On the equation \(x(x + d_1)\ldots(x + (k - 1)d_1) = y(y + d_2)\ldots(y + (mk - 1)d_2)\)

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Dedicated to the memory of Professor K G Ramanathan

Abstract. For given positive integers \(m > 2, d_1\) and \(d_2\), we consider the equation of the title in positive integers \(x, y\) and \(k \geq 2\). We show that the equation implies that \(k\) is bounded. For a fixed \(k\), we give conditions under which the equation implies that \(\max(x, y)\) is bounded.

Keywords. Exponential diophantine equations; arithmetic-geometric mean.

1. Introduction

For positive integers \(m \geq 2, d_1\) and \(d_2\), we consider the equation

\[x(x + d_1)\ldots(x + (k - 1)d_1) = y(y + d_2)\ldots(y + (mk - 1)d_2)\]  \hspace{1cm} (1)

in integers \(x > 0, y > 0\) and \(k \geq 2\). Equation (1) with \(d_1 = d_2\) was considered in [4] and [5]. It was shown in [5, Corollary 2] that equation (1) with \(d_1 = d_2 = d\) and \(m \geq 2\) implies that \(\max(x, y, k)\) is bounded by an effectively computable number depending only on \(m\) and \(d\). In this paper, we extend this result as follows:

**Theorem 1.** There exists an effectively computable number \(C\) depending only on \(d_1\) and \(d_2\) such that equation (1) with \(m = 2\) implies that either

\[\max(x, y, k) \leq C\]

or

\[k = 2, d_1 = 2d_2^2, x = y^2 + 3d_2y.\]

On the other hand, we observe that equation (1) with \(m = 2\) is satisfied whenever the latter possibility holds.

**Theorem 2.** Let \(m > 2\). Assume that equation (1) is satisfied. Then

(a) \(k\) is bounded by an effectively computable number \(C_1\) depending only on \(m, d_1\) and \(d_2\).

(b) Let \(k \leq C_1\). There exists an effectively computable number \(C_2\) depending only on \(m, d_1\) and \(d_2\) such that either

\[\max(x, y) \leq C_2\]

or

\[d_1/d_2^m\] is a product of \(m\) distinct positive integers composed of primes not exceeding \(m\).
(c) Let \( k \leq C_1 \). Then, either (2) holds or

\[ m \geq \alpha(k) \]  

where

\[ \alpha(k) = \begin{cases} 
14 & \text{for } 2 \leq k \leq 7 \\
50 & \text{for } k = 8 \\
\exp\{k \log k - (1.25475)k - \log k + 1.56577\} & \text{for } k \geq 9.
\]  

We observe from (3) and (4) that \( m \geq 14 \) for \( k \geq 2 \) and \( m \geq 2568 \) for \( k \geq 9 \), \( m \geq 17010 \) for \( k \geq 10 \), \( m \geq 125804 \) for \( k \geq 11 \). Thus, we observe from Theorem 2(a) and Theorem 2(c) that equation (1) with \( 3 \leq m \leq 13 \) implies that \( \max(x, y, k) \) is bounded by an effectively computable number depending only on \( d_1 \) and \( d_2 \). This is also the case whenever equation (1) with \( m < 2568 \) and \( k \geq 9 \) is valid. More generally, equation (1) with \( m > 2 \) and

\[ k \geq \max(10, (21 \log m)/20) \]

implies that \( \max(x, y, k) \) is bounded by an effectively computable number depending only on \( m, d_1 \) and \( d_2 \). Finally, we remark that Theorem 2(b) is applied in the proof of Theorem 2(c).

2. Lemmas

In this section, we prove lemmas for the proofs of the theorems. The lemmas are more general than required and we hope that they may be of independent interest. We start with the following extension of [5, Lemma 11]. We write \( N \) for a positive number given by

\[ N^2 = (m - 1)k \text{ with } m \geq 2. \]  

Lemma 1. Let \( \varepsilon > 0 \) and \( m \geq 2 \). There exists an effectively computable number \( C_3 \) depending only on \( \varepsilon \) such that equation (1) with \( k \geq C_3 \) and

\[ x \geq d_1 \]

implies that

\[ \log x \geq \left( \frac{1}{2} - \varepsilon \right)N. \]

Proof. We may assume that \( k \) exceeds a sufficiently large effectively computable number depending only on \( \varepsilon \). Then, by equation (1) and (6), we have

\[ (mk - 1)! d_2^{mk - 1} \leq (k!)x^k \]

which implies that

\[ x \geq e^{-1} k^{m - 1} d_2^{(mk - 1)/k}. \]  

Thus

\[ x \geq (d_1 d_2)^{1/2}, \]
On $x(x + d_1) \ldots (x + (k - 1)d_1) = y(y + d_2) \ldots (y + (m - 1)d_2)$.

otherwise we observe from (6) that

$$(d_1 x)^{1/2} \leq x < (d_1 d_2)^{1/2}$$

i.e. $x < d_2$ which contradicts (8). If all primes not exceeding $N$ divide $d_1 d_2$, we observe from (9) and Prime Number Theory that

$$\log x \geq \frac{1}{2} \log(d_1 d_2) \geq (1 - \varepsilon) N/2.$$ (10)

On the other hand, if there exists a prime $p \leq N$ such that $p \mid d_1 d_2$, then we argue $p$-adically as in [5, Lemma 1] to obtain

$$\frac{k (m - 1)}{p} < \frac{\log(x + (k - 1)d_1)}{\log p} + 2.$$ (11)

Now, we combine (11) and (5) for deriving that

$$\log(x + (k - 1)d_1) \geq (1 - \varepsilon) N/2$$

which, together with (5) and (6), implies (7).

As an immediate consequence of Lemma 1, we obtain the following extension of [5, Corollary 3].

**COROLLARY 1**

Let $\varepsilon > 0$ and $m \geq 2$. If (1) and (6) hold, then

$$\log\left(y + \left(\frac{mk - 1}{2}\right)d_2\right) \geq \left(\frac{1}{2} - \varepsilon\right) N/m \text{ for } k \geq C_3.$$ (12)

**Proof.** We apply arithmetic-geometric mean to the right hand side of (1) to derive (12) from (7) as in the proof of [5, Corollary 3].

Let $B_j = B_j(m, k)$ be given by [4, (3)-(5)]. We prove

**Lemma 2.** Let $\varepsilon > 0$ and $m \geq 2$. The equation (1) with

$$d_1 k^{m+1} \leq x^{1/2}$$ (13)

and

$$d_2 \leq y^{1 - \varepsilon}(m + 1)$$ (14)

implies that either

$$x_1 = y_2^m + B_1 d_2 y_2^{m-1} + \cdots + B_m d_2^m - \left(\frac{k + 1}{2}\right) d_1$$ (15)

where

$$x_1 = x - d_1, \ y_2 = y - d_2$$ (16)

or

$$\max(x, y, k) \leq C_4$$ (17)

for some effectively computable number $C_4$ depending only on $\varepsilon$ and $m$. 
Proof. Let $0 < \varepsilon < 1$ and $m \geq 2$. We assume (1) with (13) and (14). Then, we observe that $d_1 < x$, $d_2 < y$ and $x_1$, $y_2$ are positive integers. By (1) and (16), we have

\[(x_1 + d_1) \ldots (x_1 + kd_1) = (y_2 + d_2) \ldots (y_2 + mkd_2).\]  

(18)

We denote by $c_1, c_2, c_3$ and $c_4$ effectively computable positive numbers depending only on $\varepsilon$ and $m$. We may assume that $y_2 \geq c_1$ with $c_1$ sufficiently large, otherwise we derive from (12), (5), (14) and (1) that $\max(x, y, k) \leq c_2$. Next, we observe from Corollary 1 that

\[\log(y_2 + (mk - 1)d_2) \geq c_3 k^{1/2}.\]  

(19)

Also, we observe from Lemma 1 that

\[\log x_1 \geq c_4 k^{1/2}.\]  

(20)

Now, we follow the proof of [5, §3]. We define $A_j(m, k), B_j = B_j(m, k)$ and $H_j(m, k)$ as in [4, (2)-(5)]. Further, we define

\[F_{d_1}(x_1, k) = (x_1 + d_1) \ldots (x_1 + kd_1),\]

\[F_{d_2}(y_2, m, k) = (y_2 + d_2) \ldots (y_2 + mkd_2)\]

and

\[\Lambda_{d_2} = \Lambda_{d_2}(y_2, m, k) = y_2^m + B_1 d_2 y_2^{m-1} + \cdots + B_m d_2^m.\]  

(21)

When $d_1 = d_2 = d$, these definitions coincide with the corresponding definitions in [5].

By applying arithmetic-geometric mean to the left hand side of (1), we obtain

\[F_{d_1}(x_1, k) < \left( x_1 + \frac{1}{2} d_1 \right)^k.\]

Now, we use (18), (19), (20) and we argue as in the proof of [4, Lemma 5] to obtain

\[F_{d_1}(y_2, m, k) < (\Lambda_{d_2} + (4k^{2m-1})^{-1})^k,\]

\[F_{d_2}(y_2, m, k) > (\Lambda_{d_2} - (2k^{2m-1})^{-1})^k\]

and

\[F_{d_1}(x_1, k) > \left( x_1 + \frac{1}{2} d_1 - (4k^{2m-1})^{-1} \right)^k.\]

Finally, we utilise these estimates and [4, Lemma 3] to conclude that equation (1) implies that

\[x_1 = \Lambda_{d_2} + fd_1, \quad f = -(k + 1)/2,\]  

(22)

which, by (21), coincides with (15). \(\square\)

Lemma 3. Let $\varepsilon > 0$ and $m > 2$. There exist effectively computable numbers $C_5, C_6$ and $C_7$ depending only on $\varepsilon$ and $m$ such that equation (1) with $\max(x, y, k) \geq C_5$, (13) and (14) implies that $m \geq 14$, $k \leq C_6$ and

\[\mu d_2^m = v d_1.\]  

(23)
On \(x(x + d_1) \ldots (x + (k - 1)d_1) = y(y + d_2) \ldots (y + (mk - 1)d_2)\) for some positive integers \(\mu\) and \(\nu\) satisfying

\[
\max(\mu, \nu) \leq C_7.
\]  

\textbf{Proof.} We may assume that \(C_5 > C_4\) so that we derive from Lemma 2 that (15) is valid. Further, we re-write (15) as (22) and we substitute (22) in the left hand side of equation (18) to obtain

\[
F_{d_1}(x_1, k) = A_{d_2}^k + a_2(f, k)d_1^2A_{d_2}^{k-2} + \cdots + a_k(f, k)d_1^k
\]

where \(a_i(f, k)\) with \(1 \leq i \leq k\) are given by \([4, (44)\) and \((45)\)]. Now, we substitute (21) in (25) for writing

\[
F_{d_1}(x_1, k) = \sum_{j=0}^{mk} T_{j, d_1, d_2}(m, k)d_2^jy^{mk-j}
\]

where

\[
T_{j, d_1, d_2}(m, k) = \begin{cases} H_j(m, k) & \text{for } 0 \leq j < 2m, \\ H_j(m, k) + a_2(f, k)d_1^2d_2^{j-2m}H_{j-2m}(m, k-2) + \cdots + a_k(f, k)d_1^kd_2^{-km}H_{j-2m}(m, k-h) & \text{for } hm \leq j < (h+1)m \text{ and } 2 \leq h < k, \\ B_{m}^k + a_2(f, k)d_1^2d_2^{-2m}B_{m}^{k-2} + \cdots + a_k(f, k)d_1^kd_2^{-km} & \text{for } j = mk \end{cases}
\]

Proceeding as in the proof of \([4, (57)\) and \((58)\)], we derive that

\[
H_j(m, k) = A_j(m, k) \text{ for } 0 \leq j \leq 2m
\]  

and

\[
(H_{2m}(m, k) - A_{2m}(m, k))d_2^{2m} = \frac{k(k-1)(k+1)}{24}d_1^2.
\]  

From the explicit calculations using the method described by Glesser in \([3, \text{Appendix}]\), we derive that

\[
H_j(m, k) - A_j(m, k) > 0 \text{ for } k \geq 2, m \leq 13
\]  

where \(j = m + 1\) if \(m\) is odd and \(j = m + 2\) if \(m\) is even. By (26) and (28), we derive that \(m \geq 14\).

Since \(m > 2\), we apply a result of Balasubramanian \([4, \text{Appendix}]\) to obtain from (26) that \(k\) is bounded by an effectively computable number depending only on \(\varepsilon\) and \(m\). Finally, we take square roots on both the sides of (27) to obtain (23) satisfying (24). \(\square\)

If \(m > 2\), we show that the hypothesis (13) is not required whenever equation (1) with \(d_1 = d_2\) is satisfied. If (13) is not valid, we observe from \([5, (7)]\) that

\[
x^{1/10} < k^{m+1}
\]

which, by \([5, \text{Lemma 1}]\), implies that \(\max(x, y, k)\) is bounded by an effectively
computable absolute constant. Further, we derive from (1) and (23) that
\[ \mu^k x'(x' + 1) \ldots (x' + (k - 1)) = v^k y'(y' + 1) \ldots (y' + (mk - 1)) \tag{29} \]
where \( x' = x/d_1 \), and \( y' = y/d_2 \). Next, in view of (24) and \( k \leq C_6 \), we apply the theorem of Faltings (under suitable assumptions) to equation (29) for concluding that there are only finitely many possibilities for \( x, y \) satisfying (1). For deriving this assertion from equation (1) with (23), (24) and \( k \leq C_6 \), we shall not utilise the theorem of Faltings as it is non-effective. We shall follow an elementary approach which is valid under certain restrictions.

Let \( g = \gcd(d_1, d_2^m) \) and \( f(X) \) be a positive real valued function of a positive real variable \( X \) satisfying
\[ \lim_{X \to \infty} f(X) = \infty. \]
We derive from Lemma 3 the following result.

**Lemma 4.** Let \( m > 2 \) and \( \theta > 0 \). The equation (1) with (13), (14) and
\[ g \leq \theta \max \left( \frac{d_1}{f(d_1)}, \frac{d_2^m}{f(d_2)} \right) \tag{30} \]
implies that
\[ \max(d_1, d_2, k) \leq C_8 \tag{31} \]
where \( C_8 \) is an effectively computable number depending only on \( \epsilon, m, f \) and \( \theta \).

**Proof.** We write \( C_9, C_{10} \) and \( C_{11} \) for effectively computable numbers depending only on \( \epsilon, m, f \) and \( \theta \). By Lemma 3, we conclude that
\[ k \leq C_9 \tag{32} \]
and (23) with (24) is valid. We divide both the sides of (23) by \( g \) to derive from (24) that
\[ \max \left( \frac{d_1}{g}, \frac{d_2^m}{g} \right) \leq C_7. \tag{33} \]
By (33) and (30), we observe that
\[ \min(f(d_1), f(d_2)) \leq \theta C_7. \]
Now, by the definition of \( f \), we obtain
\[ \min(d_1, d_2) \leq C_{10} \]
which, together with (27) and (32), implies that \( \max(d_1, d_2) \leq C_{11}. \]

The assumption (30) is satisfied whenever one of the following conditions holds. (The choice of \( \theta \) and \( f \) is given in the brackets)

(i) \( d_1 \) fixed \( (\theta = f(d_1)) \)
On $x(x + d_1)(x + (k - 1)d_1) = y(y + d_2)(y + (mk - 1)d_2)$

(i) $d_2$ fixed ($\theta = f(d_2)$)
(ii) $\gcd(d_1, d_2) = 1$ ($\theta = 1, f(X) = X$)
(iii) $d_1 = d_2$ ($\theta = 1, f(X) = X$)
(iv) $d_1 \leq d_2^n \log(d_2 + 1)$ ($\theta = 1, f(X) = \log(X + 1)$)
(v) $d_1 \leq d_2^n \log(d_2 + 1)$ ($\theta = 1, f(X) = \log(X + 1)$)
(vi) $d_2^n \leq d_1 \log(d_1 + 1)$ ($\theta = 1, f(X) = \log(X + 1)$)

Therefore, equation (1) with $m > 2$, (13) and (14) implies (31) if at least one of the assumption (i)-(vi) holds. As remarked earlier, the assumption (13) is not required whenever $m > 2$ and (iv) is valid. In the next section, we prove Theorem 2(a) by showing that the assumptions (13) and (14) are not needed whenever $d_1$ and $d_2$ are fixed.

3. Proof of Theorem 2(a)

We may suppose that $y$ exceeds a sufficiently large effectively computable number depending only on $m, d_1$ and $d_2$, otherwise the assertion of Theorem 2(a) follows immediately from (12) and (5). Then (14) is satisfied and (13) is a consequence of (7). Now, as remarked at the end of the previous section, we conclude the assertion of Theorem 2(a).

4. Proofs of Theorem 2(b) and Theorem 2(c)

In this section, we shall always assume that equation (1) with $m > 2, k \leq C_1$ (34) is satisfied. Then, by equation (1), we may assume that $y_2 > y'$ where $y'$ is a sufficiently large effectively computable number depending only on $k, m, d_1, d_2$ and $y_2$ is given by (16), otherwise Theorem 2(b) and Theorem 2(c) follow immediately from (34). Then $x_1$, given by (16), is positive and (18) is valid. Also, we observe that (13) and (14) are satisfied. We put

\[ D = d_1/d_2^n, \]
\[ \phi(Y) = Y^n + B_1d_2Y^{n-1} + \cdots + B_md_2^n - \left(\frac{k + 1}{2}\right)d_1, \]
\[ L(X, Y) = (X + d_1)...(X + kd_1) - (Y + d_2)...(Y + mkd_2) \]
\[ l(Y) = L(\phi(Y), Y). \]

and

Now, we apply Lemma 2 and (18) to suppose that $l(y_2) = 0$. Then, since $y'$ is sufficiently large, we derive from (34) that

\[ l(Y) \equiv 0. \]

By (36), (37), (38) and (39), we obtain pairwise distinct integers

\[ 1 \leq \lambda_{i,j} \leq mk, \quad 1 \leq i \leq k, \quad 1 \leq j \leq m. \]
such that
\[ \phi(Y) + id_1 = (Y + \lambda_{i,1}d_2)\ldots(Y + \lambda_{i,m}d_2) \text{ for } 1 \leq i \leq k. \] (41)

We observe that (40) covers all the integers in the interval \([1, mk]\). There is no loss of generality in assuming that each \(m\)-tuple \(\{\lambda_{i,1}, \ldots, \lambda_{i,m}\}\) is such that
\[ \lambda_{i,1} < \cdots < \lambda_{i,m} \text{ for } 1 \leq i \leq k. \] (42)

Let \(\{\lambda_{i_0,1}, \ldots, \lambda_{i_0,m}\}\) be the \(m\)-tuple containing 1. Then, we observe from (42) that \(\lambda_{i_0,1} = 1\). Further, we derive from (41) that
\[ (i - i_0)d_1 = (Y + \lambda_{i,1}d_2)\ldots(Y + \lambda_{i,m}d_2) - (Y + \lambda_{i_0,1}d_2)\ldots(Y + \lambda_{i_0,m}d_2) \text{ for } 1 \leq i \leq k, i \neq i_0. \] (43)

By putting \(Y = -\lambda_{i_0,1}d_2 = -d_2\) in (43), we get
\[ (i - i_0)d_1 = (\lambda_{i,1} - 1)\ldots(\lambda_{i,m} - 1)d_2^2 \text{ for } 1 \leq i \leq k, i \neq i_0. \] (44)

We observe from (44) that
\[ i_0 = 1. \] (45)

It follows from \(B_1 = m(mk + 1)/2\), (36) and (41) that
\[ m(mk + 1)/2 = \sum_{j=1}^{m} \lambda_{i,j} \text{ for } 1 \leq i \leq k. \] (46)

Further, we set
\[ \Delta_i = (\lambda_{i,1} - 1)\ldots(\lambda_{i,m} - 1) \text{ for } 2 \leq i \leq k, \] (47)
\[ \Delta_1 = (\lambda_{1,1} - 1)\ldots(\lambda_{1,m} - 1) \] (48)
and
\[ \Omega = \prod_{i=2}^{k} \Delta_i. \] (49)

Now, we observe from (47), (48), (49), (40), (44) and (45) that
\[ \Delta_1\Omega = (mk - 1)! \] (50)
and
\[ \Omega = (k - 1)!D^{k-1}. \] (51)

**Proof of Theorem 2(b).** By (44) with \(i = 2\), (45) and (35), we conclude that \(D\) is a product of \(m\) distinct positive integers. Therefore, it suffices to show that every prime divisor of \(D\) is at most \(m\). We set
\[ \psi(Z) = Z^m + B_1Z^{m-1} + \cdots + B_m - \left(\frac{k + 1}{2}\right)D \] (52)

By (36) and (52),
\[ \frac{\phi(Y)}{d_2^n} = \psi\left(\frac{Y}{d_2}\right). \] (53)
Further, by (41) and (53), it follows that

\[ \psi(Z) + iD \equiv (Z + i_{1,1})...(Z + i_{1,m}) \text{ for } 1 \leq i \leq k. \]  

(54)

Since \( D \) is an integer, we observe from (54) that \( \psi(Z) \) is a polynomial of degree \( m \) with integer coefficients.

Let \( p \) be a prime divisor of \( D \). By (44) with \( i = 2 \) and (45), we observe that \( p < mk \). Then, we derive from (54) that

\[ \psi(-v) \equiv 0 \pmod{p} \text{ for } 1 \leq v \leq p. \]  

(55)

This implies that \( p \leq m \), since \( \psi(Z) \equiv 0 \pmod{p} \) has at most \( m \) incongruent solutions \( \pmod{p} \).

For an integer \( v > 1 \), we write \( P(v) \) for the greatest prime factor of \( v \) and we put \( P(1) = 1 \). The letter \( p \) denotes always a prime number. For the proof of Theorem 2(c), we require the following results from Prime Number Theory. The first result is a sharpening, due to Hanson [1], of a theorem of Sylvester.

**Lemma 5.** For positive integers \( k \geq 2 \) and \( n > k \), either

\[ P(n(n + 1)...(n + k - 1)) > \frac{3k}{2} \]

or

\[ (n, k) \in \{(3, 2), (8, 2), (6, 5)\}. \]

The second result is due to Rosser and Schoenfeld [2, p 65–70.] on estimates for some well-known functions in Prime Number Theory. Let

\[ \pi(x) = \sum_{p \leq x} 1 \]

\[ \vartheta(x) = \sum_{p \leq x} \log p \]

and

\[ E = -\gamma - \sum_{n=2}^{\infty} \sum_{p} (\log p)/p^n \]

where \( \gamma \) is Euler's constant. Then

**Lemma 6.** For \( x \geq 2 \), we have

\[ \pi(x) > x/(\log x) \text{ for } x \geq 17, \]  

(56)

\[ \pi(x) < 13x/(10 \log x), \]  

(57)

\[ \sum_{p \leq x} (\log p)/p > \log x + E - 1/(2 \log x), \]  

(58)

\[ \sum_{p \leq x} (\log p)/p < \log x + E + 1/(\log x) \text{ for } x \geq 32, \]  

(59)

\[ \vartheta(x) > x(1 - 1/(\log x)) \text{ for } x \geq 41, \]  

(60)

\[ \vartheta(x) < x(1 + 1/(2 \log x)). \]  

(61)
By taking $y'$ sufficiently large, we derive from Lemma 3 that
\[ m \geq 14. \tag{62} \]

Further, we apply Lemma 5 to sharpen (62) as follows.

**Lemma 7.** We have
\[ m > k. \tag{63} \]

**Proof.** By (62), we may assume that
\[ k \geq 13. \tag{64} \]

We denote by $\mu_1 < \mu_2 < \cdots < \mu_s$ the elements of $\{2^1, 2^2 - 1, \ldots, 2^m - 1\}$ which are greater than $k$. We observe that
\[ 0 \leq s \leq m - 1. \tag{65} \]

By writing $\mu_0 = k$ and $\mu_{s+1} = mk$, we divide
\[ (k, mk) - \{\mu_1, \ldots, \mu_s\} \]
into $(s + 1)$ disjoint intervals
\[ (\mu_j, \mu_{j+1}) \]
for $0 \leq j \leq s$.

Then, we find $J$ with $0 \leq J \leq s$ satisfying
\[ \mu_{j+1} - \mu_j - 1 \geq \frac{(mk - k - s - 1)}{(s + 1)}. \tag{66} \]

By (66), (65), (62) and (64), we derive that
\[ \mu_{j+1} - \mu_j - 1 \geq \frac{(13k/14) - 1}{2k/3}. \tag{67} \]

Now, we derive from (67) and Lemma 5 that the interval $(\mu_j, \mu_{j+1})$ contains an integer $\mu$ divisible by a prime $> k$. Further, we observe from (49) that $\mu$ divides $\Omega$. Therefore, we conclude from (51) and Theorem 2(b) that
\[ k < P(\mu) \leq P(D) \leq m. \]

**Lemma 8.** For $k \geq 8$, we have
\[ \log m > k - \log k - 2. \tag{68} \]

**Proof.** By (51), (63) and Theorem 2(b), we derive that
\[ w(\Omega) \leq \pi(m) \tag{69} \]

where $w(\Omega)$ denotes the number of distinct prime divisors of $\Omega$. On the other hand, we observe from (50) and (48) that
\[ w(\Omega) > \pi(mk) - m. \tag{70} \]
On $x(x + d_1) \ldots (x + (k - 1)d_1) = y(y + d_2) \ldots (y + (mk - 1)d_2)$

Further, we combine (70) and (69) for deriving that

$$\pi(mk) - m < \pi(m).$$

(71)

Now, we apply (56) and (57) in (71) for deriving that

$$\log m > k - \log k - \left(\frac{13}{10} + \frac{13 \log k}{10 \log m}\right).$$

(72)

By (72) and (63), we have

$$\log m > k - \log k - 2.6.$$  

(73)

Then, since $k \geq 8$, we observe from (73) that $m \geq 28$. Now, we derive from (72) that

$$\log m > k - \log k - 2.15.$$  

Repeating this process two more times, we obtain (68).

Proof of Theorem 2(c). By (62) and (68), we may assume that $k \geq 9$. Then, we observe from (68) that $m \geq 115$. The proof depends on comparing an upper and lower bound for $\Delta_1$.  By (48), we obtain

$$\Delta_1 < \delta_{1,2} \ldots \delta_{1,m}$$

which, by arithmetic–geometric mean and (46), implies that

$$\Delta_1 < \left(\frac{m(mk + 1)}{2(m - 1)}\right)^{m-1} < e\left(\frac{mk + 1}{2}\right)^{m-1}.$$  

(74)

By (50), (51), (63) and a consequence $P(D) \leq m$ of Theorem 2(b), we conclude that

$$\log \Delta_1 \geq \sum_{m < p \leq mk} \text{ord}_p((mk - 1)! \log p).$$

Therefore

$$\log \Delta_1 \geq (mk - 1) \sum_{m < p \leq mk} \frac{\log p}{p} - 9(mk - 1) + 9(m).$$  

(75)

Now, we apply (58), (59), (61) and (60) in (75) for deriving

$$\log \Delta_1 > (mk - 1)(\log k - 2/(\log m)) - mk + m + 1 - m/(\log m).$$  

(76)

Next, we combine (76) and (74) to obtain

$$\log m > k \log k - k - \log k - \frac{2k + 1}{\log m} + \log 2 + 1.$$  

(77)

Then, we observe from (77) and $m \geq 115$ that

$$\log m > k \log k - (1.4216)k - \log k + 1.48239.$$  

Repeated applications of (77), as in the proof of Lemma 8, yield

$$\log m > k \log k - (1.25475)k - \log k + 1.56577.$$  

$\Box$
5. Proof of Theorem 1.

Let $m = 2$. Suppose that equation (1) is satisfied. As earlier, we may assume that $y$ exceeds a sufficiently large effectively computable number depending only on $d_1$ and $d_2$. Further, the inequalities (13) and (14) are valid. Consequently, we conclude (15). Next, we argue as in the proof of Lemma 3, for deriving (27). We calculate

$$H_4(2, k) - A_4(2, k) = (4k^5 - 5k^3 + k)/90.$$  \hspace{1cm} (78)

By (27) and (78), we find that

$$D^2 = (d_1/d_2^2)^2 = 4(4k^2 - 1)/15.$$  \hspace{1cm} (79)

In particular, we observe that $k$ is bounded by an effectively computable number depending only on $d_1$ and $d_2$. Further, as in the proof of Theorem 2(b), we show that $D$ is an integer satisfying $P(D) = 2$. Then, we conclude from (79) that $D = k = 2$, which together with (15), implies that $x = y^2 + 3d_2y$.

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References