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On infinitesimal σ-fields generated by random processes

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Abstract. It is proved that the infinitesimal look-ahead and look-back σ-fields of a random process disagree at atmost countably many time instants.

Keywords. Random processes; look-ahead σ-field; look-back σ-field; Markov processes.

Let $X(t), t \geqslant 0$, be a Polish space-valued random process defined on a probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}$ is the P-completion of a countably generated σ-field $\mathscr{F}_0$. Let $\xi(t+)$ (the 'look-ahead' σ-field at t) and $\xi(t-)$ (the 'look-back' σ-field at t) denote the P-completions of $\cap_{s>t}\sigma(X(y), t \leqslant y \leqslant s)$ and $\cap_{s<t}\sigma(X(y), s \leqslant y \leqslant t)$ respectively. The aim of this note is to prove the following fact:

Theorem 1. *$\xi(t+) = \xi(t-)$ for all but at most countably many $t \geqslant 0$.*

We shall prove this through a sequence of lemmas. Let $\{A_n, n \geqslant 1\}$ be a dense collection of sets in $\mathscr{F}_0$ which generates $\mathscr{F}_0$.

Lemma 1. *The space of all P-complete sub-σ-fields of $\mathscr{F}$ forms a metric space under the metric*

$$\mathrm{d}(\mathscr{G}_1, \mathscr{G}_2) = \sum_n 2^{-n} E[|P(A_n/\mathscr{G}_1) - P(A_n/\mathscr{G}_2)|].$$

The proof is easy.

Remark. This topology was first introduced in Cotter [3].

For $t \geqslant 0$, let

$$\mathscr{F}(t) = \sigma(X(s), 0 \leqslant s \leqslant t), \quad \mathscr{F}(t+) = \bigcap_{s>t} \mathscr{F}(s), \quad \mathscr{F}(t-) = \bigvee_{s<t} \mathscr{F}(s),$$

all completed with respect to P. For $n \geqslant 1$, $t \geqslant 0$, define

$$h_n(t) = E[(E[I_{A_n}/\mathscr{F}(t)])^2].$$

Lemma 2. (i) *$t \to h_n(t)$ is bounded nondecreasing for all n.*
(ii) $\lim_{s\downarrow\downarrow t} h_n(s) = E[(E[I_{A_n}/\mathscr{F}(t+)])^2]$.
(iii) $\lim_{s\uparrow\uparrow t} h_n(s) = E[(E[I_{A_n}/\mathscr{F}(t-)])^2]$.

Proof. (i) follows from the conditional Jensen's inequality and (ii), (iii) follow from the convergence theorems for regular martingales and reversed martingales [2]. □

From (i) above, it follows that each $h_n(\cdot)$ has at most a countable set of points of discontinuity. Let $D \subset [0, \infty)$ be the at most countable set of points where one or more of the $h_n(\cdot)$'s is discontinuous.

Lemma 3. For $t \notin D, \mathscr{F}(t+) = \mathscr{F}(t-)$.

Proof. For $t \notin D$, Lemma 2 (ii), (iii) imply that

$$E[(E[I_{A_n}/\mathscr{F}(t+)])^2] = E[(E[I_{A_n}/\mathscr{F}(t-)])^2], \quad n \geqslant 1.$$

Thus

$$E[(E[I_{A_n}/\mathscr{F}(t+)] - E[(E[I_{A_n}/\mathscr{F}(t-)])^2] = 0, \quad n \geqslant 1,$$

implying $E[I_{A_n}/\mathscr{F}(t+)] = E[I_{A_n}/\mathscr{F}(t-)]$ a.s., $n \geqslant 1$. The claim follows from Lemma 1. □

COROLLARY 1

$\xi(t+) \subset \xi(t-)$ *for all but at most countably many* t.

Proof. Let $\{r_m\}$ be an enumeration of rationals in $[0, \infty)$. Define $\mathscr{F}^m(t)$, $\mathscr{F}^m(t+)$, $\mathscr{F}^m(t-)$, $h_n^m(t)$ as in (1)–(4) resp. with $X(r_m + .)$ replacing $X(\cdot)$, $m \geqslant 1$. The foregoing results hold for each $X(r_m + .)$ as well. For every rational $r \geqslant 0$, let

$$D_r = \{t > r \mid P\text{-completion of } \bigvee_{\varepsilon>0} \sigma(X_s, r \leqslant s \leqslant t - \varepsilon) \neq$$

$$P\text{-completion of } \bigcap_{\varepsilon>0} \sigma(X_s, r \leqslant s \leqslant t + \varepsilon)\},$$

$$\bar{D} = \cup D_r.$$

Then by Lemma 3, D_r and therefore $\bar{D}$ is at most countable. Fix $t \notin \bar{D}$ and let $\{r_{m(i)}\}$ be a collection of rationals increasing to t. Then for $i \geqslant 1$,

$$\bigvee_{\varepsilon>0} \sigma(X_s, r_{m(i)} \leqslant s \leqslant t - \varepsilon) = \bigcap_{\varepsilon>0} \sigma(X_s, r_{m(i)} \leqslant s \leqslant t + \varepsilon)$$

on P-completion. Thus

$$\xi(t+) \subset P\text{-completion of } \bigvee_{\varepsilon>0} \sigma(X_i, r_{m(i)} \leqslant s \leqslant t - \varepsilon), \quad i \geqslant 1,$$

and hence

$$\xi(t+) \subset P\text{-completion of } \bigcap_i \bigvee_{\varepsilon>0} \sigma(X_s, r_{m(i)} \leqslant s \leqslant t - \varepsilon) \subset \xi(t-).$$

The claim follows. □

Proof of Theorem 1. It suffices to consider $t \in [0, T]$ for some finite $T > 0$. Applying the above corollary to the process $X(T-t)$, $t \in [0, T]$, we conclude that $\xi(t-) \subset \xi(t+)$ except at most countably many t. Combine this with the corollary to conclude. □

COROLLARY 2

If $X(\cdot)$ is a Markov process, then $\xi(t+) = \xi(t-) = \xi(t)$ ($\triangleq$ the P-completion of $\sigma(X(t))$), at all but at most countably many t.

Proof. Note that $\xi(t) \subset \xi(t+) \cap \xi(t-)$ for all $t \geqslant 0$. Let $t \geqslant 0$ be such that $\xi(t+) = \xi(t-)$. Since $X(\cdot)$ is Markov, $\xi(t+)$, $\xi(t-)$ are conditionally independent given $X(t)$. Thus $\xi(t+)$ is conditionally independent of itself given $X(t)$, implying $\xi(t+) \subset \xi(t)$. Similarly $\xi(t-) \subset \xi(t)$. □

It is conjectured that the conclusions of Corollary 2 hold even in absence of the Markov property. If true, this result will have important implications in stochastic control theory [1]. We conclude with an example to show that one cannot improve on Theorem 1 in general.

Example Let $\Omega = [0,1]^\infty$, $\mathscr{F}_0 =$ the product Borel σ-field, $P =$ the product Lebesgue measure and $\mathscr{F}$ the product σ-field completed with respect to P. Let $w = (w_1, w_2, \cdots)$ denote a typical element of Ω. Let $\{r_n\}$ be an enumeration of rationals in (0, 1). Define an R^∞-valued process $X(t) = [X_1(t), X_2(t), \cdots]$, $t \in [0, 1]$ as follows: $[X_1(t), X_2(t), \cdots]$ evaluated at the sample point $[w_1, w_2, \cdots]$ is given by

$$X_{2i}(t) = w_{2i}[(t - r_i)^+], i \geqslant 1,$$

$$X_{2i-1}(t) = w_{2i-1}[(t - r_i)^-], \quad i \geqslant 1,$$

for $t \in [0, 1]$. Then it is easy to see that $\xi(t+)$, $\xi(t-)$, $\xi(t)$ are the P-completions of $\Pi_n G_n^+(t)$, $\Pi_n G_n^-(t)$, $\Pi_n G_n^0(t)$ respectively, where $G_n^+(t)$, $G_n^-(t)$, $G_n^0(t)$ are as described below: Let $\mathfrak{B} =$ the Borel σ-field of [0, 1], $\beta =$ the trivial σ-field $\{\phi, [0, 1]\}$ on [0, 1]. Then for $n \geqslant 1$,

$$G_{2n}^+(t) = \mathfrak{B} \text{ if } t \geqslant r_n, = \beta \text{ otherwise},$$

$$G_{2n-1}^+(t) = \mathfrak{B} \text{ if } t < r_n, = \beta \text{ otherwise},$$

$$G_{2n}^-(t) = \mathfrak{B} \text{ if } t > r_n, = \beta \text{ otherwise},$$

$$G_{2n-1}^-(t) = \mathfrak{B} \text{ if } t \leqslant r_n, = \beta \text{ otherwise},$$

$$G_{2n}^0(t) = \mathfrak{B} \text{ if } t > r_n, = \beta \text{ otherwise},$$

$$G_{2n-1}^0(t) = \mathfrak{B} \text{ if } t < r_n, = \beta \text{ otherwise}.$$

It follows that $\xi(t+) = \xi(t-) = \xi(t)$ for t irrational in [0, 1] whereas $\xi(t+) \neq \xi(t-) \neq \xi(t)$ for $t \in \{r_n, n \geqslant 1\}$.

Remark: One may replace $X(\cdot)$ above by a real valued process without altering the conclusions by virtue of the isomorphism theorem for Polish spaces (Theorem 2.12, p. 14, of [4]).

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References

[1] Borkar V S, On extremal solutions to stochastic control problems II. *Appl. Math. Optim.* **28** (1993) 49–56
[2] Chow Y S and Teicher H, *Probability theory – independence, interchangeability, martingales* (New York: Springer Verlag) (1978)
[3] Cotter K D, Similarity of information and behaviour with a pointwise convergence topology, *J. Math. Econ.* **15** (1986) 25–38
[4] Parthasarathy K R, *Probability measures on metric spaces* (New York: Academic Press) (1967)