field K over $\mathbb{Q}$ of discriminant $d = d_K < 0$ in $\mathbb{Z}$, then $\Gamma := PSL(2; R)$ is a discontinuous group of homeomorphisms $w \mapsto \gamma\langle w\rangle$ of $\mathbb{H}_3$ onto $\mathbb{H}_3$, with a fundamental domain $\mathscr{F}$. On $\mathbb{H}_3$, we have the $PSL(2; \mathbb{C})$-invariant volume element $dv = y^{-3} dx_1 dx_2 dy$. Let $\Gamma_\infty := \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma\right\}$ and $\Gamma'_\infty := \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty\right\}$. For $w \in \mathbb{H}_3$ and $\gamma \in \Gamma$, we write $\gamma\langle w\rangle = z(\gamma\langle w\rangle) + jy(\gamma\langle w\rangle)$.

For n in $R^* := \{2a/\sqrt{d} | a \in R\}$, $w \in \mathbb{H}_3$ and s in $\mathbb{C}$ with $\mathrm{Re}(s) > 2$, the Poincaré series $P(w, s; n)$ is defined by

$$P(w, s; n) = \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma} y(\gamma\langle w\rangle)^s \exp[-2\pi|n| y(\gamma\langle w\rangle) + 2\pi i \mathrm{Re}(z(\gamma\langle w\rangle)\bar{n}.)]$$

For $n = 0$, $P(w, s; 0)$ is precisely the Eisenstein series denoted by $E(w, s)$. While $P(w, s; n)$ is real analytic on $\mathbb{H}_3$ and invariant under Γ, it is in $L^2(\mathscr{F})$ only for $n \neq 0$; on the other hand, it is an eigenfunction of the Laplacian Δ on $\mathbb{H}_3$ only for $n = 0$, with eigenvalue $s(s-2)$.

The Eisenstein series $E(w, s)$ can be continued meromorphically for all s in $\mathbb{C}$, the only singularity being a simple pole at $s = 2$. The Poincaré series $P(w, s; n)$, for $n \neq 0$, have a meromorphic continuation for all $\mathrm{Re}(s) > 1$ and are holomorphic there except possibly at a finite number of points s_l; the finitely many $\mu_l := s_l(2 - s_l)$ in $(0, 1)$ are called the 'exceptional' eigenvalues (in the spectrum of Δ). The Eisenstein series $E(w, 1 + it)$ for t in $\mathbb{R}$ together with similar series for the cusps of $\mathscr{F}$ different from ∞ 'span' the continuous spectrum of Δ; further, within $[1, \infty)$, there is also the discrete spectrum of Δ giving rise to the Maass waveforms $u_l (l = 1, 2, \ldots)$ which are eigenfunctions for Δ in $L^2(\Gamma\backslash\mathbb{H}_3)$ with $\Delta u_l + \lambda_l u_l = 0$ and $\lambda_l \geqslant 1$. For these u_l as well as the eigenfunctions corresponding to μ_l, we have at the infinite cusp, the Fourier expansion

$$u_l(w) = \sum_{0 \neq m \in R^*} \rho_l(m) y K_{i\chi_l}(2\pi|m|y)\exp[2\pi i \operatorname{Re}(\bar{m}z)] \tag{1}$$

where $K_\nu(\cdot)$ is the Macdonald function of order ν and $\chi_l := (\mu_l - 1)^{1/2}$ or $(\lambda_l - 1)^{1/2}$. If w' is the order of the group $R^\times$ of units of R, then the index $[\Gamma_\infty : \Gamma'_\infty] = w'/2$. In the case of fields K with class number 1, we know from ([1], Theorem 4.12) that the Eisenstein series $E(w, s)$ has the Fourier expansion

$$E(w,s) = \frac{w'}{2} y^s + \frac{\pi w'}{(s-1)\sqrt{|d_K|}} \frac{\zeta_K(s-1)}{\zeta_K(s)} y^{2-s} + \frac{2\pi^s}{\sqrt{|d_K|}\Gamma(s)\zeta_K(s)} \tag{2}$$

$$\times \sum_{0 \neq m \in R^*} |m|^{s-1} \sigma_{1-s}\left(\frac{m}{2}\sqrt{d_K}\right) y K_{s-1}(2\pi|m|y)\exp[2\pi i \operatorname{Re}(\bar{m}z)]$$

where ζ_K is the Dedekind zeta function for K and $\sigma_\nu(m) := \sum_{t|m} |t|^{2\nu}$. We note that the expansion (2) is valid even for $s = 1 + ir$, for r in $\mathbb{R}$. Let us write $\langle f, g \rangle := \int_{\mathscr{F}} f\bar{g}\,dv$ for f, g (measurable on $\mathscr{F}$) and $v(\mathscr{F}) := \int dv$.

Our object is to obtain an estimate for the Fourier coefficients $\rho_l(m)$ of u_l, in terms of m and χ_l. This estimate can be deduced from an asymptotic formula for

$$A(X) := \sum_{|\chi_l| \leqslant X} \frac{|\rho_l(m)|^2}{1/|\Gamma(1 - i\chi_l)|^2} \tag{3}$$

More specifically, we shall prove the following.

Theorem. *For $A(X;m) = A(X)$ defined by* (3), *we have, as $X \to \infty$,*

$$A(X;m) = \frac{v(\mathscr{F}_K)}{\pi^3 |d_K| c_1} X^3 + O(X^{5/2}|m|^{1+\varepsilon}) \tag{4}$$

for any $\varepsilon > 0$ with the O-constant depending on ε, where $c_1 := 2\int_0^\infty \frac{u}{\sinh \pi u} du$. As a consequence, we obtain

$$\rho_l(m) = \mathcal{O}(\chi_l^{3/4} e^{\pi\chi_l/2} |m|^{1/2+\varepsilon}) \tag{5}$$

and O-constant depending at most on ε.

Remark. Surprisingly, the contribution to $A(X)$ from the exceptional eigenvalues is very much under control here (unlike in the problem of sums of Kloosterman sums).

The proof of the theorem follows Kuznetsov [3] and at the same time, hopefully helps in clearing a few obscurities. It rests, of course, on an identity arising from the computation of the inner product

$$\langle P(., 2 + it; m), P(., 2 + it; n) \rangle \text{ for } t \text{ in } \mathbb{R} \text{ and } m, n \neq 0 \text{ in } R^* \tag{6}$$

in two different ways, namely via the Parseval relation for $L^2(\mathscr{F})$ and also using the

Fourier expansion of the Poincaré series involved. We proceed to give the details of the proof wherein, for the sake of simplicity, we shall restrict ourselves to the case when the field K has class number 1.

Let us first establish that for $\mathrm{Re}\, s > 2$, r in $\mathbb{R}$ and $m \neq 0$ in R^*,

$$\langle P(.,s;m), E(.,1+ir)\rangle = \frac{2^{1+2ir}\pi^{3/2}}{\sqrt{|d_K|}(4\pi m)^{s-1+ir}} \frac{\Gamma(s-1+ir)\Gamma(s-1-ir)\sigma_{ir}\left(\frac{m}{2}\sqrt{d_K}\right)}{\Gamma(1-ir)\Gamma(s-\frac{1}{2})\zeta_K(1-ir)} \tag{7}$$

The left hand side is precisely

$$\int_{\mathscr{F}} \sum_{\gamma\in\Gamma'_\infty\backslash\Gamma} (y(\gamma\langle w\rangle))^s \exp[-2\pi|m|y(\gamma\langle w\rangle)+2\pi i \mathrm{Re}(z(\gamma\langle w\rangle)\bar{m})]\overline{E(w,1+ir)}\mathrm{d}v$$

$$= \int_{\Gamma'_\infty\backslash \mathbb{H}_3} y^s \exp[-2\pi|m|y]\overline{E(w,1+ir)}\exp[2\pi i\mathrm{Re}(z\bar{m})]\mathrm{d}v$$

$$= \int_0^\infty y^{s-3}\exp[-2\pi|m|y]\mathrm{d}y \int_{R\backslash\mathbb{C}}\int \overline{E(w,1+ir)\exp[-2\pi i \mathrm{Re}(z\bar{m})}\mathrm{d}x_1\mathrm{d}x_2$$

$$= \int_0^\infty y^{s-3}\exp[-2\pi|m|y] \frac{2\pi^{1-ir}|m|^{-ir}\sigma_{ir}\left(\frac{m}{2}\sqrt{d_K}\right)}{\sqrt{|d_K|}\Gamma(1-ir)\zeta_K(1-ir)} yK_{ir}(2\pi|m|y)\mathrm{d}y, \text{ by } (2)$$

$$= \frac{2\sqrt{\pi}\Gamma(s-1+ir)\Gamma(s-1-ir)}{|d_K|^{1/2}(4\pi|m|)^{s-1}\Gamma(s-1/2)} \frac{\pi^{1-ir}}{|m|^{ir}} \frac{\sigma_{ir}\left(\frac{m}{2}\sqrt{d_K}\right)}{\Gamma(1-ir)\zeta_K(1-ir)}$$

which proves (7). Using now (7) with $m, n \neq 0$ in R^*,

$$\int_{-\infty}^\infty \langle P(.,s;m), E(.,1+ir)\rangle \overline{\langle P(.,s;n), E(.,1+ir)\rangle}\mathrm{d}r$$

$$= \frac{4\pi^3(4\pi)^{2-s-\bar{s}}}{|m|^{s-1}|n|^{\bar{s}-1}|d_K|} \times \int_{-\infty}^\infty \left|\frac{n}{m}\right|^{ir} \frac{|\Gamma(s-1+ir)|^2|\Gamma(s-1-ir)|^2\sigma_{ir}\left(\frac{m}{2}\sqrt{d_K}\right)\overline{\sigma_{ir}\left(\frac{n}{2}\sqrt{d_K}\right)}}{|\Gamma(1-ir|^2|\Gamma(s-1/2)|^2|\zeta_K(1+ir)|^2}\mathrm{d}r. \tag{8}$$

On the other hand, we know from Sarnak [4], that, for t in $\mathbb{R}$,

$$\langle P(.,2+it;m), u_l(\cdot)\rangle\overline{\langle P(.,2+it,n), u_l(\cdot)\rangle} = \frac{|d_K|\rho_l(m)\overline{\rho_l(n)}}{16\pi|m|^{1+it}|n|^{1-it}} \frac{|\Gamma(1+i(t+\chi_l))|^2|\Gamma(1+i(t-\chi_l))|^2}{|\Gamma(3/2+it)|^2}. \tag{9}$$

Using (8) and (9), the Parseval relation gives us immediately that the inner product in (6) is nothing but

$$\frac{|d_K|}{16\pi|m|^{1+it}|n|^{1-it}}\sum_{l=1}^{\infty}\rho_l(m)\overline{\rho_l(n)}\frac{|\Gamma(1+i(t+\chi_l))|^2\Gamma(1+i(t-\chi_l))|^2}{|\Gamma(3/2+it)|^2}$$
$$+\frac{4\pi^3|\Gamma(3/2+it)|^{-2}}{(4\pi)^2|d_K||m|^{1+it}|n|^{1-it}}\times\int_{-\infty}^{\infty}\left|\frac{n}{m}\right|^{ir}$$
$$\times\frac{|\Gamma(1+i(t+r))|^2|\Gamma(1+i(t-r))|^2\sigma_{ir}\left(\frac{m}{2}\sqrt{d_K}\right)\overline{\sigma_{ir}\left(\frac{n}{2}\sqrt{d_K}\right)}}{|\Gamma(1-ir)|^2|\zeta_K(1-ir)|^2}\frac{\mathrm{d}r}{4\pi} \tag{10}$$

Using the Fourier expansion of the Poincaré series $P(.,2+it,.)$ from [4], we see that the inner product in (6) also equals

$$\frac{\delta_{m,n}v(\mathscr{F}_R)}{4\pi^2(|m|+|n|)^2}+\sum_{\substack{0\neq c\in R\\ c \bmod \pm 1}}\frac{S(m,n;c)}{|c|^{4+2it}}\int_0^{\infty}\frac{\mathrm{d}y}{y^{1+2it}}\int_{\mathbf{R}^2}$$
$$\times\exp\left[-2\pi\left(|n|y+\frac{|m|}{y|c|^2(1+|z|^2)}\right)+2\pi i\,\mathrm{Re}\left(-\frac{z\bar{m}}{c^2y(1+|z|^2)}-y\bar{n}z\right)\right.$$
$$\left.\times\frac{\mathrm{d}x_1\mathrm{d}x_2}{(1+|z|^2)^{2+it}}\right] \tag{11}$$

where

$$S(m,n;c):=\sum_{\substack{a \bmod c\\ a\bar{a}\equiv 1 (\bmod c)}}\exp[2\pi i\,\mathrm{Re}((\bar{m}a+\bar{n}\bar{a})/c)].$$

For the Kloosterman sums $S(m,m;c)$ we have from [2], the estimate similar to Weil's, viz. for every $\varepsilon>0$,

$$|S(m,m;c)|\leqslant|c|^{1+\varepsilon}|(m\sqrt{d_K},c)|\sigma_0(c/(\dot{m}\sqrt{d_K},c)) \tag{12}$$

On the same lines as in Kuznetsov [3], let us introduce

$$H(r,t):=\frac{|\Gamma(1+i(t+r))|^2|\Gamma(1+i(t-r))|^2}{\pi|\Gamma(1-ir)|^2}$$

and

$$h_Y(r)=\int_0^Y\cosh(\pi t)H(r,t)\mathrm{d}t,\ \text{for } Y>0 \text{ and } r \text{ in } \mathbb{C} \text{ with } |Imr|<1.$$

Moreover, let us multiply the two equal expressions (10) and (11) for the inner product in (6) with $m=n$, throughout by

$$\frac{16\pi^2|m|^2}{|d_K|}\left(\frac{1}{4}+t^2\right)=\frac{16\pi^2|m|^2}{|d_K|}\frac{\cosh(\pi t)|\Gamma(3/2+it)|^2}{\pi}$$

and then integrate both the resulting expressions with respect to t from 0 to Y. We then obtain

$$\sum_l \frac{|\rho_l(m)|^2}{1/|\Gamma(1-i\chi_l)|^2} h_Y(\chi_l) + \frac{\pi}{|d_K|^2}\int_0^\infty \cosh(\pi t)\mathrm{d}t \int_{-\infty}^{\infty} \frac{H(r,L)\left|\sigma_{ir}\left(\frac{m}{2}\sqrt{d_K}\right)\right|^2}{|\zeta_K(1+ir)|^2}\mathrm{d}r =$$

$$= \frac{v(\mathcal{F}_R)}{|d_K|\pi}\left(\frac{1}{3}Y^3 + \frac{1}{4}Y\right) + \frac{16\pi|m|^2}{|d_K|} \sum_{\substack{0 \neq c \in R \\ c \bmod \pm 1}} |c|^{-1} S(m,m;c) \times$$

$$\times \int_0^Y \int_0^\infty \int_{\mathbf{R}^2} \left(\frac{1}{4} + t^2\right)(y^2|\delta_1|)^{-it} \tag{13}$$

$$\exp[-2\pi|m|(y + 1/(y|\delta_1|)) - 2\pi i \operatorname{Re}(\bar{m}z(y + 1/(y\delta_1)))]\mathrm{d}t\frac{\mathrm{d}y}{y}\frac{\mathrm{d}x_1\mathrm{d}x_2}{(1+x_1^2+x_2^2)^2}$$

using the abbreviation δ_1 for $c^2(1+x_1^2+x_\alpha^2)$ and recalling that $z = x_1 + ix_2$.

To derive an asymptotic formula for $A(X)$ as X lends to infinity, we first need to estimate $h_Y(r)$ before employing (13). We see actually that

$$h_Y(r) = \int_0^Y \frac{t^2 - r^2}{2r}\left(\frac{1}{\sinh(\pi t - \pi r)} - \frac{1}{\sinh(\pi t + \pi r)}\right)\mathrm{d}t$$

and then it is not difficult to find that indeed

$$h_\infty(r) = 2\int_0^\infty \frac{u}{\sinh(\pi u)}\mathrm{d}u$$

(which positive constant we denote by c_1). Again, as in [2], we have for $1 \leqslant r \leqslant Y - \log Y$,

$$c_1 > h_Y(r) = c_1 + O(Y^{-\pi/2} + \exp(-\pi r)) \tag{14}$$

Further, for $r \geqslant Y + \log Y$,

$$h_Y(r) \ll (r - Y)\exp(-\pi(r - Y)). \tag{15}$$

We next estimate the series

$$\frac{16\pi|m|^2}{|d_K|}\sum_c |c|^{-4} S(m,m;c) I(c)$$

on the right hand side of (13), denoting the relevant integral therein by $I(c)$. This integral may be rewritten as

$$I(c) = \int_0^\infty \frac{\mathrm{d}y}{y}\int_0^\infty \frac{r\mathrm{d}r}{(1+r^2)^2}\int_0^{2\pi}$$

$$\times \exp[-2\pi|m|\delta(y + y^{-1}) - 2\pi i \operatorname{Re}(re^{i\theta}\bar{m}\delta(y + y^{-1}\exp(-2i\arg c)))]\mathrm{d}\theta \times$$

$$\times \int_0^Y \left(\frac{1}{4} + t^2\right)\exp[-2it\log y]\,\mathrm{d}t. \tag{16}$$

(after replacing y/δ by y, with $\delta := 1/|\delta_1|^{1/2} = 1/(|c|(1+r^2)^{1/2})$. The integral with respect to t occurring in (16) is trivially $O(Y^3)$; however, for $y \neq 1$, it can be seen to be $O(Y^2/|\log y|)$, on using integration by parts. The integration with respect to y can be broken up into integration over $A_1 := (0, 1/w_Y)$, $A_2 := [1/w_Y, w_Y]$ and $A_3 := (w_Y, \infty)$ writing w_Y for $\exp(1/Y^\varepsilon)$, for any fixed $\varepsilon > 0$. We are then led to the estimation

$$
\begin{aligned}
I(c) &\ll Y^2 \int_0^\infty \frac{r\,dr}{(1+r^2)^2} \left(\int_{A_1} + \int_{A_3} \right) \exp[-2\pi|m|\delta(y+y^{-1})] \frac{dy}{y|\log y|} + \\
&\quad + Y^3 \int_0^\infty \frac{r\,dr}{(1+r^2)^2} \int_{A_2} \exp[-2\pi|m|\delta(y+y^{-1})] \frac{dy}{y} \\
&\ll Y^2 \int_0^\infty \frac{r}{(1+r^2)^2} Y^\varepsilon \left(\frac{|c|(1+r^2)^{1/2}}{|m|} \right)^{1-\beta} \\
&\quad \times dr + Y^3 \int_0^\infty \frac{r}{(1+r^2)^2} Y^{-\varepsilon} \left(\frac{|c|(1+r^2)^{1/2}}{|m|} \right)^{1-\beta} dr,
\end{aligned}
$$

for any β in $(0, 1)$. Choosing now $\varepsilon = 1/2$, we obtain that

$$
I(c) \ll \frac{Y^{5/2}}{(|m|/|c|)^{1-\beta}} \int_0^\infty \frac{r\,dr}{(1+r^2)^{(3+\beta)/2}} \ll Y^{5/2}(|c|/|m|)^{1-\beta}
$$

for any β in $(0, 1)$. This together with the estimate (12) implies, for the second term on the right hand side of (13), the bound, for any β in $(0, 1)$:

$$
\frac{16\pi|m|^2}{|d_K|} \sum_{\substack{0 \neq c \in R \\ c \bmod \pm 1}} |c|^{-4} S(m, m; c) I(c) \ll Y^{5/2} \sigma_{(1+\beta)/2}(m\sqrt{d_K}). \tag{17}
$$

For the contribution to the first term on the left hand side of (13) from all the exceptional $\chi_l := (\mu_l - 1)^{1/2}$, we derive the following estimate, namely

$$
\sum_{\substack{l \\ \chi_l = (\mu_l - 1)^{1/2}}} |\rho_l(m)|^2 h_Y(\chi_l) |\Gamma(1 - i\chi_l)|^2 \ll |m|^{1+\varepsilon} \text{ for any } \varepsilon > 0. \tag{18}
$$

For this purpose, we note that these χ_l are all purely imaginary and further know from Sarnak ([4], Theorem 3.1) that $|\chi_l| \leqslant 1/2$; hence $H(\chi_l, t)$ and $h_Y(\chi_l)$ are well-defined. The asymptotic formula $\Gamma(x+iy) \sim \sqrt{2\pi}\exp[-\pi|y|/2]|y|^{x-1/2}$ for $|y| \to \infty$ implies that $H(\chi_l, t) \sim 2\pi|t|^2 \exp[-2\pi|t|]/|\Gamma(1 - i\chi_l)|^2$ as $t \to \infty$. Consequently, for every $Y > 0$, we have

$$
h_Y(\chi_l)/\{1/|\Gamma(1 - i\chi_l)|^2\} \ll \int_0^Y t^2 \exp[-2\pi t] \cosh(\pi t) dt < 4/\pi^3.
$$

Now, for $Y \geqslant 1$,

$$
\min_{\chi_l = (\mu_l - 1)^{1/2}} \frac{h_Y(\chi_l)}{1/|\Gamma(1 - i\chi_l)|^2} \geqslant \min_{\chi_l = (\mu_l - 1)^{1/2}} \frac{h_1(\chi_l)}{1/|\Gamma(1 - i\chi_l)|^2} =: C(>0)
$$

and so

$$C \sum_{\substack{l \\ \chi_l=(\mu_l-1)^{1/2}}} |\rho_l(m)|^2 \leqslant \sum_l \frac{|\rho_l(m)|^2 h_1(\chi_l)}{1/|\Gamma(1-i\chi_l)|^2} \ll |m|^{1+\varepsilon}, \quad \text{by (13) and (17).}$$

This proves (18).

For the proof of our theorem, we shall, in the light of (18), totally ignore the presence of the μ_l's, in the sequel.

Setting $Y = X + \log X$ in (13), the left hand side of (13) is $\geqslant (c_1 - c_2 X^{-\pi/2})A(X) - c_3 \sum_{|\chi_l| \leqslant X} |\rho_l(m)|^2 |\exp[-\pi\chi_l]|\Gamma(1-i\chi_l)|^2$ for certain positive constants c_2 and c_3, on using (14) as well as the positivity of the terms on the left hand side of (13). Hence, for any $\varepsilon > 0$, we obtain, with a constant $c_4 > 0$,

$$\frac{\pi 3|d_K|}{v(\mathscr{F}_R)} c_1 A(X) \leqslant X^3 + c_4 X^{5/2}|m|^{1+\varepsilon}. \tag{19}$$

We next substitute $Y = X - \log X$ in (13) and noting that $Y + \log Y \approx X$ and $Y - \log Y \geqslant X - 2\log X$, we get by (14) that

$$\begin{aligned} &\sum_{\chi_l \leqslant X - 2\log X} |\rho_l(m)|^2 \frac{(c_1 + O((X-\log X)^{-\pi/2} + \exp(-\pi\chi_l))}{1/|\Gamma(1-i\chi_l)|^2} + \\ &+ \sum_{X - 2\log X < \chi_l \leqslant X} |\rho_l(m)|^2 h_Y(\chi_l) + \\ &+ O\left(\sum_{\chi_l > X} \exp(-\pi(\chi_l - X + \log X)) \frac{|\rho_l(m)|^2 (\chi_l - X + \log X)}{1/|\Gamma(1-i\chi_l)|^2} \right) + I_1 = \\ &= \frac{v(\mathscr{F}_R)}{\pi |d_K|} \left\{ \frac{(X-\log X)^3}{3} + \frac{X - \log X}{4} \right\} + O((X-\log X)^{5/2}|m|^{1+\varepsilon}) \end{aligned}$$

where I_1 denotes the second term on the left hand side of (13). From (14), (15) and the estimate $|\zeta_K(1+ir)|^{-1} = O(|r|^\varepsilon)$ for every $\varepsilon > 0$ as $|r| \to \infty$, we see that $I_1 = O(X^{1+\varepsilon}|m|^\varepsilon)$, while the right hand side of the equality above is $\geqslant \frac{v(\mathscr{F}_R)}{\pi 3|d_K|} X^3 - c_5 X^{5/2}|m|^{1+\varepsilon}$ for a positive constant c_5. But now, by (13) and (17),

$$\begin{aligned} \sum_{X - 2\log X \leqslant \chi_l \leqslant X} |\rho_l(m)|^2 \frac{h_Y(\chi_l)}{1/|\Gamma(1-i\chi_l)|^2} &= O(X^3 - (X - 2\log X)^3) + \\ &+ O(X^{5/2}|m|^{1+\varepsilon}) \\ &= O(X^{5/2}|m|^{1+\varepsilon}) \end{aligned}$$

Therefore

$$c_1 A(X) \geqslant c_1 A(X - 2\log X) \geqslant \frac{v(\mathscr{F}_R)}{3|d_K|\pi} X^3 - c_6 X^{5/2}|m|^{1+\varepsilon},$$

for a positive constant c_6. This together with (19) proves the Theorem. The estimate (5) follows on applying the usual difference argument to (4).

Remarks (*i*) In principle, it should be possible to derive the estimate (5) through the general theory of automorphic forms on $GL(2)$ but our proof is elementary.

(ii) An estimate similar to (5) with $|m|^{1+\varepsilon}$ in place of $|m|^{1/2+\varepsilon}$ may be obtained in the case of Maass cusp forms on 4-dimensional hyperbolic space (the order of integral Hurwitz quaternions replacing the ring R).

Appendix

In this appendix we indicate how one can proceed to generalise the result of the theorem above to higher-dimensional hyperbolic spaces. We will utilise the exposition in [5] for our basic set up in this case.

Let q be a non-degenerate quadratic form on a k-dimensional vector space E over $\mathbf{Q}$ and $\mathscr{C}(q)$, the associated Clifford algebra (see [5]), identifying $\mathbf{Q}$ and E with their canonical images in $\mathscr{C}(q)$; for $k=0$, $\mathscr{C}(q):=\mathbf{Q}$. Taking an orthogonal basis $\{e_1,\ldots,e_k\}$ for E over $\mathbf{Q}$ with respect to q, we have

$$e_p^2 = q(e_p)(p=1,2,\ldots k), \quad e_l e_m = -e_m e_l (1 \leqslant l \neq m \leqslant k).$$

For any subset $(M=\{e_{v_1},\ldots,e_{v_r}|v_1<v_2<\cdots<v_r\}$ of the basis $\{e_1,\ldots,e_k\}$, define $e_M := e_{v_1}\cdots e_{v_r}$ and $e_\phi := 1$ for the empty set ϕ. Then these e_M form a basis for $\mathscr{C}(q)$ over $\mathbf{Q}$. We have three $\mathbf{Q}$-linear involutions $x\mapsto x'$, $x\mapsto\bar{x}$ and $x\mapsto x^*$ on $\mathscr{C}(q)$ reducing to the identity map on $\mathbf{Q}$ such that for any e_M as above $e'_M=(-1)^r e_M$, $\bar{e}_M=(-1)^{(r^2+r)/2}e_M$ and $e^*_M=(-1)^{(r^2-r)/2}e_M$. Further for any x,y in $\mathscr{C}(q)$,

$$\overline{xy}=\bar{y}\bar{x} \text{ and } (xy)^*=y^*x^*.$$

We have a *trace* map $\mathrm{tr}:\mathscr{C}(q)\to\mathscr{C}(q)$ defined by $\mathrm{tr}(x)=x+\bar{x}$. When $q=-I_k$, the negative of the unit quadratic form I_k on E, we have on $V_{k+1}:=\mathbf{Q}.1\oplus E\subset\mathscr{C}(q)$, a scalar product $\langle v,w\rangle:=\frac{1}{2}\mathrm{tr}(v\bar{w})$ for all v,w in V_{k+1}, so that $\{1,e_1,\ldots,e_k\}$ becomes an orthonormal basis for V_{k+1}. For any $x=\sum_M\lambda_M e_M$ in $\mathscr{C}(q)$ with $\lambda_M\in\mathbf{Q}$ or more generally for $x=\sum_M\lambda_M\ e_M=\sum_M\lambda_M e_M\otimes 1$ in $\mathscr{C}(q)\otimes\mathbf{R}$ with λ_M in $\mathbf{R}$, we know that $|x|:=\left(\sum_M\lambda_M^2\right)^{1/2}$ defines the euclidean norm of x. Further we have $|v|^2=v\bar{v}=\bar{v}v$ whenever $v\neq 0$ in $\mathscr{C}(q)\otimes\mathbf{R}$ has the property that there exists a $\mathbf{Q}$-linear automorphism φ_v: $V_{k+1}\to V_{k+1}$ such that $vx=\varphi_v(x)v'$ for all x in V_{k+1}. We denote the algebra $\mathscr{C}(q)\otimes\mathbf{R}$ and the vector space $V_{k+1}\otimes\mathbf{R}$ resulting by base change respectively from $\mathscr{C}(q)$ and V_{k+1}, also by $\mathscr{C}(q)$ and V_{k+1} again. Given a lattice L in V_{k+1}, the dual lattice $L^\#$ is defined by $L^\#:=\{y\in V_{k+1}|\langle x,y\rangle\in\mathbf{Z}$ for every x in $L\}$.

Let $\mathbf{H}^{k+2}$ be the $(k+2)$-dimensional hyperbolic space given by the upper half-space

$$\{w=x_0+x_1e_1+x_2e_2+\cdots+x_ke_k+x_{k+1}e_{k+1}| \\ x_0,x_1,x_2,\ldots,x_{k+1}\in\mathbf{R}, x_{k+1}>0\}.$$

We write $w=Z+re_{k+1}$ or more precisely, for $P=(x_0,x_1,\ldots,x_{k+1})$ corresponding to $w=x_0+x_1e_1+\cdots+x_ke_k+x_{k+1}e_{k+1}$ in $\mathbf{H}^{k+2}$, $Z=Z(P):=x_0+x_1e_1+\cdots+x_ke_k$ and $r=r(P)=x_{k+1}>0$. Then $|W(P)|^2:=|Z(P)|^2+(r(P))^2$ or simply $|w|^2:=|Z|^2+r^2$.

We have on $\mathbf{H}^{k+2}$, a Riemannian metric $ds^2 = x_{k+1}^{-2}(dx_0^2 + \cdots + dx_{k+1}^2)$ and associated volume element $dv := x_{k+1}^{-(k+2)} dx_0 \Lambda dx_1 \Lambda dx_{k+1}$; the corresponding Laplace-Beltrami operator Δ is given by

$$\Delta := x_{k+1}^2 \left(\frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_{k+1}^2} \right) - k x_{k+1} \frac{\partial}{\partial x_{k+1}},$$

For the Clifford algebra $\mathscr{C}(q)$ over the base field $K = \mathbf{R}$ or $\mathbf{Q}$ and $q = -I_K$, the *Vahlen group* $SV_k(K)$ is defined by

$$SV_k(K) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \left| \begin{array}{ll} (i) & \alpha, \beta, \gamma, \delta \in \mathscr{C}(q) \text{ with } \alpha\delta^* - \beta\gamma^* = 1 \\ (ii) & \alpha\beta^* = \beta\alpha^*, \gamma\delta^* = \delta\gamma^* \\ (iii) & \alpha\bar{\alpha},\ \beta\bar{\beta},\ \gamma\bar{\gamma},\ \delta\bar{\delta} \in K \\ (iv) & \alpha\bar{\gamma}, \beta\bar{\delta} \in V_{k+1} \\ (v) & \alpha x \bar{\beta} + \beta \bar{x} \bar{\alpha}, \gamma x \bar{\delta} + \delta \bar{x} \bar{\gamma} \in K, \forall x \in V_{k+1} \\ (vi) & \alpha x \bar{\delta} + \beta \bar{x} \bar{\gamma} \in V_{k+1}, \forall x \in V_{k+1} \end{array} \right. \right\}$$

For $K = \mathbf{R}$, $SV_0 = SL_2(\mathbf{R})$ and $SV_1 = SL_2(\mathbf{C})$. Let J be a $\mathbf{Z}$-order in the $\mathbf{Q}$-algebra $\mathscr{C}(q)$ (i.e. a subring containing 1 and a $\mathbf{Q}$-basis of $\mathscr{C}(q)$, with the underlying additive group finitely generated) which is stable under the involutions $*$ and $'$ of $\mathscr{C}(q)$. By $\Gamma := SV_k(J)$, we mean the subgroup $\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k(\mathbf{Q}) \mid \alpha, \beta, \gamma, \delta \in J \right\}$. The Vahlen group $SV_k(\mathbf{R})$ acts on $\mathbf{H}^{k+2}$ as orientation preserving isometries through the maps

$$P \mapsto \sigma P := (\alpha P + \beta)(\gamma P + \delta)^{-1} \text{ for } \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k(\mathbf{R}).$$

For any $w = w(P) = Z(P) + r(P) e_{k+1} \in \mathbf{H}^{k+2}$, we have correspondingly for $w(\sigma P) = Z(\sigma P) + r(\sigma P) e_{k+1}$,

$$Z(\sigma P) = \frac{(\alpha Z + \beta)\overline{(\gamma Z + \delta)} + \alpha \bar{\gamma} r^2}{|\gamma Z + \delta|^2 + |\gamma|^2 r^2}, \quad \text{with } Z := Z(P), \quad r := r(P)$$

$$= \alpha \gamma^{-1} - (\gamma^*)^{-1} \frac{\overline{(\gamma Z + \delta)}}{|\gamma Z + \delta|^2 + |\gamma|^2 r^2} \quad \text{whenever } \gamma \neq 0,$$

and $r(\sigma P) = r(P)/(|\gamma Z + \delta|^2 + |\gamma|^2 r^2)$. By Γ'_∞, we mean the subgroup $\left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}$. We also assume, for simplicity, that Γ has only one cusp viz. ∞, for the above action on $\mathbf{H}^{k+2}$.

If $\Lambda := J \cap V_{k+1}$ for a given $\mathbf{Z}$-order J in $\mathscr{C}(q)$ over $\mathbf{Q}$, then Λ is a lattice in V_{k+1}. Then, for any μ in the dual lattice $\Lambda^\#$ and s in $\mathbf{C}$ with $\mathrm{Re}(s) > k+1$, the Poincaré series $U_\mu(., s)$ is defined by

$$U_\mu(P, s) := \sum_{\sigma \in \Gamma'_\infty \backslash \Gamma} r(\sigma P)^s e(i|\mu| r(\sigma P) + \langle Z(\sigma P), \mu \rangle) \tag{1}$$

where $e(\theta) := \exp(2\pi i \theta)$ for $\theta \in \mathbf{C}$ and $i = \sqrt{-1} \in \mathbf{C}$ (with $\arg i = \pi/2$). (see [5]). This series converges absolutely, uniformly on compact subsets of $\mathbf{H}^{k+2} \times \{s\in\mathbf{C} \mid \mathrm{Re}(s) > k+1\}$ and $U_0(P,s)$ is actually an Eisenstein series. If $\mu \neq 0$ in $\Lambda^{\#}$ and $\mathrm{Re}(s) > k+1$, $U_\mu(.,s) \in L^2(\Gamma\backslash\mathbf{H}^{k+2})$. For $\mathrm{Re}(s) > k+1$ again, $U_\mu(.,s)$ satisfies the differential equation

$$(-\Delta - s(k+1-s))U_\mu(.,s) = 2\pi|\mu|(2s-k)U_\mu(.,s+1) \tag{2}$$

which implies immediately that $U_\mu(.,s)$ has a meromorphic continuation to the domain given by $\mathrm{Re}(s) > k$; further, it has *no* pole at $s = k+1$ and indeed, from ([5], Theorem 10.1) it even follows that $U_\mu(.,s)$ is holomorphic in s for $\mathrm{Re}(s) > k + 1/2$. The possible poles of $U_\mu(.,s)$ in $((k+1)/2, k+1)$ correspond to the values of $s > (k+1)/2$ for which $s(k+1-s)$ is an ("exceptional") eigenvalue of $-\Delta$.

The proof of the proposed generalization rests as before, on an identity arising from the computation of the inner product

$$\langle U_\mu(.,k+1+it), U_\mu(.,k+1+it)\rangle \quad \text{for } \mu \neq 0 \text{ in } \Lambda^{\#},\quad t\in\mathbf{R}$$

in two different ways, namely through the Parseval relation in $L^2(\Gamma\backslash\mathbf{H}^{k+2})$ or by using the Fourier expansion of the given Poincaré series as described below.

Let $C_{\mu,\nu}$, for given $\mu,\nu \in \Lambda^{\#}$, denote the number of α in J such that $\alpha\bar{\alpha} = 1$ and $\varphi_\alpha^*(\mu) = \nu$ where φ_α^* is the map dual (with respect to $\langle,\rangle$) to the map φ_α defined by $\alpha x = \varphi_\alpha(x)\alpha'$ for every x in $V_{k+1}\otimes\mathbf{R}$. We note that $C_{\mu,\nu}$ is $O(1)$ for all μ,ν. For all s in $\mathbf{C}$ with $\mathrm{Re}\, s > k+1$, we have from [5] the following Fourier expansion

$$\begin{aligned} v(\mathscr{F}_\Lambda)U_\mu(w,s) = \sum_{\nu\in\Lambda^{\#}} e(\langle Z,\nu\rangle)\Bigg\{ & C_{\mu,\nu} r^s \exp(-2\pi|\mu|r)\, v(\mathscr{F}_\Lambda) \\ & + r^{k+1-s}\sum_{\gamma\bar{\gamma}\neq 0}\frac{S(\mu,\nu;\gamma)}{(\gamma\bar{\gamma})^s}\int_{V_{k+1}}\frac{\exp\left[\dfrac{-2\pi|\mu|}{r|\gamma|^2(1+|Z|^2)}\right]}{(1+|Z|^2)^s}\, e\left(-\left\langle\frac{(\gamma^*)^{-1}\bar{Z}\gamma^{-1}}{r(1+|Z|^2)},\mu\right\rangle \right. \\ & \left. - r\langle Z,\nu\rangle\right)\mathrm{d}Z\Bigg\} \end{aligned} \tag{3$_A$}$$

where $v(\mathscr{F}_\Lambda)$ is the volume of fundamental domain $\mathscr{F}_\Lambda = V_{k+1}/\Lambda$, $\delta_{..}$ denotes the Kronecker delta and the generalized Kloosterman sum $S(\mu,\nu,\gamma)$ for $\gamma \neq 0$ in J is defined by

$$S(\mu,\nu;\gamma) := \sum_{(x,y)\in D(\gamma)} e(\langle x\gamma^{-1},\mu\rangle + \langle\gamma^{-1}y,\nu\rangle)$$

with $D(\gamma) := \{(\alpha,\delta) \mid \sigma = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ for fixed γ lie in distinct double cosets of Γ modulo $\Gamma'_\infty\}$. For $(\alpha,\delta)\in D(\gamma)$, we have $\alpha\delta^* - 1 \in J\gamma^*$ by definition.

Let $G(r,\nu)$ denote the series occurring as the coefficient of $e(\langle Z,\nu\rangle)$ in the Fourier expansion $(3)_A$. For $\mu,\nu \neq 0$, the integral in $G(r,\nu)$ may be seen to be $O((r|\nu|)^{-N})$ for any $N\in\mathbf{N}$ with the O-constant independent of γ while, for $\mu = 0$, the integral can be evaluated ([5], (9.10) et seq.) and is well-behaved at infinity as a function of r. Therefore the series $\sum_\nu G(r,\nu)e(\langle Z,\nu\rangle)$ is absolutely convergent for $\mathrm{Re}(s) > k + 1/2$ since, from [5], we know that the Linnik – Selberg series $\mathbf{Z}(\mu,\nu,s) := \sum_\gamma \dfrac{S(\mu,\nu;\gamma)}{(\gamma\bar{\gamma})^s}$ converges absolutely for $\mathrm{Re}(s) > k + 1/2$. We may thus obtain the continuation of $U_\mu(.,s)$ for $\mathrm{Re}(s) > k + 1/2$. Since, for any μ in $\Lambda^\#$, the growth of $U_\mu(.,s)$ as r tends to infinity, is governed by that of $(\max(r^{\mathrm{Re}(s)}, r^{\mathrm{Re}((k+1/2)-s)})) \times \exp(-2\pi|\mu|r)$, the inner product $\langle U_\mu(.,s_1), U_\gamma(.,\bar{s}_2)\rangle$ is well-defined for $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2 > k + 1/2$ whenever at least one of μ, ν is not zero. We now proceed to compute this inner product first for $\mu \neq 0$, $\nu = 0$. In fact unfolding the same yields

$$\langle U_\mu(.,s_1), E(.,\bar{s}_2)\rangle = \frac{2\pi^{s_2}|\mu|^{s_2-(k+1)/2}}{\Gamma(s_2)} \sum_{\gamma\bar{\gamma}\neq 0} \frac{\overline{S(0,\mu;\gamma)}}{(\gamma\bar{\gamma})^{s_2}} \times$$

$$\times \int_0^\infty r^{s_1-(k+3)/2} \exp(-2\pi|\mu|r) K_{s_2-(k+1)/2}(2\pi|\mu|r)\mathrm{d}r$$

If we now set (formally) $s_2 = (k+1)/2 - i\tau$ (with $\tau \in \mathbf{R}$) and $s_1 = k + 1 + it$ (with real t), then we obtain

$$\langle U_\mu(.,k+1+it), \quad E(.,(k+1)/2 + i\tau)\rangle = \frac{2\pi^{(k+1)/2-i\tau}|\mu|^{-i\tau}}{\Gamma((k+1)/2 - i\tau)} \times$$

$$\times \sum_{\gamma\bar{\gamma}\neq 0} \frac{\overline{S(0,\mu;\gamma)}}{(\gamma\bar{\gamma})^{(k+1)/2+i\tau}} \int_0^\infty r^{(k-1)/2+it} \exp(-2\pi|\mu|r) K_{i\tau}(2\pi|\mu|r)\mathrm{d}r$$

$$= \frac{2\pi^{(k+1)/2-i\tau}|\mu|^{-i\tau}}{\Gamma\left(\frac{k+1}{2} - i\tau\right)} \times$$

$$\times \sum_{\gamma\bar{\gamma}\neq 0} \frac{\overline{S(0,\mu;\gamma)}\sqrt{\pi}\,\Gamma((k+1)/2 + i(t+\tau))\Gamma((k+1)/2 + i(t-\tau))}{(\gamma\bar{\gamma})^{(k+1)/2+i\tau}(4\pi|\mu|)^{(k+1)/2+it}\Gamma((k+2)/2) + it)}.$$

since

$$\int_0^\infty \exp(-\alpha x) K_\nu(\alpha x) x^{s-1} \mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma(s+\nu)\Gamma(s-\nu)}{(2\alpha)^s \Gamma(s+1/2)} (\mathrm{Re}(\alpha) > 0, \mathrm{Re}(s) > |\mathrm{Re}(\nu)|).$$

Thus

$$\langle U_\mu(.,k+1+it), E(.,(k+1)/2 + i\tau)\rangle =$$

$$= \frac{2\pi^{(k+2)/2-i\tau}|\mu|^{-it}}{(4\pi|\mu|)^{(k+1)/2+it}} \frac{\Gamma\left(\frac{k+1}{2} + i(t+\tau)\right)\Gamma\left(\frac{k+1}{2} + i(t-\tau)\right)}{\Gamma\left(\frac{k+1}{2} - i\tau\right)\Gamma\left(\frac{k+2}{2} + it\right)}$$

$$\sum \frac{\overline{S(0,\mu,\gamma)}}{(\gamma\bar{\gamma})^{(k+1/2)-i\tau}}$$

$$= \frac{2\pi^{(k+2)/2-i\tau}|\mu|^{-it}}{(4\pi|\mu|)^{(k+1)/2+it}} \frac{\Gamma\left(\frac{k+1}{2} + i(t+\tau)\right)\Gamma\left(\frac{k+1}{2} + i(t-\tau)\right)}{\Gamma\left(\frac{k+1}{2} - i\tau\right)\Gamma\left(\frac{k+2}{2} + it\right)}$$

$$\bar{\mathbf{Z}}\left(0, \mu, \frac{k+1}{2} + i\tau\right).$$

We recall that for $\mu \neq 0$ in $\Lambda^{\#}$, $U_\mu(.,s)$ has a meromorphic continuation for $\mathrm{Re}(s) > k$ and is actually holomorphic for $\mathrm{Re}(s) > k + 1/2$; its finitely many poles may come from s_l in $\mathbf{R}$ such that $\mu_l := s_l(k+1-s_l)$ is an "exceptional" eigenvalue for $-\Delta$ i.e. $0 < s_l \leqslant k + 1/2$ (cf. [5]). In addition to the eigenfunctions v_l corresponding to these 'exceptional" eigenvalues, the discrete spectrum for $-\Delta$ in $L^2(\Gamma\backslash\mathbf{H}^{k+2})$ consisting of $\{0\} \cup \{\lambda_j | j = 1, 2, \ldots\}$ gives rise to corresponding eigenfunctions $u_0 =$ constant and $\{u_1, u_2, \ldots\}$ respectively. If we define

$$\tilde{\chi}_l := i\sqrt{(k+1)^2/4 - \mu_l}, \quad \chi_j := \sqrt{\lambda_j - (k+1)^2/4}$$

corresponding respectively to eigenvalues μ_l that are "exceptional" and to "non-exceptional" eigenvalues λ_j, then any eigenfunction u_ρ for $\rho = i\tilde{\chi}_l$ or $i\chi_j$ corresponding to μ_l or λ_j has the Fourier expansion

$$u_\rho(w) = b_\rho(0) r^{(k+1)/2 - s} + \sum_{0 \neq \mu \in \Lambda^{\#}} a_\rho(\mu) r^{(k+1)/2} K_\rho(2\pi|\mu| r) e(\langle \mu, Z\rangle)$$

with a constant $b_\rho(0)$ possibly non-zero for $\rho = i\tilde{\chi}_l$.

For any eigenfunction u_ρ, we see that

$$\langle u_\rho, U_\mu(., k+1+it)\rangle = 2\sqrt{\pi}(4\pi|\mu|)^{(k+1/2)-(k+1-it)} v(\mathscr{F}_\Lambda) a_\rho(\mu) \times \\ \times \frac{\Gamma(k+1-it-(k+1)/2-\rho)\Gamma(k+1-it-((k+1)/2-\rho))}{\Gamma(k+1-it-k/2)}.$$

By analytic continuation, the Parseval relation now gives us

$$\langle U_\mu(., k+1+it), U_\mu(., k+1+it)\rangle = \\ = \frac{v^2(\mathscr{F}_\Lambda)}{(4\pi)^k |\mu|^{k+1} |\Gamma(k/2+1+it)|^2} \sum_\rho |a_\rho(\mu)|^2 \left|\Gamma\left(\frac{k+1}{2} - it + \rho\right)\right|^2 \times \\ \times \left|\Gamma\left(\frac{k+1}{2} + it - \rho\right)\right|^2 + \frac{4\pi^{k+2}}{(4\pi|\mu|)^{k+1}} \times \\ \times \int_{-\infty}^{\infty} \frac{\left|\Gamma\left(\frac{k+1}{2} + i(t+\tau)\right)\right|^2 \left|\Gamma\left(\frac{k+1}{2} + i(t-\tau)\right)\right|^2}{\left|\Gamma\left(\frac{k+1}{2} - i\tau\right)\right|^2 \left|\Gamma\left(\frac{k+2}{2} + it\right)\right|^2} \\ \times \left|\mathbf{Z}\left(0, \mu, \frac{k+1}{2} + i\tau\right)\right|^2 \frac{d\tau}{4\pi} \qquad (4)_A$$

where ρ is summed over the non-zero part of the discrete spectrum of $-\Delta$. This inner product, on using the Fourier expansion of $U_\mu(., k+1+it)$ (and unfolding) becomes

$$\int_{\Gamma'_\infty \backslash \mathbf{H}^{k+2}} U_\mu(w, k+1+it) r^{k+1-it} \exp(-2\pi|\mu| r - 2\pi i\langle Z, \mu\rangle) \frac{dZ\,dr}{r^{k+2}} \qquad (5)_A$$

$$= v(\mathscr{F}_\wedge)C_{\mu,\mu}\int_0^\infty \exp(-4\pi|\mu|r)r^k dr + \int_0^\infty \frac{r^{k+1-2it}}{r^{k+2}}\exp(-2\pi|\mu|r)dr \times$$

$$\times \sum_{\gamma\bar{\gamma}\neq 0}\frac{S(\mu,\mu,\gamma)}{(\gamma\bar{\gamma})^{k+1+it}}\int_{V_{k+1}}\frac{\exp\left(-\frac{2\pi|\mu|}{r|\gamma|^2(1+|Z|^2)}\right)}{(1+|Z|^2)^{k+1+it}}$$

$$\exp\left(-\left\langle\frac{(\gamma^*)^{-1}\bar{Z}\gamma^{-1}}{r(1+|Z|^2)},\mu\right\rangle - r\langle Z,\mu\rangle dZ.\right.$$

As in Kuznetsov [3] we now define

$$H_k(\tau,t): \frac{\left|\Gamma\left(\frac{k+1}{2}+i(t+\tau)\right)\right|^2\left|\Gamma\left(\frac{k+1}{2}+i(t-\tau)\right)\right|^2}{\pi\left|\Gamma\left(\frac{k+1}{2}-i\tau\right)\right|^2}$$

and for $\tau\in\mathbf{C}$ with $|Im(\tau)|<A$ (a constant) and given parameter $Y>0$, we set

$$h_Y(\tau) := \textstyle\int_0^Y \sin(\pi(\kappa_k+it)\exp(\pi i(\kappa_k-1/2))H_k(\tau,t)dt \tag{6}$$

(κ_k being 0 or 1/2 according as k is even or odd). For real τ, we note that $H_k(\tau,t)=\varphi_k(\tau,t)H_{2\kappa_k}(\tau,t)$ where $H_0(\tau,t)$ is precisely the kernel $H(\tau,t)$ in [3],

$$\cosh \pi t H_1(\tau,t) = \frac{t^2-\tau^2}{2\tau}\left(\frac{1}{\sinh(\pi(t-\tau))}-\frac{1}{\sinh(r(t+\tau))}\right) \text{ and}$$

$$\varphi_k(\tau,t) = \begin{cases}\prod_{j=1}^{k/2}\frac{((j+1/2)^2+(t+\tau)^2)((j+1/2)^2+(t-\tau)^2)}{((j+1/2)^2+\tau^2)} & (k\,\text{even})\\ \prod_{j=1}^{(k-1)/2}\frac{(j^2+(t+\tau)^2)(j^2+(t-\tau)^2)}{(j^2+\tau^2)} & (k\,\text{odd})\ .\end{cases}$$

Further, for τ real and for $t\geqslant 0$, we readily see that

$$\varphi_k(\tau,t)\geqslant c_k := \begin{cases}\prod_{j=1}^{k/2}(j+1/2)^2 & (k\,\text{even})\\ \prod_{j=1}^{k/2} j^2 & (k\,\text{odd})\ .\end{cases} \tag*{$(7)_A$}$$

Multiplying both sides of the equation $(4)_A=(5)_A$ by

$$\frac{(4\pi|\mu|)^{k+1}}{\pi(v(\mathscr{F}))^2}\exp(\pi i(\chi_k-1/2))\sin(\pi(\chi_k+it))\left|\Gamma\left(\frac{k+2}{2}+it\right)\right|^2$$

and then integrating both sides with respect to t over $(0,Y)$, we get

$$4\sum_{\rho}\frac{|a_{\rho}(\mu)|^{2}}{\left|1/\Gamma\left(\frac{k+1}{2}-\rho\right)\right|^{2}}h_{Y}(-i\rho)+\frac{4\pi^{k+1}}{v^{2}(\mathscr{F}_{\Lambda})}\int_{0}^{Y}\exp(\pi i(\chi_{k}-\tfrac{1}{2}))\sin(\pi(\chi_{k}+it))\times$$

$$\times\int_{-\infty}^{\infty}H_{k}(\tau,t)\left|Z\left(0,\mu,\frac{k+1}{2}+i\tau\right)\right|^{2}\frac{d\tau}{4\pi}dt=\frac{1}{\pi}\frac{\Gamma(k+1)}{v(\mathscr{F}_{\Lambda})}C_{\mu,\mu}\int_{0}^{Y}P_{k}(t)dt$$

$$+\frac{(4\pi|\mu|)^{k+1}}{\pi v^{2}(\mathscr{F}_{\Lambda})}\int_{0}^{Y}\sum_{\gamma\bar{\gamma}\neq 0}\frac{S(\mu,\mu;\gamma)}{(\gamma\bar{\gamma})^{k+1+it}}P_{k}(t)dt\int_{0}^{\infty}\int_{V_{k+1}}r^{k+1-2it}$$

$$\times\exp\left(-2\pi|\mu|\left(r+\frac{1}{r|\gamma|^{2}(1+|Z|^{2})}\right)\right)\times e\left(-\left\langle\frac{(\gamma^{*})^{-1}\bar{Z}\gamma^{-1}}{r(1+|Z|^{2})},\mu\right\rangle\right)$$

$$\times e(-r\langle Z,\mu\rangle)\frac{dr}{r^{k+2}}dZ \tag{8}$$

where $P_{k}(t)=\frac{1}{\pi}\exp(\pi i)\left(\chi_{k}-\frac{1}{2}\right)\sin(\pi(\chi_{k}+it))\left|\Gamma\left(\frac{k+2}{2}+it\right)\right|^{2}$ is a monic polynomial in t of degree $k+1$.

We are now led to the estimation of

$$I(\gamma):=\int_{0}^{Y}\frac{P_{k}(t)dt}{(\gamma\bar{\gamma})^{it}}\int_{0}^{\infty}\int_{V_{k+1}}\exp\left[-2\pi\left|\mu\left(r+\frac{1}{r\delta_{1}}\right)-\right.\right.$$

$$\left.-2\pi i\left(\left\langle\frac{\gamma^{*-1}\bar{Z}\gamma^{-1}}{r(1+|Z|^{2})},\mu\right\rangle+r\langle Z,\mu\rangle\right)\right]$$

$$\frac{dr}{r^{1+2it}}\frac{dZ}{(1+|Z|^{2})^{k+1+it}} \tag{9}$$

with $\delta_{1}:=|\gamma|^{2}(1+|Z|^{2})$ and then of the series

$$\sum_{\gamma\bar{\gamma}\neq 0}\frac{S(\mu,\mu,\gamma)}{(\gamma\bar{\gamma})^{k+1}}I(\gamma).$$

We have, for $k>0$,

$$I(\gamma)\ll Y^{k+3/2}(|\mu|/|\gamma|)^{-1+\beta}\ \forall\beta\in(0,1). \tag{10}$$

We now proceed to estimate

$$h_{Y}(\tau):=\begin{cases}\int_{0}^{Y}\sinh\pi t H_{k}(\tau,t)dt & \text{for } k \text{ even}\\ \int_{0}^{Y}\cosh\pi t H_{k}(\tau,t)dt & \text{for } k \text{ odd}\end{cases} \tag{6$'$}$$

first for τ satisfying $1\leqslant\tau\leqslant Y-\log Y$ with Y large. In fact,

$$h_{Y}(\tau)=h_{\infty}(\tau)-\int_{Y}^{\infty}\begin{matrix}\sin\mathrm{h}\\ \cos\mathrm{h}\end{matrix}\,\pi t H_{k}(\tau,t)dt.$$

For odd $k\geqslant 1$,

$$\int_{Y}^{\infty}\cosh\pi t.H_{k}(\tau,t)dt=\int_{Y}^{\infty}\frac{t^{2}-\tau^{2}}{2\tau}\left(\frac{1}{\sinh(\pi(t-\tau))}-\frac{1}{\sinh(\pi(t+\tau))}\right)\varphi_{k}(\tau,t)dt$$

$$\ll \frac{1}{\tau \prod_{j=1}^{k-1} (j^2+\tau^2)} \left\{ \int_{Y-\tau}^{\infty} \frac{u(u+2\tau)}{\sinh \pi\mu} \sum_{l,m=0}^{k-1} u^l (u+2\tau)^m du + \right.$$

$$\left. + \exp(-\pi\tau) \int_Y^\infty \exp(-\pi t) t^{2k-2} dt \right\}$$

$$\ll \frac{1}{\tau^k} \int_{Y-\tau}^{\infty} u^{2k} \tau^k \exp(-\pi u) + \exp(-\pi\tau) \ll Y^{-(\pi-\varepsilon)} + \exp(-\pi\tau)$$

for every $\varepsilon > 0$.

The same estimate

$$h_\infty(\tau) - h_Y(\tau) \ll Y^{(\pi-\varepsilon)} + \exp(-\pi\tau) \text{ for } 1 \leqslant \tau \leqslant Y - \log Y \tag{11}$$

holds also for even $k \geqslant 0$.

On the other hand, for $\tau \geqslant Y + \log Y$, we claim that

$$h_Y(\tau) \ll \exp(-(\pi - \varepsilon)(\tau - Y) \text{ for every } \varepsilon > 0. \tag{12}$$

It is easy to get lower bounds for $h_\infty(\tau)$ for all $\tau \geqslant 1$. Actually, from $(7)_A$,

$$h_\infty(\tau) \geqslant c_k \begin{cases} \int_0^\infty \sinh(\pi t) H_0(\tau, t) dt & \text{for } k \text{ even} \\ \int_0^\infty \cosh(\pi t) H_1(\tau, t) dt & \text{for } k \text{ odd} \end{cases} \tag*{$(13)_A$}$$

The integral in $(13)_A$ for even k is

$$\geqslant \frac{1}{2} \int_0^\infty \sinh(\pi t) \exp(\pi\tau) \exp(\pi(t+\tau) - \pi|\tau - t|) dt \geqslant (1 - e^{-\pi})/2\pi$$

while, for odd k, the integral equals $2\int_0^\infty u/\sinh(\pi u) du$ which is clearly independent of τ. We thus obtain

$$h_\infty(\tau) \geqslant c_1' = \min\left(\frac{1 - \exp(-\pi)}{2\pi}, \quad 2\int_0^\infty \frac{u}{\sinh \pi u} du \right) \tag*{$(13)'$}$$

Now, as before, let us define

$$A(X; \mu) = A(X) := \sum_{|\rho| \leqslant X} \frac{|a_\rho(\mu)|^2}{1 \Big/ \left| \Gamma\left(\frac{k+1}{2} - \rho \right) \right|^2}$$

We need only an analogue of Weil's classical estimate for the generalized Kloosterman sums $S(\mu, \mu; \gamma)$ as well as an appropriate estimate for $|\mathbf{Z}(0, \mu, k + 1/2 + i\tau)|^2$ (due to Elstrodt-Grunewald-Mennicke and unpublished as yet) in order to prove the following formula for $A(X)$, carrying out the remaining steps exactly as in the case $k = 1$:

For $k \geqslant 2$ and any $\varepsilon > 0$, we have

$$A(X; \mu) = A(X) = B_k X^{k+2} + O(X^{k+3/2} |\mu|^{k+\varepsilon})$$

as X tends to infinity, with an explicit constant B_k depending on k and the O-constant depending at most on ε and k.

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