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On the equation $x(x+d_1)\dots(x+(k-1)d_1) = y(y+d_2)\dots(y+(mk-1)d_2)$

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Dedicated to the memory of Professor K G Ramanathan

Abstract. For given positive integers $m \geqslant 2, d_1$ and d_2, we consider the equation of the title in positive integers x, y and $k \geqslant 2$. We show that the equation implies that k is bounded. For a fixed k, we give conditions under which the equation implies that $\max(x, y)$ is bounded.

Keywords. Exponential diophantine equations; arithmetic-geometric mean.

1. Introduction

For positive integers $m \geqslant 2$, d_1 and d_2, we consider the equation

$$x(x+d_1)\dots(x+(k-1)d_1) = y(y+d_2)\dots(y+(mk-1)d_2) \tag{1}$$

in integers $x > 0$, $y > 0$ and $k \geqslant 2$. Equation (1) with $d_1 = d_2$ was considered in [4] and [5]. It was shown in [5, Corollary 2] that equation (1) with $d_1 = d_2 = d$ and $m \geqslant 2$ implies that $\max(x, y, k)$ is bounded by an effectively computable number depending only on m and d. In this paper, we extend this result as follows:

Theorem 1. *There exists an effectively computable number C depending only on d_1 and d_2 such that equation* (1) *with $m = 2$ implies that either*

$$\max(x, y, k) \leqslant C$$

or

$$k = 2,\ d_1 = 2d_2^2,\ x = y^2 + 3d_2 y.$$

On the other hand, we observe that equation (1) with $m = 2$ is satisfied whenever the latter possibility holds.

Theorem 2. *Let $m > 2$. Assume that equation* (1) *is satisfied. Then*
(a) *k is bounded by an effectively computable number C_1 depending only on m, d_1 and d_2.*
(b) *Let $k \leqslant C_1$. There exists an effectively computable number C_2 depending only on m, d_1 and d_2 such that either*

$$\max(x, y) \leqslant C_2 \tag{2}$$

or

d_1/d_2^m is a product of m distinct positive integers composed of primes not exceeding m.

(c) *Let* $k \leqslant C_1$. *Then, either* (2) *holds or*

$$m \geqslant \alpha(k) \tag{3}$$

where

$$\alpha(k) = \begin{cases} 14 \textit{ for } 2 \leqslant k \leqslant 7 \\ 50 \textit{ for } k = 8 \\ \exp\{k \log k - (1{\cdot}25475)k - \log k + 1{\cdot}56577\} \textit{ for } k \geqslant 9. \end{cases} \tag{4}$$

We observe from (3) and (4) that $m \geqslant 14$ for $k \geqslant 2$ and $m \geqslant 2568$ for $k \geqslant 9$, $m \geqslant 17010$ for $k \geqslant 10$, $m \geqslant 125804$ for $k \geqslant 11$. Thus, we observe from Theorem 2(a) and Theorem 2(c) that equation (1) with $3 \leqslant m \leqslant 13$ implies that $\max(x, y, k)$ is bounded by an effectively computable number depending only on d_1 and d_2. This is also the case whenever equation (1) with $m < 2568$ and $k \geqslant 9$ is valid. More generally, equation (1) with $m > 2$ and

$$k \geqslant \max(10,(21 \log m)/20)$$

implies that $\max(x, y, k)$ is bounded by an effectively computable number depending only on m, d_1 and d_2. Finally, we remark that Theorem 2(b) is applied in the proof of Theorem 2(c).

2. Lemmas

In this section, we prove lemmas for the proofs of the theorems. The lemmas are more general than required and we hope that they may be of independent interest. We start with the following extension of [5, Lemma 1]. We write N for a positive number given by

$$N^2 = (m-1)k \text{ with } m \geqslant 2. \tag{5}$$

Lemma 1. *Let* $\varepsilon > 0$ *and* $m \geqslant 2$. *There exists an effectively computable number* C_3 *depending only on* ε *such that equation* (1) *with* $k \geqslant C_3$ *and*

$$x \geqslant d_1 \tag{6}$$

implies that

$$\log x \geqslant \left(\frac{1}{2} - \varepsilon\right) N. \tag{7}$$

Proof. We may assume that k exceeds a sufficiently large effectively computable number depending only on ε. Then, by equation (1) and (6), we have

$$(mk-1)!\, d_2^{mk-1} \leqslant (k!) x^k$$

which implies that

$$x \geqslant e^{-1} k^{m-1} d_2^{(mk-1)/k}. \tag{8}$$

Thus

$$x \geqslant (d_1 d_2)^{1/2}, \tag{9}$$

otherwise we observe from (6) that

$$(d_1 x)^{1/2} \leqslant x < (d_1 d_2)^{1/2}$$

i.e. $x < d_2$ which contradicts (8). If all primes not exceeding N divide $d_1 d_2$, we observe from (9) and Prime Number Theory that

$$\log x \geqslant \frac{1}{2}\log(d_1 d_2) \geqslant (1-\varepsilon)N/2. \tag{10}$$

On the other hand, if there exists a prime $p \leqslant N$ such that $p \nmid d_1 d_2$, then we argue p-adically as in [5, Lemma 1] to obtain

$$\frac{k}{p}(m-1) < \frac{\log(x+(k-1)d_1)}{\log p} + 2. \tag{11}$$

Now, we combine (11) and (5) for deriving that

$$\log(x+(k-1)d_1) \geqslant (1-\varepsilon)N/2$$

which, together with (5) and (6), implies (7). □

As an immediate consequence of Lemma 1, we obtain the following extension of [5, Corollary 3].

COROLLARY 1

Let $\varepsilon > 0$ *and* $m \geqslant 2$. *If* (1) *and* (6) *hold, then*

$$\log\left(y+\left(\frac{mk-1}{2}\right)d_2\right) \geqslant \left(\frac{1}{2}-\varepsilon\right)N/m \text{ for } k \geqslant C_3. \tag{12}$$

Proof. We apply arithmetic-geometric mean to the right hand side of (1) to derive (12) from (7) as in the proof of [5, Corollary 3]. □

Let $B_j = B_j(m,k)$ be given by [4, (3)–(5)]. We prove

Lemma 2. *Let* $\varepsilon > 0$ *and* $m \geqslant 2$. *The equation* (1) *with*

$$d_1 k^{m+1} \leqslant x^{1/2} \tag{13}$$

and

$$d_2 \leqslant y^{(1-\varepsilon)/(m+1)} \tag{14}$$

implies that either

$$x_1 = y_2^m + B_1 d_2 y_2^{m-1} + \cdots + B_m d_2^m - \left(\frac{k+1}{2}\right)d_1 \tag{15}$$

where

$$x_1 = x - d_1,\ y_2 = y - d_2 \tag{16}$$

or

$$\max(x, y, k) \leqslant C_4 \tag{17}$$

for some effectively computable number C_4 *depending only on* ε *and* m.

Proof. Let $0 < \varepsilon < 1$ and $m \geqslant 2$. We assume (1) with (13) and (14). Then, we observe that $d_1 < x$, $d_2 < y$ and x_1, y_2 are positive integers. By (1) and (16), we have

$$(x_1 + d_1)\dots(x_1 + kd_1) = (y_2 + d_2)\dots(y_2 + mkd_2). \tag{18}$$

We denote by c_1, c_2, c_3 and c_4 effectively computable positive numbers depending only on ε and m. We may assume that $y_2 \geqslant c_1$ with c_1 sufficiently large, otherwise we derive from (12), (5), (14) and (1) that $\max(x, y, k) \leqslant c_2$. Next, we observe from Corollary 1 that

$$\log(y_2 + (mk - 1)d_2) \geqslant c_3 k^{1/2}. \tag{19}$$

Also, we observe from Lemma 1 that

$$\log x_1 \geqslant c_4 k^{1/2}. \tag{20}$$

Now, we follow the proof of [5, § 3]. We define $A_j(m, k), B_j = B_j(m, k)$ and $H_j(m, k)$ as in [4, (2)–(5)]. Further, we define

$$F_{d_1}(x_1, k) = (x_1 + d_1)\dots(x_1 + kd_1),$$

$$F_{d_2}(y_2, m, k) = (y_2 + d_2)\dots(y_2 + mkd_2)$$

and

$$\Lambda_{d_2} = \Lambda_{d_2}(y_2, m, k) = y_2^m + B_1 d_2 y_2^{m-1} + \cdots + B_m d_2^m. \tag{21}$$

When $d_1 = d_2 = d$, these definitions coincide with the corresponding definitions in [5].

By applying arithmetic-geometric mean to the left hand side of (1), we obtain

$$F_{d_1}(x_1, k) < \left(x_1 + \frac{k+1}{2} d_1\right)^k.$$

Now, we use (18), (19), (20) and we argue as in the proof of [4, Lemma 5] to obtain

$$F_{d_2}(y_2, m, k) < (\Lambda_{d_2} + (4k^{2m-1})^{-1})^k,$$

$$F_{d_2}(y_2, m, k) > (\Lambda_{d_2} - (2k^{2m-1})^{-1})^k$$

and

$$F_{d_1}(x_1, k) > \left(x_1 + \frac{k+1}{2} d_1 - (4k^{2m-1})^{-1}\right)^k.$$

Finally, we utilise these estimates and [4, Lemma 3] to conclude that equation (1) implies that

$$x_1 = \Lambda_{d_2} + f d_1, \; f = -(k+1)/2, \tag{22}$$

which, by (21), coincides with (15). □

Lemma 3. *Let* $\varepsilon > 0$ *and* $m > 2$. *There exist effectively computable numbers* C_5, C_6 *and* C_7 *depending only on* ε *and* m *such that equation* (1) *with* $\max(x, y, k) \geqslant C_5$, (13) *and* (14) *implies that* $m \geqslant 14$, $k \leqslant C_6$ *and*

$$\mu d_2^m = \nu d_1 \tag{23}$$

for some positive integers μ and ν satisfying

$$\max(\mu, \nu) \leqslant C_7. \tag{24}$$

Proof. We may assume that $C_5 > C_4$ so that we derive from Lemma 2 that (15) is valid. Further, we re-write (15) as (22) and we substitute (22) in the left hand side of equation (18) to obtain

$$F_{d_1}(x_1, k) = \Lambda_{d_2}^k + a_2(f,k)d_1^2\Lambda_{d_2}^{k-2} + \cdots + a_k(f,k)d_1^k \tag{25}$$

where $a_i(f,k)$ with $1 \leqslant i \leqslant k$ are given by [4, (44) and (45)]. Now, we substitute (21) in (25) for writing

$$F_{d_1}(x_1, k) = \sum_{j=0}^{mk} T_{j,d_1,d_2}(m,k)d_2^j y_2^{mk-j}$$

where

$$T_{j,d_1,d_2}(m,k) = \begin{cases} H_j(m,k) \text{ for } 0 \leqslant j < 2m, \\ H_j(m,k) + a_2(f,k)d_1^2 d_2^{-2m} H_{j-2m}(m,k-2) + \cdots \\ \quad + a_h(f,k)d_1^h d_2^{-hm} H_{j-hm}(m,k-h) \text{ for} \\ \quad hm \leqslant j < (h+1)m \text{ and } 2 \leqslant h < k, \\ B_m^k + a_2(f,k)d_1^2 d_2^{-2m} B_m^{k-2} + \cdots + \\ \quad a_k(f,k)d_1^k d_2^{-km} \text{ for } j = mk \end{cases}$$

Proceeding as in the proof of [4, (57) and (58)], we derive that

$$H_j(m,k) = A_j(m,k) \text{ for } 0 \leqslant j \leqslant 2m \tag{26}$$

and

$$(H_{2m}(m,k) - A_{2m}(m,k))d_2^{2m} = \frac{k(k-1)(k+1)}{24}d_1^2. \tag{27}$$

From the explicit calculations using the method described by Glesser in [3, Appendix], we derive that

$$H_j(m,k) - A_j(m,k) > 0 \text{ for } k \geqslant 2,\ m \leqslant 13 \tag{28}$$

where $j = m+1$ if m is odd and $j = m+2$ if m is even. By (26) and (28), we derive that $m \geqslant 14$.

Since $m > 2$, we apply a result of Balasubramanian [4, Appendix] to obtain from (26) that k is bounded by an effectively computable number depending only on ε and m. Finally, we take square roots on both the sides of (27) to obtain (23) satisfying (24). □

If $m > 2$, we show that the hypothesis (13) is not required whenever equation (1) with $d_1 = d_2$ is satisfied. If (13) is not valid, we observe from [5, (7)] that

$$x^{1/10} < k^{m+1}$$

which, by [5, Lemma 1], implies that $\max(x, y, k)$ is bounded by an effectively computable absolute constant. Further, we derive from (1) and (23) that

$$\mu^k x'(x'+1)\dots(x'+(k-1)) = \nu^k y'(y'+1)\dots(y'+(mk-1)) \tag{29}$$

where $x' = x/d_1$, and $y' = y/d_2$. Next, in view of (24) and $k \leqslant C_6$, we apply the theorem of Faltings (under suitable assumptions) to equation (29) for concluding that there are only finitely many possibilities for x, y satisfying (1). For deriving this assertion from equation (1) with (23), (24) and $k \leqslant C_6$, we shall not utilise the theorem of Faltings as it is non-effective. We shall follow an elementary approach which is valid under certain restrictions.

Let $g = \gcd(d_1, d_2^m)$ and $f(X)$ be a positive real valued function of a positive real variable X satisfying

$$\lim_{X\to\infty} f(X) = \infty.$$

We derive from Lemma 3 the following result.

Lemma 4. *Let* $m > 2$ *and* $\theta > 0$. *The equation* (1) *with* (13), (14) *and*

$$g \leqslant \theta \max\left(\frac{d_1}{f(d_1)}, \frac{d_2^m}{f(d_2)}\right) \tag{30}$$

implies that

$$\max(d_1, d_2, k) \leqslant C_8 \tag{31}$$

where C_8 *is an effectively computable number depending only on* ε, m, f *and* θ.

Proof. We write C_9, C_{10} and C_{11} for effectively computable numbers depending only on ε, m, f and θ. By Lemma 3, we conclude that

$$k \leqslant C_9 \tag{32}$$

and (23) with (24) is valid. We divide both the sides of (23) by g to derive from (24) that

$$\max\left(\frac{d_1}{g}, \frac{d_2^m}{g}\right) \leqslant C_7. \tag{33}$$

By (33) and (30), we observe that

$$\min(f(d_1), f(d_2)) \leqslant \theta C_7.$$

Now, by the definition of f, we obtain

$$\min(d_1, d_2) \leqslant C_{10}$$

which, together with (27) and (32), implies that $\max(d_1, d_2) \leqslant C_{11}$. □

The assumption (30) is satisfied whenever one of the following conditions holds. (The choice of θ and f is given in the brackets)

(i) d_1 fixed $\qquad (\theta = f(d_1))$

(ii) d_2 fixed $(\theta = f(d_2))$
(iii) $\gcd(d_1, d_2) = 1$ $(\theta = 1, f(X) = X)$
(iv) $d_1 = d_2$ $(\theta = 1, f(X) = X)$
(v) $d_1 \leqslant d_2^m/\log(d_2+1)$ $(\theta = 1, f(X) = \log(X+1))$
(vi) $d_2^m \leqslant d_1/\log(d_1+1)$ $(\theta = 1, f(X) = \log(X+1))$

Therefore, equation (1) with $m > 2$, (13) and (14) implies (31) if at least one of the assumption (i)–(vi) holds. As remarked earlier, the assumption (13) is not required whenever $m > 2$ and (iv) is valid. In the next section, we prove Theorem 2(a) by showing that the assumptions (13) and (14) are not needed whenever d_1 and d_2 are fixed.

3. Proof of Theorem 2(a)

We may suppose that y exceeds a sufficiently large effectively computable number depending only on m, d_1 and d_2, otherwise the assertion of Theorem 2(a) follows immediately from (12) and (5). Then (14) is satisfied and (13) is a consequence of (7). Now, as remarked at the end of the previous section, we conclude the assertion of Theorem 2(a). $\square$

4. Proofs of Theorem 2(b) and Theorem 2(c)

In this section, we shall always assume that equation (1) with

$$m > 2, \quad k \leqslant C_1 \tag{34}$$

is satisfied. Then, by equation (1), we may assume that $y_2 > y'$ where y' is a sufficiently large effectively computable number depending only on k, m, d_1, d_2 and y_2 is given by (16), otherwise Theorem 2(b) and Theorem 2(c) follow immediately from (34). Then x_1, given by (16), is positive and (18) is valid. Also, we observe that (13) and (14) are satisfied. We put

$$D = d_1/d_2^m, \tag{35}$$

$$\phi(Y) = Y^m + B_1 d_2 Y^{m-1} + \cdots + B_m d_2^m - \binom{k+1}{2} d_1, \tag{36}$$

$$L(X, Y) = (X+d_1)\dots(X+kd_1) - (Y+d_2)\dots(Y+mkd_2) \tag{37}$$

and

$$l(Y) = L(\phi(Y), Y). \tag{38}$$

Now, we apply Lemma 2 and (18) to suppose that $l(y_2) = 0$. Then, since y' is sufficiently large, we derive from (34) that

$$l(Y) \equiv 0. \tag{39}$$

By (36), (37), (38) and (39), we obtain pairwise distinct integers

$$1 \leqslant \lambda_{i,j} \leqslant mk, \quad 1 \leqslant i \leqslant k, \quad 1 \leqslant j \leqslant m, \tag{40}$$

such that

$$\phi(Y) + id_1 \equiv (Y + \lambda_{i,1} d_2)\ldots(Y + \lambda_{i,m} d_2) \text{ for } 1 \leqslant i \leqslant k. \tag{41}$$

We observe that (40) covers all the integers in the interval $[1, mk]$. There is no loss of generality in assuming that each m-tuple $\{\lambda_{i,1}\ldots,\lambda_{i,m}\}$ is such that

$$\lambda_{i,1} < \cdots < \lambda_{i,m} \text{ for } 1 \leqslant i \leqslant k. \tag{42}$$

Let $\{\lambda_{i_0,1},\ldots,\lambda_{i_0,m}\}$ be the m-tuple containing 1. Then, we observe from (42) that $\lambda_{i_0,1} = 1$. Further, we derive from (41) that

$$(i - i_0)d_1 \equiv (Y + \lambda_{i,1} d_2)\ldots(Y + \lambda_{i,m} d_2) - (Y + \lambda_{i_0,1} d_2)\ldots(Y + \lambda_{i_0,m} d_2) \text{ for } 1 \leqslant i \leqslant k, i \neq i_0. \tag{43}$$

By putting $Y = -\lambda_{i_0,1} d_2 = -d_2$ in (43), we get

$$(i - i_0)d_1 = (\lambda_{i,1} - 1)\ldots(\lambda_{i,m} - 1)d_2^m \text{ for } 1 \leqslant i \leqslant k, i \neq i_0. \tag{44}$$

We observe from (44) that

$$i_0 = 1. \tag{45}$$

It follows from $B_1 = m(mk+1)/2$, (36) and (41) that

$$m(mk+1)/2 = \sum_{j=1}^{m} \lambda_{i,j} \text{ for } 1 \leqslant i \leqslant k. \tag{46}$$

Further, we set

$$\Delta_i = (\lambda_{i,1} - 1)\ldots(\lambda_{i,m} - 1) \text{ for } 2 \leqslant i \leqslant k, \tag{47}$$

$$\Delta_1 = (\lambda_{1,2} - 1)\ldots(\lambda_{1,m} - 1) \tag{48}$$

and

$$\Omega = \prod_{i=2}^{k} \Delta_i. \tag{49}$$

Now, we observe from (47), (48), (49), (40), (44) and (45) that

$$\Delta_1 \Omega = (mk - 1)! \tag{50}$$

and

$$\Omega = (k-1)!\,D^{k-1}. \tag{51}$$

Proof of Theorem 2(*b*). By (44) with $i = 2$, (45) and (35), we conclude that D is a product of m distinct positive integers. Therefore, it suffices to show that every prime divisor of D is at most m. We set

$$\psi(Z) = Z^m + B_1 Z^{m-1} + \cdots + B_m - \left(\frac{k+1}{2}\right) D \tag{52}$$

By (36) and (52),

$$\frac{\phi(Y)}{d_2^m} \equiv \psi\left(\frac{Y}{d_2}\right). \tag{53}$$

Further, by (41) and (53), it follows that

$$\psi(Z)+iD\equiv(Z+\lambda_{i,1})\ldots(Z+\lambda_{i,m}) \text{ for } 1\leqslant i\leqslant k. \tag{54}$$

Since D is an integer, we observe from (54) that $\psi(Z)$ is a polynomial of degree m with integer coefficients.

Let p be a prime divisor of D. By (44) with $i=2$ and (45), we observe that $p<mk$. Then, we derive from (54) that

$$\psi(-v)\equiv 0 (\operatorname{mod} p) \text{ for } 1\leqslant v\leqslant p. \tag{55}$$

This implies that $p\leqslant m$, since $\psi(Z)\equiv 0(\operatorname{mod} p)$ has at most m incongruent solutions $\operatorname{mod} p$. □

For an integer $v>1$, we write $P(v)$ for the greatest prime factor of v and we put $P(1)=1$. The letter p denotes always a prime number. For the proof of Theorem 2(c), we require the following results from Prime Number Theory. The first result is a sharpening, due to Hanson [1], of a theorem of Sylvester.

Lemma 5. *For positive integers $k\geqslant 2$ and $n>k$, either*

$$P(n(n+1)\ldots(n+k-1))>3k/2$$

or

$$(n,k)\in\{(3,2),(8,2),(6,5)\}.$$

The second result is due to Rosser and Schoenfeld [2, p 65–70.] on estimates for some well-known functions in Prime Number Theory. Let

$$\pi(x)=\sum_{p\leqslant x}1$$

$$\vartheta(x)=\sum_{p\leqslant x}\log p$$

and

$$E=-\gamma-\sum_{n=2}^{\infty}\sum_{p}(\log p)/p^n$$

where γ is Euler's constant. Then

Lemma 6. *For $x\geqslant 2$, we have*

$$\pi(x)>x/(\log x) \textit{ for } x\geqslant 17, \tag{56}$$

$$\pi(x)<13x/(10\log x), \tag{57}$$

$$\sum_{p\leqslant x}(\log p)/p>\log x+E-1/(2\log x), \tag{58}$$

$$\sum_{p\leqslant x}(\log p)/p<\log x+E+1/(\log x) \textit{ for } x\geqslant 32, \tag{59}$$

$$\vartheta(x)>x(1-1/(\log x)) \textit{ for } x\geqslant 41, \tag{60}$$

$$\vartheta(x)<x(1+1/(2\log x)). \tag{61}$$

By taking y' sufficiently large, we derive from Lemma 3 that

$$m \geqslant 14. \tag{62}$$

Further, we apply Lemma 5 to sharpen (62) as follows.

Lemma 7. *We have*

$$m > k. \tag{63}$$

Proof. By (62), we may assume that

$$k \geqslant 13. \tag{64}$$

We denote by $\mu_1 < \mu_2 < \cdots < \mu_s$ the elements of $\{\lambda_{1,2} - 1, \ldots, \lambda_{1,m} - 1\}$ which are greater than k. We observe that

$$0 \leqslant s \leqslant m - 1. \tag{65}$$

By writing $\mu_0 = k$ and $\mu_{s+1} = mk$, we divide

$$(k, mk) - \{\mu_1, \ldots, \mu_s\}$$

into $(s + 1)$ disjoint intervals

$$(\mu_j, \mu_{j+1}) \text{ for } 0 \leqslant j \leqslant s.$$

Then, we find J with $0 \leqslant J \leqslant s$ satisfying

$$\mu_{J+1} - \mu_J - 1 \geqslant (mk - k - s - 1)/(s + 1). \tag{66}$$

By (66), (65), (62) and (64), we derive that

$$\mu_{J+1} - \mu_J - 1 \geqslant (13k/14) - 1 > 2k/3. \tag{67}$$

Now, we derive from (67) and Lemma 5 that the interval (μ_J, μ_{J+1}) contains an integer μ divisible by a prime $> k$. Further, we observe from (49) that μ divides Ω. Therefore, we conclude from (51) and Theorem 2(b) that

$$k < P(\mu) \leqslant P(D) \leqslant m. \qquad \square$$

Lemma 8. *For* $k \geqslant 8$, *we have*

$$\log m > k - \log k - 2. \tag{68}$$

Proof. By (51), (63) and Theorem 2(b), we derive that

$$w(\Omega) \leqslant \pi(m) \tag{69}$$

where $w(\Omega)$ denotes the number of distinct prime divisors of Ω. On the other hand, we observe from (50) and (48) that

$$w(\Omega) > \pi(mk) - m. \tag{70}$$

Further, we combine (70) and (69) for deriving that

$$\pi(mk)-m<\pi(m). \tag{71}$$

Now, we apply (56) and (57) in (71) for deriving that

$$\log m > k - \log k - \left(\frac{13}{10}+\frac{13\log k}{10\log m}\right). \tag{72}$$

By (72) and (63), we have

$$\log m > k-\log k - 2{\cdot}6. \tag{73}$$

Then, since $k \geqslant 8$, we observe from (73) that $m \geqslant 28$. Now, we derive from (72) that

$$\log m > k - \log k - 2{\cdot}15.$$

Repeating this process two more times, we obtain (68). □

Proof of Theorem 2(*c*). By (62) and (68), we may assume that $k \geqslant 9$. Then, we observe from (68) that $m \geqslant 115$. The proof depends on comparing an upper and lower bound for Δ_1. By (48), we obtain

$$\Delta_1 < \lambda_{1,2}\ldots\lambda_{1,m}$$

which, by arithmetic – geometric mean and (46), implies that

$$\Delta_1 < \left(\frac{m(mk+1)}{2(m-1)}\right)^{m-1} < e\left(\frac{mk+1}{2}\right)^{m-1}. \tag{74}$$

By (50), (51), (63) and a consequence $P(D) \leqslant m$ of Theorem 2(b), we conclude that

$$\log \Delta_1 \geqslant \sum_{m<p\leqslant mk} \mathrm{ord}_p((mk-1)!)\log p.$$

Therefore

$$\log \Delta_1 \geqslant (mk-1)\sum_{m<p\leqslant mk}\frac{\log p}{p} - \vartheta(mk-1)+\vartheta(m). \tag{75}$$

Now, we apply (58), (59), (61) and (60) in (75) for deriving

$$\log \Delta_1 > (mk-1)(\log k - 2/(\log m)) - mk + m + 1 - m/(\log m). \tag{76}$$

Next, we combine (76) and (74) to obtain

$$\log m > k\log k - k - \log k - \frac{2k+1}{\log m} + \log 2 + 1. \tag{77}$$

Then, we observe from (77) and $m \geqslant 115$ that

$$\log m > k \log k - (1{\cdot}4216)k - \log k + 1{\cdot}48239.$$

Repeated applications of (77), as in the proof of Lemma 8, yield

$$\log m > k\log k - (1{\cdot}25475)k - \log k + 1{\cdot}56577.$$ □

5. Proof of Theorem 1.

Let $m = 2$. Suppose that equation (1) is satisfied. As earlier, we may assume that y exceeds a sufficiently large effectively computable number depending only on d_1 and d_2. Further, the inequalities (13) and (14) are valid. Consequently, we conclude (15). Next, we argue as in the proof of Lemma 3, for deriving (27). We calculate

$$H_4(2, k) - A_4(2, k) = (4k^5 - 5k^3 + k)/90. \tag{78}$$

By (27) and (78), we find that

$$D^2 = (d_1/d_2^2)^2 = 4(4k^2 - 1)/15. \tag{79}$$

In particular, we observe that k is bounded by an effectively computable number depending only on d_1 and d_2. Further, as in the proof of Theorem 2(b), we show that D is an integer satisfying $P(D) = 2$. Then, we conclude from (79) that $D = k = 2$, which together with (15), implies that $x = y^2 + 3d_2 y$. □

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References

[1] Hanson D, On a theorem of Sylvester and Schur, *Can. Math. Bull.* **16** (1973) 195–199

[2] Rosser B and Schoenfeld L, Approximate formulas for some functions of prime numbers, *Illinois. J. Math.* **6** (1962) 64–94

[3] Saradha N and Shorey T N, The equations $(x+1)\ldots(x+k) = (y+1)\ldots(y+mk)$ with $m = 3, 4$, *Indagationes Math.* **2**(4) (1991) 489-510

[4] Saradha N and Shorey T N, On the equation $(x+1)\ldots(x+k) = (y+1)\ldots(y+mk)$, *Indagationes Math.* 3 (1992) 79–90

[5] Saradha N and Shorey T N, On the equation $x(x+d)\ldots(x+(k-1)d) = y(y+d)\ldots(y+(mk-1)d)$, *Indagationes Math.* **3** (1992) 237–242