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Lambert series and Ramanujan

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Abstract. Lambert series are of frequent occurrence in Ramanujan's work on elliptic functions, theta functions and mock theta functions. In the present article an attempt has been made to give a critical and up-to-date account of the significant role played by Lambert series and its generalizations in further development and a better understanding of the works of Ramanujan in the above and allied areas.

Keywords. Basic hypergeometric series; Lambert series; elliptic functions; mock theta functions.

1. Introduction

The topic of this article called the *Lambert series* is a very well-known class of series, both as analytic function and number theories. Our present interest in these series emerges out of the works of Ramanujan who has profusely and elegantly used these series in a variety of contexts in his mystic research work. A number of mathematicians, later, in an attempt to unravel the mysticism underlying some of the unproved identities of Ramanujan, have also used these series with advantage.

To appreciate the effectiveness of choice of the use of these series by Ramanujan and subsequently, by other mathematicians in proving a variety of identities, one must examine how beautifully they enter into the theory of numbers, theory of Weierstrass's elliptic functions and the basic hypergeometric theory.

We, now present our work starting with the definition and then the impact of Lambert series on the development of the above areas.

2. Lambert series

The series

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \tag{2.1}$$

was considered by Lambert [19] in connection with the convergence of power series. If the series $\sum_1^{\infty} a_n$ converges, then the Lambert series (2.1) converges for all values of

x except for $x = \pm 1$; otherwise it converges for those values of x for which the series $\sum_1^\infty a_n x^n$ converges. The Lambert series is used in certain problems of number theory. Thus, for $|x| < 1$ the sum $\phi(x)$ of the series (2.1) can be represented as a power series

$$\phi(x) = \sum_1^\infty \alpha_n x^n \tag{2.2}$$

where

$$\alpha_n = \sum_{k|n} a_k,$$

and the summation is over all divisors k of n. In particular, if $a_n = 1$, then $\alpha_n = \tau(n)$, the number of divisors of n. The behaviour of $\phi(x)$ (with suitable a_n) as $x \uparrow 1$ is used (see for example [20]) in the problem of Hardy and Ramanujan on obtaining an asymptotic formula for the number of unbounded partitions of a natural number.

Lambert series also occur in the expansion of Eisenstein series, a particular kind of modular form.

We list below a variety of familiar functions which can be expressed in terms of a Lambert series [18, pp. 448–451]:

(i) If $a_n = \mu(n)$, where $\mu(n)$ is the Möbius function, then

$$z = \sum_1^\infty \mu(n) \frac{z^n}{1 - z^n}.$$

(ii) For $|z| < 1$, $\dfrac{z}{(1-z)^2} = \sum_1^\infty \phi(n) \dfrac{z^n}{1 - z^n}$,

where $\phi(n)$ is the Euler function defined as the number of positive integers not exceeding n that are relatively prime to n.

(iii) If $\sum_{n=1}^\infty a_n z^n / 1 - z^n = f(z)$ and $\sum_{n=1}^\infty a_n z^n = g(z)$, then one can easily see that

$$f(z) = \sum_{m=1}^\infty g(z^m).$$

(iv) For $a_n = (-1)^{n-1}$; $a_n = n$; $a_n = (-1)^{n-1} n$; $a_n = 1/n$; $a_n = (-1)^{n-1}/n$; $a_n = \alpha^n$; we have for $|z| < 1$, the remarkable identities, (each summation is for $n = 1$ to ∞),

a) $\sum (-)^{n-1} \dfrac{z^n}{1 - z^n} = \sum \dfrac{z^n}{1 + z^n}$

b) $\sum n \dfrac{z^n}{1 - z^n} = \sum \dfrac{z^n}{(1 - z^n)^2}$

c) $\sum (-)^{n-1} \dfrac{nz^n}{1 - z^n} = \sum \dfrac{z^n}{(1 + z^n)^2}$

d) $\sum \dfrac{1}{n} \dfrac{z^n}{1 - z^n} = \sum \log\left(\dfrac{1}{1 - z^n}\right)$

e) $\sum \frac{(-)^{n-1}}{n} \frac{z^n}{1-z^n} = \sum \log(1+z^n)$

f) $\sum \alpha^n \frac{z^n}{1-z^n} = \sum \frac{\alpha z^n}{1-\alpha z^n}$,

respectively.

(v) In the identities (iv) (d) and (e) above, with the series of Logarithms (for which we take the principal values) one easily sees that they are equivalent to

$$\prod_1^\infty (1-z^n) = e^{-\omega} \text{ with } \omega = \sum \frac{1}{n} \frac{z^n}{1-z^n},$$

$$\prod_1^\infty (1+z^n) = e^{\omega} \text{ with } \omega = \sum \frac{(-)^{n-1}}{n} \frac{z^n}{1-z^n}.$$

(vi) An interesting numerical example is furnished by the Fibonacci sequence of numbers defined by $u_0 = 0$, $u_i = 1$ and for every $n > 1$, $u_n = u_{n-1} + u_{n-2}$, namely

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

We then have

$$\sum_{k=1}^{\infty} \frac{1}{u_{2k}} = 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{21} + \frac{1}{55} + \cdots$$

$$= \sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right],$$

where $L(x)$ is the sum of Lambert series $\Sigma x^n/1 - x^n$. The proof is based on the fact that

$$u_v = \frac{\alpha^v - \beta^v}{\alpha - \beta}, \quad (v = 0, 1, 2, \ldots)$$

where α and β are the roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

(vii) Lastly, an important illustration is provided by the identity

$$\log \Gamma_q(x) = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}},$$

where $0 < q < 1$, $x > 0$ and $\Gamma_q(x)$ is Jackson's q-analogue of gamma function defined as $(q)_\infty (1-q)^{1-x}/(q^x)_\infty$.

The integral transform

$$F(x) = \int_0^\infty \frac{ta(t)}{e^{xt}-1} dt,$$

known as the Lambert transform is the continuous analogue of the Lambert series (under the correspondence $ta(t)\leftrightarrow a_n$, $e^{-x}\leftrightarrow x$). The following inversion formula holds: Suppose that

$$a(t)\in L(0,\infty)$$

and that

$$\lim_{t\to +0} a(t)t^{1-\delta}=0,\quad \delta>0.$$

If also $\tau>0$ and if the function $a(t)$ is continuous at $t=\tau$, then one has

$$\tau a(\tau)=\lim_{k\to\infty}\frac{(-1)^k}{k!}\left(\frac{k}{\tau}\right)^{k+1}\sum_{n=1}^{\infty}\mu(n)n^kF^{(k)}\left(\frac{nk}{\tau}\right),$$

where $\mu(n)$ is the Möbius function (Widder [30]).

Still another case of particular interest is the Lambert series obtained by taking

$$\begin{aligned}F(x)&=\sum_{n=1}^{\infty}a_n\frac{x^n}{1-x^n}\\&=\sum_{n=1}^{\infty}a_n\sum_{m=1}^{\infty}x^{mn}\\&=\sum_{N=1}^{\infty}b_Nx^N,\end{aligned}$$

where $b_N=\sum_{n|N}a_n$.

The relation between $\{a_n\}$ and $\{b_n\}$ is equivalent to

$$\xi(s)f(s)=g(s),$$

where $f(s)$ and $g(s)$ are respectively the Dirichlet series associated with $\{a_n\}$ and $\{b_n\}$. Hence, if $f(s)=\sum_1^{\infty}a_nn^{-s}$ and $g(s)=\sum_1^{\infty}b_nn^{-s}$, then

$$F(x)=\sum_{n=1}^{\infty}a_n\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}b_nx^n,$$

if and only if

$$\xi(s)f(s)=g(s).$$

3. Lambert series and certain arithmetical functions

We begin with an identity founded by E.T. Bell, namely

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right]=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}. \tag{3.1}$$

Bell obtained the formula (or its arithmetical equivalent) while looking for quadratic forms which represent all integers with at most a finite number of exceptions. An immediate equivalent of (3.1) is

$$\sum_{s=1}^{n-1}\left[\sum_{d|s} N_d(n-s)\right]=\zeta_2(n)-\zeta_0(n), \tag{3.2}$$

where $N_a(m)$ is the sum of the divisors of m not greater than m/a, and $\zeta_r(n)$ denotes the sum of rth powers of the divisors of n. Another consequence of (3.1) mentioned by Bell is that the number of representations of any integer $n>0$ in the form $wx+xy+yz+zu$, where $w,x,z,u>0, y\geqslant 0$, is

$$\zeta_2(n)-n\zeta_0(n),$$

a classical result stated without proof by Liouville, in 1867.

Bailey [12] gave three proofs for (3.1). Two of the proofs are probably as elementary as one could expect.

In the first proof, he began by proving the formula

$$\frac{z}{1-q}+\frac{2z^2}{1-q^2}+\frac{3z^3}{1-q^3}+\cdots=\frac{z}{(1-q)(1-z)}+\frac{(1-q)z^2}{(1-q^2)(1-z)(1-qz)}+$$
$$+\frac{(1-q)(1-q^2)z^3}{(1-q^3)(1-z)(1-qz)(1-q^2z)}+\cdots \tag{3.3}$$

where $|z|<1$, $|q|<1$.

Denoting the right hand side by $F(z)$, and writing

$$(a)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),\quad (a)_0=1,$$

we have, on simplification

$$F(z)-F(qz)=\frac{z}{(1-z)^2}.$$

It follows that

$$F(z)=\frac{z}{(1-z)^2}+\frac{qz}{(1-qz)^2}+\frac{q^2z}{(1-q^2z)^2}+\cdots$$
$$=\sum_{r=0}^{\infty}\frac{q^rz}{(1-q^rz)^2}=\sum_{r=0}^{\infty}\sum_{n=1}^{\infty}n(q^rz)^n=\sum_{n=1}^{\infty}\frac{nz^n}{1-q^n},$$

and this proves (3.3).

Now differentiating (3.3) with respect to z, we obtain the formula

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}z^n}{(1-q^n)(z)_n}\left[\frac{1}{1-z}+\frac{1}{1-qz}+\cdots+\frac{1}{1-q^{n-1}z}\right]=\sum_{n=1}^{\infty}\frac{n^2z^n}{1-q^n}, \tag{3.4}$$

which reduces to (3.1) when $z=q$.

Another result given by Bailey, in this context, was the identity

$$(1-a)\sum_{n=1}^{\infty}\frac{(1-aq^{2n})q^n}{(1-q^n)^2(1-aq^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right.$$
$$\left.+\frac{a}{1-a}+\frac{aq}{1-aq}+\cdots+\frac{aq^{n-1}}{1-aq^{n-1}}\right]=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}, \tag{3.5}$$

which reduces to Bell's identity when $a=0$.

Bell pointed out that if in (3.5) one expands in powers of q and equates coefficients of like powers of q, one gets a curious polynomial in a, which must vanish identically. Expressing this identically vanishing polynomial arithmetically, one can get theorems on numbers of representations and also interesting recurrence formulae for functions of divisors. By giving 'a' special values in the polynomial, for example $a=-1$ or $a=\rho$, where ρ is a primitive rth root of unity, one obtains further theorems relating to restricted representations.

In view of Bell's remarks on the number-theoretic interest of (3.5), Bailey [13] gave elementary algebraic proof for it. Being quite instructive we give below in some details this proof.

Denote the left hand side of (3.5) by $F(a)$. Then

$$F(a)=\sum_{n=1}^{\infty}\left[\frac{q^n}{(1-q^n)^2}-\frac{aq^n}{(1-aq^n)^2}\right]\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}+\right.$$
$$\left.+\frac{a}{1-a}+\frac{aq}{1-aq}+\cdots+\frac{aq^{n-1}}{1-aq^{n-1}}\right],$$

and, on simplification, it can be shown that

$$F(a)-F(aq)=0.$$

Thus

$$F(a)=F(aq)=\cdots=F(aq^n).$$

Making $n\to\infty$, we get

$$F(a)=F(0)=\sum_{a=1}^{\infty}\frac{n^2q^n}{1-q^n},$$

Bell's identity being used in the final step.

In a letter to Bailey, Bell later mentioned still another identity, namely

$$8\sum_{\substack{m>1\\ m\,\mathrm{odd}}}\frac{q^m(1+q^{2m})}{(1-q^{2m})^2}\left[\frac{1}{1-q^2}+\frac{1}{1-q^4}+\cdots+\frac{1}{1-q^{2m-2}}\right]$$
$$=\sum_{m\,\mathrm{odd}}\left[\frac{(m^2-1)q^m}{1-q^{2m}}+\frac{4(m-1)q^m(1+q^{2m})}{(1-q^{2m})^2}\right]. \tag{3.6}$$

He was led to this identity from the theorem that, if m is odd, then the number of representations of m in the form

$$m = 2wx + xy + yz + zu + ux,$$

where $w, x, y, z > 0$, $u \geqslant 0$, is

$$\frac{1}{8}[\xi_2(m) - (4m+1)\xi_0(m) + 4\xi_1(m)],$$

$\xi_r(n)$ being the sum of the rth powers of all divisors of n. Bailey proved that (3.6) can be easily deduced from (3.5). Put $a = -1$ in (3.5) to obtain

$$2\sum_{n=1}^{\infty} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[-\frac{1}{2} + \frac{1}{1-q^n} + \frac{1+q^2}{1-q^2} + \frac{1+q^4}{1-q^4} + \cdots + \frac{1+q^{2n-2}}{1-q^{2n-2}}\right]$$
$$= \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^n},$$

Changing the sign of q and subtracting, we find that

$$4\sum_{n\,\mathrm{odd}} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[-\frac{1}{2} + \frac{1}{1-q^{2n}} + \frac{1+q^2}{1-q^2} + \frac{1+q^4}{1-q^4} + \cdots + \frac{1+q^{2n-2}}{1-q^{2n-2}}\right]$$
$$= \sum_{n\,\mathrm{odd}} \frac{2n^2 q^n}{1-q^{2n}},$$

and so

$$8\sum_{n>1,\mathrm{odd}} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[\frac{1}{1-q^2} + \frac{1}{1-q^4} + \cdots + \frac{1}{1-q^{2n-2}}\right]$$
$$= \sum_{n\,\mathrm{odd}} \frac{2n^2 q^n}{1-q^{2n}} + 4\sum_{n\,\mathrm{odd}} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[\frac{1}{2} - \frac{1}{1-q^{2n}} + n - 1\right]$$
$$= \sum_{n\,\mathrm{odd}}\left[\frac{2(n^2-1)q^n}{1-q^{2n}} + \frac{4(n-1)q^n(1+q^{2n})}{(1-q^{2n})^2} - \frac{8q^{3n}}{(1-q^{2n})^3}\right].$$

Since

$$\sum_{n\,\mathrm{odd}} \frac{8q^{3n}}{(1-q^{2n})^3} = 4\sum_{n\,\mathrm{odd}}\sum_{m=1}^{\infty} m(m+1)q^{m(2m+1)}$$
$$= 4\sum_{m=1}^{\infty} \frac{m(m+1)q^{2m+1}}{1-q^{2(2m+1)}} = \sum_{n\,\mathrm{odd}} \frac{(n^2-1)q^n}{1-q^{2n}},$$

(3.6) follows.

It may be further noted, as a consequence of the above identities, that

$$\sum_{n=1}^{\infty} \xi_2(n)q^n = \sum_{1}^{\infty} \frac{n^2 q^n}{1-q^n},$$

$$\sum_{n=1}^{\infty} \xi_0(n)q^n = \sum_{1}^{\infty} \frac{q^n}{1-q^n}$$

$$\sum_{n=1}^{\infty} \xi_1(n)q^n = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2},$$

$$\sum_{n=1}^{\infty} n\xi_0(n)q^n = \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)^2}.$$

The above illustrations are enough to show how elegantly the Lambert series are related to various arithmetical functions.

4. Lambert series and Weierstrass's elliptic functions

We now pass on to exhibit the relationship of the Lambert series with the classical Weierstrass elliptic function.

Ramanujan [21] gave two formulae

$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5\prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{(1-x^n)^6}, \tag{4.1}$$

$$\prod_{n=1}^{\infty} \frac{(1-x^n)^5}{1-x^{5n}} = 1 - 5\left(\frac{x}{1-x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6} - \cdots\right), \tag{4.2}$$

where $p(n)$ is the number of partitions of n. In addition Ramanujan stated the formula

$$x\prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{1-x^n} = \frac{x}{(1-x)^2} - \frac{x^2}{(1-x^2)^2} - \frac{x^3}{(1-x^3)^2} + \frac{x^4}{(1-x^4)^2} + \frac{x^6}{(1-x^6)^2} - \cdots, \tag{4.3}$$

and used it to prove (4.1). Bailey [15] showed that the proofs of (4.2) and (4.3) depend only on well-known formulae in elliptic functions which we, today, believe would be familiar to Ramanujan.

It is well known that

$$\mathscr{P}(u) = -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \operatorname{cosec}^2 \frac{\pi u}{2\omega_1} - 2\left(\frac{\pi}{\omega_1}\right)^2 \sum_{1}^{\infty} \frac{sq^{2s}}{1-q^{2s}} \cos\frac{s\pi u}{\omega_1},$$

where $\mathscr{P}(u)$ is the Weierstrass's elliptic function.

It is also easy to see that

$$\sum_{-\infty}^{\infty} \frac{aq^n}{(1-aq^n)^2} = \sum_{n=1}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} + \frac{q^n/a}{(1-q^n/a)^2}\right]$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m(a^m + a^{-m})q^{mn} = \sum_{m=1}^{\infty} \frac{m(a^m + a^{-m})q^m}{1-q^m}, \tag{4.4}$$

where $|q| < |a| < 1/|q|$. If we write $a = \exp(\pi i u/\omega_1)$, we get

$$\sum_{-\infty}^{\infty} \frac{aq^{2n}}{(1-aq^{2n})^2} = \frac{\eta\omega_1}{\pi^2} - \frac{\omega_1^2}{\pi^2}\mathscr{P}(u).$$

Thus, if $b = \exp(\pi i v/\omega_1)$, we have

$$\sum_{-\infty}^{\infty} \left[\frac{aq^{2n}}{(1-aq^{2n})^2} - \frac{bq^{2n}}{(1-bq^{2n})^2}\right] = \frac{\omega_1^2}{\pi^2}[\mathscr{P}(v) - \mathscr{P}(u)] = \frac{\omega_1^2}{\pi^2}\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}.$$

We therefore find, after a little reduction and changing q^2 into q, that

$$\sum_{-\infty}^{\infty} \left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right] = a\Pi\left[\begin{matrix} ab, q/ab, b/a, qa/b, q, q, q, q; \\ a, a, q/a, q/a, b, b, q/b, q/b \end{matrix}\right] \qquad (4.5)$$

where

$$\Pi\left[\begin{matrix} a_1, a_2, \ldots, \\ b_1, b_2, \ldots \end{matrix}\right] = \prod_{n=0}^{\infty} \frac{(1-a_1q^n)(1-a_2q^n)\cdots}{(1-b_1q^n)(1-b_2q^n)\cdots}.$$

The formula (4.5) is thus equivalent to the well-known expression for $\mathscr{P}(u) - \mathscr{P}(v)$ in terms of σ-functions.

For the proofs of both (4.2) and (4.3) we take $b = a^2$ in (4.5). First take $a = x$, $b = x^2$, $q = x^5$, and we get

$$\sum_{-\infty}^{\infty} \left[\frac{x^{5n+1}}{(1-x^{5n+1})^2} - \frac{x^{5n+2}}{(1-x^{5n+2})^2}\right] = x\prod_{1}^{\infty}\frac{(1-x^{5n})^5}{1-x^n},$$

which is (4.3).

Next take $a = \omega$, $b = \omega^2$, where $\omega = \exp(2\pi i/5)$, and the product on the right of (4.5) becomes

$$\frac{\omega(1-\omega^3)(1-\omega)}{(1-\omega)^2(1-\omega^2)^2}\prod_{1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}.$$

The left hand side of (4.5) is, by (4.4)

$$\frac{\omega}{(1-\omega)^2} - \frac{\omega^2}{(1-\omega^2)^2} + \sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}(\omega^m + \omega^{4m} - \omega^{2m} - \omega^{3m})$$

$$= \frac{\omega(1-\omega^3)(1-\omega)}{(1-\omega)^2(1-\omega^2)^2} + \sum_{m=1}^{\infty}\frac{mq^m\omega^m(1-\omega^m)(1-\omega^{2m})}{1-q^m}.$$

If we denote $\omega(1-\omega)(1-\omega^2)$ by A, it is easily seen that

$$\omega^m(1-\omega^m)(1-\omega^{2m})$$

has the values $0, A, -A, -A, A$ when m has the forms $5n$, $5n+1$, $5n+2$, $5n+3$, $5n+4$. Thus

$$\prod_{1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}} = 1 + B\sum_{n=0}^{\infty}\left[\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}} - \frac{(5n+2)q^{5n+2}}{1-q^{5n+2}} - \frac{(5n+3)q^{5n+3}}{1-q^{5n+3}} + \frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right],$$

where

$$B = -(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$$
$$= -5.$$

This completes the proof of (4.2). It is interesting to note that both (4.2) and (4.3) are merely particular cases of the formula expressing $\mathscr{P}(u) - \mathscr{P}(2u)$ in terms of σ-functions, when the $\mathscr{P}$-functions are replaced by the corresponding Fourier series.

This singular illustration picked out of Ramanujan's own work, to my mind is sufficient to bring home the close relationship of the Lambert series with the elliptic function theory with which Ramanujan is believed to have been well-acquainted. Bailey's proof also suggests as to how Ramanujan, himself, might have derived (4.2) and (4.3). One could, of course, cite many more such examples (see § 7 ahead).

5. Lambert series and basic hypergeometric series

This aspect of study of the Lambert series and basic hypergeometric series is, probably, the most effective tool, in handling a large number of queer looking unproved identities of Ramanujan. Perhaps, Ramanujan must have gone through the elements of these functions, although, it is not likely, that he may have used them to discover some of his remarkable identities mentioned, without proof, and now deducible directly from the transformation theory of basic hypergeometric series [1, 2, 3, 5, 9].

In what follows we shall use the following notation:

If

$$|q| < 1, \quad (a;q^k)_n = (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)})$$
$$(a;q^k)_0 = 1 \text{ and } (a;q^k)_\infty = \prod_0^\infty (1-aq^{kn}),$$

(for $k = 1$, we simply write $(a)_n$), and define a generalised basic hypergeometric series for $|z| < 1$, $|q| < 1$

$$_{r+1}\phi_r\left[\begin{matrix} a_1,\ldots\ldots, a_{r+1}; z \\ b_1,\ldots\ldots, b_r \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n,\ldots\ldots(a_{r+1};q)_n z^n}{(q;q)_n(b_1;q)_n\ldots\ldots(b_r;q)_n}.$$

Also, a generalized bilateral basic hypergeometric series is defined as

$$_r\Psi_r\left[\begin{matrix} a_1,\ldots\ldots, a_r; q^k; z \\ b_1,\ldots\ldots, b_r \end{matrix}\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1;q^k)_n,\ldots\ldots(a_r;q^k)_n z^n}{(b_1;q^k)_n\ldots\ldots(b_r;q^k)_n},$$

convergent for

$$\left|\frac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r}\right| < |z| < 1.$$

To illustrate the elegance and effectiveness of basic hypergeometric methods in proving and generalising a variety of q-identities, compared to other methods, alternative proofs are given here through these functions, of some of the key identities already proved in § 3 and § 4 by different methods.

The formula (3.1) can also be deduced from the basic analogue of Gauss's theorem for the sum of the hypergeometric series with unit argument. This analogue is

$$\sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(q)_n(d)_n}\left(\frac{d}{bc}\right)^n = \prod_{n=0}^{\infty}\left[\frac{(1-dq^n/b)(1-dq^n/c)}{(1-dq^n)(1-dq^n/bc)}\right]. \tag{5.1}$$

Differentiating with respect to d, and dividing by $(1-b)(1-c)$, we find that

$$\sum_{n=1}^{\infty} \frac{(bq)_{n-1}(cq)_{n-1}}{(q)_n(d)_n}\left(\frac{d}{bc}\right)^{n-1}\left[\frac{1}{1-d}+\frac{1}{1-qd}+\cdots\frac{1}{1-q^{n-1}d}\right]$$
$$= \prod_{n=0}^{\infty}\left[\frac{(1-dq^n/b)(1-dq^n/c)}{(1-dq^n)(1-dq^n/bc)}\right]$$
$$\times \sum_{n=0}^{\infty} \frac{q^n(1-d^2q^{2n}/bc)}{(1-dq^n)(1-dq^n/b)(1-dq^n/c)(1-dq^n/bc)}. \tag{5.2}$$

Setting b and c tend to unity (assuming that $|q|<1$, $|d|<1$), we obtain

$$\sum_{n=1}^{\infty} \frac{(q)_{n-1}d^n}{(1-q^n)(d)_n}\left[\frac{1}{1-d}+\frac{1}{1-qd}+\cdots\frac{1}{1-q^{n-1}d}\right] = \sum_{n=0}^{\infty}\frac{dq^n(1+dq^n)}{(1-dq^n)^3}. \tag{5.3}$$

When $d=q$, this gives

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right] = \sum_{n=1}^{\infty}\frac{q^n(1+q^n)}{(1-q^n)^3}$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^2q^{mn} = \sum_{m=1}^{\infty}\frac{m^2q^m}{1-q^m},$$

which is Bell's identity (3.1).

For general values of d, (5.3) similarly gives (3.4). It is evident that further results can be obtained, in a manner similar to the above from other formulae which sum basic series.

Thus, from the formula

$${}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & d, & e, & f; aq/def \\ & \sqrt{a}, & -\sqrt{a}, & aq/d, & aq/e, & aq/f \end{matrix}\right]$$
$$= \prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/ef)(1-aq^n/df)(1-aq^n/de)}{(1-aq^n/d)(1-aq^n/e)(1-aq^n/f)(1-aq^n/def)}\right] \tag{5.4}$$

we can deduce, among other results, the formula

$$(1-a)\sum_{n=1}^{\infty}\frac{(1-aq^{2n})q^n}{(1-q^n)^2(1-aq^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right.$$
$$\left.+\frac{a}{1-a}+\frac{aq}{1-aq}+\cdots+\frac{aq^{n-1}}{1-aq^{n-1}}\right] = \sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}. \tag{5.5}$$

For $a=0$ this again reduces to Bell's identity. For $a=q$ and $a=-q$, we obtain the formulae

$$(1-q)\sum_{n=1}^{\infty}\frac{(1-q^{2n+1})q^n}{(1-q^n)^2(1-q^{n+1})^2}\left[\frac{1+q}{1-q}+\frac{1+q^2}{1-q^2}+\cdots+\frac{1+q^n}{1-q^n}\right]$$
$$=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}. \tag{5.6}$$

$$(1+q)\sum_{n=1}^{\infty}\frac{(1+q^{2n+1})q^n}{(1-q^n)^2(1+q^{n+1})^2}\left[\frac{1+q^2}{1-q^2}+\frac{1+q^4}{1-q^4}+\cdots+\frac{1+q^{2n}}{1-q^{2n}}\right]$$
$$=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}. \tag{5.7}$$

In 1952, Bailey [14] showed that even (4.3), follows easily from the known sum of a well-poised basic bilateral hypergeometric series. The right hand side of (4.3) is

$$\sum_{n=0}^{\infty}\left[\frac{x^{5n+1}}{(1-x^{5n+1})^2}-\frac{x^{5n+2}}{(1-x^{5n+2})^2}-\frac{x^{5n+3}}{(1-x^{5n+3})^2}+\frac{x^{5n+4}}{(1-x^{5n+4})^2}\right]$$
$$=\sum_{n=-\infty}^{\infty}\left[\frac{x^{5n+1}}{(1-x^{5n+1})^2}-\frac{x^{5n+3}}{(1-x^{5n+3})^2}\right]$$
$$=\sum_{n=-\infty}^{\infty}\frac{x^{5n+1}(1-x^2)(1+x^{5n+2})(1-x^{5n+2})}{(1-x^{5n+1})^2(1-x^{5n+3})^2}$$
$$=\frac{x(1-x^2)(1-x^4)}{(1-x)^2(1-x^3)^2}\,{}_6\Psi_6\left[\begin{matrix}x^7, & -x^7, & x, & x, & x^3, & x^3; \\ x^2, & -x^2, & x^8, & x^8, & x^6, & x^6\end{matrix}\; x^5\right]$$
$$=\frac{x(1-x^2)(1-x^4)}{(1-x)^2(1-x^3)^2}\times$$
$$\times\prod_{n=1}^{\infty}\left[\frac{(1-x^{5n+4})(1-x^{5n+2})(1-x^{5n})^4(1-x^{5n-2})(1-x^{5n-4})}{(1-x^{5n-1})^2(1-x^{5n-3})^2(1-x^{5n+3})^2(1-x^{5n+1})^2}\right]$$
$$=x\prod_{n=1}^{\infty}\frac{(1-x^{5n})^5}{1-x^n},$$

where $q=x^5$ in the bilateral series. This completes the proof.

The other formula (4.2), namely

$$\frac{[(1-x)(1-x^2)(1-x^3)\ldots]^5}{(1-x^5)(1-x^{10})(1-x^{15})\ldots}$$
$$=1-5\left(\frac{x}{1-x}-\frac{2x^2}{1-x^2}-\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{6x^6}{1-x^6}-\cdots\right), \tag{5.8}$$

which was stated by Ramanujan in one of his unpublished manuscript, but was also given elsewhere, can be expressed in the form that, if

$$f(x)=x^{1/5}\frac{(1-x)(1-x^4)(1-x^6)(1-x^9)\ldots}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\ldots},$$

$$\frac{df}{fdx}=\frac{1}{5x}\prod_{n=1}^{\infty}\frac{(1-x^n)^5}{1-x^{5n}}.$$

A very complicated proof of this result was given by Darling whereas a short but more sophisticated proof was given by Mordell. Bailey [14] showed that (5.8) can also be deduced from the same sum of a bilateral series. In fact, from this sum we have

$$ {}_6\Psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, \omega\sqrt{a}, \omega^2\sqrt{q}, \omega^3\sqrt{a}, \omega^4\sqrt{a}; \\ \sqrt{a}, -\sqrt{a}, \omega^4 q\sqrt{a}, \omega^3 q\sqrt{a}, \omega^2 q\sqrt{a}, \omega q\sqrt{a}\end{matrix} q\right]$$
$$=\prod_{n=1}^{\infty}\frac{(1-q^n)(1-aq^n)(1-q^n/a)(1-q^{5n})(1-q^n\sqrt{a})(1-q^n/\sqrt{a})}{(1-q^{5n}a^{5/2})(1-q^{5n}/a^{5/2})},$$

where $\omega=\exp(2\pi i/5)$. Now, if $a\to 1$, the product on the right becomes

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}.$$

The series on the left is

$$1+\frac{1-a^{5/2}}{(1-a)(1-a^{1/2})}\sum_{n=1}^{\infty}\left[\frac{(1-aq^{2n})(1-a^{1/2}q^n)q^n}{1-a^{5/2}q^{5n}}-\frac{(q^{2n}-a)(q^n-a^{1/2})q^n}{a^{5/2}-q^{5n}}\right]. \tag{5.9}$$

When $a\to 1$, (5.9) becomes

$$1-5\sum_{n=1}^{\infty}\left[\frac{q^n-2q^{2n}-3q^{3n}+4q^{4n}+4q^{6n}-3q^{7n}-2q^{8n}+q^{9n}}{(1-q^{5n})^2}\right].$$

Now

$$\sum_{n=1}^{\infty}\frac{q^{an}}{(1-q^{5n})^2}=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(m+1)q^{(5m+a)n}=\sum_{m=0}^{\infty}\frac{(m+1)q^{5m+a}}{1-q^{5m+a}},$$

and so we have

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}=1-5\sum_{m=0}^{\infty}\left[\frac{(m+1)q^{5m+1}}{1-q^{5m+1}}-\frac{2(m+1)q^{5m+2}}{1-q^{5m+2}}-\right.$$
$$-\frac{3(m+1)q^{5m+3}}{1-q^{5m+3}}+\frac{4(m+1)q^{5m+4}}{1-q^{5m+4}}+\frac{4mq^{5m+1}}{1-q^{5m+1}}-$$
$$\left.-\frac{3mq^{5m+2}}{1-q^{5m+2}}-\frac{2mq^{5m+3}}{1-q^{5m+3}}+\frac{mq^{5m+4}}{1-q^{5m+4}}\right]$$
$$=1-5\sum_{m=0}^{\infty}\left[\frac{(5m+1)q^{5m+1}}{1-q^{5m+1}}-\frac{(5m+2)q^{5m+2}}{1-q^{5m+2}}-\right.$$
$$\left.-\frac{(5m+3)q^{5m+3}}{1-q^{5m+3}}+\frac{(5m+4)q^{5m+4}}{1-q^{5m+4}}\right].$$

This completes the proof of (4.2).

The above applications provide a *prima facie* evidence of the elegance with which the basic hypergeometric series may be utilized in the derivation and generalisation of a variety of q-identities involving Lambert series.

6. Ramanujan and a generalised Lambert series

We now turn to numerous identities given by Ramanujan in which Lambert series and its generalisations find a mention. A series of the form

$$\sum_{n=-\infty}^{\infty} (-1)^{\varepsilon n} q^{\lambda n^2} R(q^n), \tag{6.1}$$

where $\varepsilon = 0$ or 1, $\lambda > 0$, and $R(x)$ is a rational function of x, will be called a generalised Lambert series.

In the 'Lost' Notebook [7] Ramanujan gave a number of identities involving series of the type (6.1). I mention below eleven of them, namely

$$[C(q)]^{-3} = \frac{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}}} \tag{6.2}$$

$$= \frac{\sum_{n=0}^{\infty} q^{5n^2+4n} \frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1} \frac{1+q^{5n+3}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty} q^{5n^2+2n} \frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3} \frac{1+q^{5n+4}}{1-q^{5n+4}}}. \tag{6.3}$$

$$(q^5;q^5)_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}} \tag{6.4}$$

$$(q^5;q^5)_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}} \tag{6.5}$$

$$(q^5;q^5)_\infty^2 \frac{G(q)^2}{H(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}} \tag{6.6}$$

$$(q^5;q^5)_\infty^2 \frac{H(q)^2}{G(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}} \tag{6.7}$$

$$C(q) = \frac{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}} \tag{6.8}$$

$$C(q)^2 = \frac{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}}}{\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}} \tag{6.9}$$

$$C(q)^2 = \frac{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}, \tag{6.10}$$

$$(q^5;q^5)_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{10n+1}} \tag{6.11}$$

$$(q^5;q^5)_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{10n+3}}, \tag{6.12}$$

where the Rogers-Ramanujan functions

$$G(q) = [(q;q^5)_\infty (q^4;q^5)_\infty]^{-1}$$
$$H(q) = [(q^2;q^5)_\infty (q^3;q^5)_\infty]^{-1}$$

and $C(q)$ is the Ramanujan's continued function with its value equal to $C(q) = G(q)/H(q)$.

To prove the equivalences (6.2–6.12) let us consider the famous Ramanujan's sum of a bilateral basic hypergeometric series ${}_1\Psi_1$, namely

$$\begin{aligned} {}_1\Psi_1(a;b;q;z) &= \sum_{-\infty}^{\infty} [a]_n z^n/[b]_n \\ &= \frac{[b/a;q]_\infty [az;q]_\infty [q/az;q]_\infty [q;q]_\infty}{[q/a;q]_\infty [b/az;q]_\infty [b;q]_\infty [z;q]_\infty}. \end{aligned} \tag{6.13}$$

Hardy [17] had called this sum as a remarkable formula with many parameters. It is noteworthy that more than half-a-dozen different proofs of this interesting sum have been given by different mathematicians from time to time.

Thus,

$$\frac{1}{(1-\alpha)} {}_1\Psi_1(\alpha;\alpha q^\mu;q^\mu;z); \quad (|q^\mu| < |\alpha|, |z| < 1) \tag{6.14}$$

$$\begin{aligned} &= \sum_{n=-\infty}^{\infty} \frac{z^n}{1-\alpha q^{\mu n}} \\ &= \frac{[q^\mu;q^\mu]_\infty^2 [z\alpha;q^\mu]_\infty [q^\mu/\alpha z;q^\mu]_\infty}{[q^\mu/\alpha;q^\mu]_\infty [q^\mu/z;q^\mu]_\infty [\alpha;q^\mu]_\infty [z;q^\mu]_\infty}. \end{aligned} \tag{6.15}$$

We now proceed to prove that

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} (\alpha z)^n q^{\mu n^2} \frac{(1-z\alpha q^{2\mu n})}{(1-\alpha q^{\mu n})(1-zq^{\mu n})} \\ &= \sum_{-\infty}^{\infty} \frac{z^n}{(1-\alpha q^{\mu n})}; \quad |\alpha| < 1, |z| < 1. \end{aligned} \tag{6.16}$$

We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{z^n}{1-\alpha q^{\mu n}} &= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{\infty} q^{m\mu n}\alpha^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} z^{n+m} q^{m\mu(m+n)}\alpha^m + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} z^n \alpha^{m+n+1} q^{n\mu(m+n+1)} \\ &= \sum_{m=0}^{\infty} z^m\alpha^m q^{\mu m^2}\sum_{n=0}^{\infty} z^n q^{nm\mu} + \sum_{n=0}^{\infty} z^n\alpha^{n+1}q^{n\mu(n+1)}\sum_{m=0}^{\infty} q^{mn\mu}\alpha^m \\ &= \sum_{m=0}^{\infty}\frac{(z\alpha)^m q^{\mu m^2}}{1-zq^{m\mu}} + \alpha\sum_{n=0}^{\infty}\frac{(\alpha z)^n q^{n^2\mu+n\mu}}{1-\alpha q^{n\mu}}.\end{aligned}$$

$$\therefore \sum_{n=0}^{\infty}\frac{z^n}{1-\alpha q^{\mu n}} = \sum_{n=0}^{\infty}\frac{(\alpha z)^n q^{\mu n^2}(1-z\alpha q^{2n\mu})}{(1-zq^{n\mu})(1-\alpha q^{n\mu})}. \tag{6.17}$$

Using (6.17), twice, we get (6.16).

Thus, we get by using (6.14) and (6.15) in (6.16), the main result in the equivalent form ($|q^\mu| < |z|, |\alpha| < 1$)

$$\begin{aligned}&\sum_{-\infty}^{\infty} q^{\mu n^2}\frac{(1-\alpha z q^{2\mu n})(\alpha z)^n}{(1-\alpha q^{\mu n})(1-zq^{\mu n})} \\ &= \frac{[q^\mu;q^\mu]_\infty^2[\alpha z;q^\mu]_\infty[q^\mu/\alpha z;q^\mu]_\infty}{[q^\mu/\alpha;q^\mu]_\infty[q^\mu/z;q^\mu]_\infty[\alpha;q^\mu]_\infty[z;q^\mu]_\infty}\end{aligned} \tag{6.18}$$

The cases $\alpha = q^i, i = 1,2,3,4; z = q^j, 0 < j < 4; \mu = 5$ were considered by Andrews [8]. It may be further remarked that the results of Andrews can also follow directly, as very special cases of the Bailey's sum of a well-poised ${}_6\Psi_6$-series.

Following Andrews [8], to prove (6.2) and (6.3), take in the main result (6.18), $\mu = 5$, $z = \alpha = q^2$, to get the numerator of (6.2) and $\mu = 5$, $z = \alpha = q$, to get the denominator of (6.2). Dividing the two expressions and using the product equivalent for $C(q)$, one gets (6.3).

In (6.15) take

(a) $\mu = 5$, $\alpha = q^2$, $z = q$, to obtain (6.4).
(b) $\mu = 5$, $\alpha = q$, $z = q^3$, to obtain (6.5).
(c) $\mu = 5$, $\alpha = q$, $z = q$, to obtain (6.6).
(d) $\mu = 5$, $\alpha = q^2$, $z = q^2$, to obtain (6.7).

To obtain

(e) (6.8) use (6.4) and (6.5)
(f) (6.9) use (6.5) and (6.6)
(g) (6.10) use (6.4) and (6.7).

To obtain (6.11), in (6.15) let $q \to q^2$, and take $\mu = 5$, $\alpha = q$, $z = q^4$, and finally, to obtain (6.12), in (6.15) let $q \to q^2$ and take $\mu = 5$, $\alpha = q^3$, $z = q^2$. In (6.15) if we take

$\mu = 4$, $z = \alpha = q$ and also $z = q^2$, $\alpha = q$ the resulting expressions give

$$\frac{\sum_{-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}}}{\sum_{-\infty}^{\infty} \frac{q^{2n}}{1-q^{4n+1}}} = \prod_{n\geqslant 0} \frac{(1-q^{4n+2})^4}{(1-q^{4n+1})^2(1-q^{4n+3})^2}$$

$$= \left[1 + \frac{q}{1+}\,\frac{q+q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^5}{1+}\cdots\right]^2,$$

which gives an expression for the square of another continued fraction identity of Ramanujan found in the 'Lost' Notebook and does not seem to have been mentioned earlier.

Again, in (6.15), taking $\mu = 6, \alpha = q^2, z = q$ and also $\mu = 6, \alpha = q^2, z = q^2$, the resulting expressions give

$$\sum_{-\infty}^{\infty} \frac{q^n}{1-q^{6n+2}} \Big/ \sum_{-\infty}^{\infty} \frac{q^{2n}}{1-q^{6n+2}} = \prod_{n\geqslant 0} \frac{(1-q^{6n+3})^2}{(1-q^{6n+1})(1-q^{6n+5})}$$

$$= 1 + \frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+}\cdots,$$

a new representation for another known continued fraction due to Watson [29], Gordon [11] and Andrews [8].

Similarly, in (6.15) taking $\mu = 8, \alpha = q^5, z = q$ and also $\mu = 8, \alpha = q^3, z = q^3$, we get on taking the quotient of the resulting identities an expression representing another continued fraction

$$1 + \frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+}\,\frac{q^8}{1+}\cdots,$$ given in the 'Lost' note book.

Denis [16], in 1988 used (6.15) and (6.16) in the form

$$\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{\mu n+j}} = \frac{(q^\mu;q^\mu)_\infty^2 (q^{i+j};q^\mu)_\infty (q^{\mu-i-j};q^\mu)_\infty}{(q^j;q^\mu)_\infty (q^{\mu-j};q^\mu)_\infty (q^i;q^\mu)_\infty (q^{\mu-i};q^\mu)_\infty} \tag{6.19}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{\mu n+j}} = \sum_{n=-\infty}^{\infty} q^{\mu n^2+(i+j)n} \frac{(1-q^{2\mu n+i+j})}{(1-q^{\mu n+i})(1-q^{\mu n+j})} \tag{6.20}$$

and listed some very interesting new results of different types deducible from them.

To mention a few we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})} = (q^5;q^5)_\infty^2 G(q)$$

$$(\mu = 5, i = 1, j = 2), \tag{6.21}$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})} = (q^5;q^5)_\infty^2 H(q)$$

$$(\mu = 5, i = 1, j = 3), \tag{6.22}$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})} = (q^5;q^5)^2_\infty G(q)^2/H(q)$$
$$(\mu=5, i=1, j=1), \qquad (6.23)$$

which is $(3.11)_R$ of Andrews [8],

$$\sum_{n=-\infty}^{\infty} q^{5n^2+4n}\frac{(1+q^{5n+2})}{(1-q^{5n+2})} = [q^5;q^5]^2_\infty H(q)^2/G(q)$$
$$(\mu=5, i=2, j=2) \qquad (6.24)$$

which is $(3.12)_R$ of Andrews [8],

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+5n}(1-q^{20n+5})}{(1-q^{10n+2})(1-q^{10n+3})} = (q^5;q^5)^2_\infty H(q)$$
$$(\mu=10, i=2, j=3), \qquad (6.25)$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+5n}(1-q^{20n+5})}{(1-q^{10n+4})(1-q^{10n+1})} = (q^5;q^5)^2_\infty G(q)$$
$$(\mu=10, i=4, j=1), \qquad (6.26)$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+5})(1-q^{10n+2})} = \sum_{n=-\infty}^{\infty} \frac{q^{5n}}{1-q^{10n+2}}$$
$$= [-q^5;q^5]^2_\infty [q^7;q^{10}]_\infty [q^3;q^{10}]_\infty G(q^2)$$
$$(\mu=10, i=5, j=2) \qquad (6.27)$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+3})(1-q^{10n+4})} = \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{10n+4}}$$
$$= [q^{10};q^{10}]^2_\infty H(q^2)$$
$$(\mu=10, i=3, j=4). \qquad (6.28)$$

$$\sum_{-\infty}^{\infty} \frac{q^{7n^2+5n}(1-q^{14n+5})}{(1-q^{7n+2})(1-q^{7n+3})} = \sum_{-\infty}^{\infty} \frac{q^{2n}}{1-q^{7n+3}}$$
$$= [q^7;q^7]^2_\infty/[q^4;q^7]_\infty [q^3;q^7]_\infty$$
$$(\mu=7, i=2, j=3) \qquad (6.29)$$

$$\sum_{-\infty}^{\infty} \frac{q^{7n^2+4n}(1-q^{14n+4})}{(1-q^{7n+1})(1-q^{7n+3})} = \sum_{-\infty}^{\infty} \frac{q^{n}}{1-q^{7n+3}}$$
$$= [q^7;q^7]^2_\infty/[q;q^7]_\infty [q^6;q^7]_\infty$$
$$(\mu=7, i=1, j=3) \qquad (6.30)$$

$$\sum_{-\infty}^{\infty} q^{7n^2+6n}\frac{(1+q^{7n+3})}{(1-q^{7n+3})} = \sum_{-\infty}^{\infty} \frac{q^{3n}}{1-q^{7n+3}}$$
$$= \frac{[q^7;q^7]^2_\infty [q^6;q^7]_\infty [q;q^7]_\infty}{[q^3;q^7]^2_\infty [q^4;q^7]^2_\infty}$$
$$(\mu=7, i=1, j=3) \qquad (6.31)$$

$$\frac{\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+5})(1-q^{10n+2})}}{\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+3})(1-q^{10n+4})}} = \frac{\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{10n+2})}}{\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{10n+4}}}$$

$$= \frac{[q^3;q^{10}]_\infty [q^7;q^{10}]_\infty}{(q^5;q^5)_\infty} C(q^2) \tag{6.32}$$

(follows from (6.27) and (6.28)).

A different type of double series representations can be obtained if, for example, one considers the following identity of Verma and Jain [28] for Rogers–Ramanujan type of identities related to modulus 57:

$$(-q;q)_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^6;q^6)_{n+2r} q^{2n^2+12nr+24r^2+6n+24r}}{(q^6;q^6)_r (q^2;q^2)_n (q^2;q^2)_{2n+6r+2}(-q^3;q^3)_{2r+1}}$$
$$= (q^{57};q^{57})_\infty (q^{54};q^{57})_\infty (q^3;q^{57})_\infty / (q;q)_\infty .$$

If we use (6.15) for $\mu = 57$, $\alpha = q^{27} = z$ and equate the products in above identity, we get the interesting representation

$$\left[\frac{[(q^{27};q^{57})_\infty (q^{30};q^{57})_\infty]^2}{[q^{57};q^{57}]_\infty}\right]^{-1} (q^2;q^2)_\infty \times$$
$$\times \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^6;q^6)_{n+2r} q^{2n^2+12nr+24r^2+6n+24r}}{(q^6;q^6)_r (q^2;q^2)_n (q^2;q^2)_{2n+6r+2}(-q^3;q^3)_{2r+1}}$$
$$= \sum_{-\infty}^{\infty} q^{57n^2+54n}(1+q^{57n+27})/(1-q^{57n+27}). \tag{6.33}$$

Similar identities, even, for quotients of two double series, as in (6.33), corresponding to other modulii, can be written down from various other results of Verma and Jain [28]. Also, if one scans the L J Slater's [23] list of identities one can give scores of other representations on the above lines, for different modulii.

Very recently, in 1989, Andrews and Garvan [10] have proved the truth of certain identities which are equivalent to the mock theta conjectures. In fact, they have considered two families of fifth order mock theta functions of Ramanujan and have shown that the identities in each family are equivalent (i.e. if one is true all five are true and if one is false all five are false). They have used the method of generalised Lambert series to prove them, and this enabled them to reduce all their computations to simple manipulations of these series. I sketch, below their method of proof, by deducing only one of the ten Ramanujan's unproved identities, involving mock theta functions of the fifth order.

Ramanujan had stated, without proof, that

$$M_1(q) \equiv \chi_0(q) - 2 - 3\Phi(q) + A(q) = 0$$
$$M_3(q) \equiv \varphi_0(-q) + \Phi(q) - (q^5;q^5)_\infty G(q^2)H(q)/H(q^2) = 0 \tag{6.34}$$

where χ_0, φ_0 are two of the ten fifth order mock theta functions and $\Phi(q)$, $A(q)$, $G(q)$

and $H(q)$ are certain auxilliary functions associated with them. (For definitions of all these functions one is referred to the original work of Andrews and Garvan [10]). They proceed to prove that

$$\begin{aligned} &M_1(q^2)+3M_3(q^2)\\ &=\chi_0(q^2)+3\varphi_0(-q^2)-2+A(q^2)-\frac{3(q^{10};q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}\\ &=\theta_4(0,-q)G(-q)+\theta_4(0,q)G(q)+A(q^2)-\\ &\quad -3(q^{10};q^{10})_\infty G(q^4)H(q^2)/H(q^4)\\ &=\frac{1}{(q^{10};q^{10})_\infty}\left[\sum_{-\infty}^{\infty}\frac{(-q^2)^n}{1-(-1)^n q^{5n+1}}+\sum_{-\infty}^{\infty}\frac{(-q^2)^n}{1+q^{5n+1}}+\right.\\ &\quad\left.+\sum_{-\infty}^{\infty}\frac{q^{2n}}{1-q^{10n+2}}-3\sum_{-\infty}^{\infty}\frac{(-q^2)^n}{1-q^{10n+2}}\right], \end{aligned} \tag{6.35}$$

where we have replaced each term by its generalised Lambert series equivalent. It may further be remarked that each sum in (6.35) can be put by (6.19) equal to several ${}_1\Psi_1$-series each with base q^{10}. Andrews and Garvan proved that (6.35) equals

$$\frac{2}{(q^{10};q^{10})_\infty}\sum_{-\infty}^{\infty} q^{4n+2}/(1-q^{10n+6}) \tag{6.36}$$

which is shown to be identically equal to zero (writing its equivalent infinite products). This proves the desired identity.

Since Ramanujan is believed to have an adequate background of the theory of elliptic functions and had also discovered the sum of a ${}_1\Psi_1$ series, it is reasonable to expect that he must have been conversant with the Lambert series and its bilateral extension.

The above method of proof, through Lambert series, given by Andrews and Garvan to establish the truth of Ramanujan's "unproven" identities, hence, is indicative of how Ramanujan, himself, might have arrived at the truth of his conjectures. Andrews and Garvan, thus, give credence to and help to unfold the mysticism popularly associated with Ramanujan's ingeniuty and creative thinking process.

It would, therefore, indeed be a marvellous achievement, if one could prove the numerous other "unproven" identities of Ramanujan also, in the spirit—"how Ramanujan might have proved them".

Nevertheless, the various other techniques (which might not have been known to Ramanujan) used, later, to prove his results have not only helped us to knit together the scores of his scattered identities in one net but also have helped to classify and generalise them to get a better insight in the ideas underlying their structure. However, these sophisticated methods, fall short in projecting the greatness of Ramanujan's genius, vision and intuitive power.

7. Certain other miscelleaneous identities and Lambert series

I next mention two recent applications of Lambert series, one each in sorting theory and Weierstrass's elliptic function theory.

Uchimura [25] in his recent works has considered the polynomial $U_m(x)$ which is defined as follows:

$$U_1(x) = x,$$
$$U_m(x) = mx^m + (1 - x^m)U_{m-1}(x) \quad (m \geqslant 2).$$

For instance:

$$U_1(x) = x,$$
$$U_2(x) = x + 2x^2 - x^3,$$
$$U_3(x) = x + 2x^2 + 2x^3 - x^4 - 2x^5 + x^6,$$
$$U_4(x) = x + 2x^2 + 2x^3 + 3x^4 - 3x^5 - x^6 - 2x^7 + x^8 + 2x^9 - x^{10}.$$

It is clear that $U_m(x)$ is a polynomial of degree $m(m+1)/2$.

The polynomial $U_m(x)$ has its origin in the analysis of the data structure called "heap" [24].

He proved the following two theorems.

Theorem 1. *Let*

$$U_m(x) = \sum_{n=1}^{m(m+1)/2} a_n^{(m)} x^n.$$

Then $a_n^{(m)} = d(n)$ for any $n \leqslant m$, where the divisor function $d(n)$ is the number of divisors of n, including 1 *and n.*

Theorem 2.

$$\sum_{n=1}^{\infty} nx^n \prod_{j=n+1}^{\infty} (1 - x^j)$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n(n+1)/2}}{(1-x)(1-x^2)\cdots(1-x^{n-1})(1-x^n)^2}$$
$$= \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}. \tag{7.1}$$

Uchimura [26] later generalized (7.1) in the form that, for any positive integer m,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1} q^{k(k+1)/2}}{1 - q^{k+m-1}} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k=1}^{n} \frac{q^k}{1-q^k} \Bigg/ \begin{bmatrix} k+m-1 \\ k \end{bmatrix}, \tag{7.2}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{k(k+1)/2}}{(1-q)(1-q^2)\cdots(1-q^k)(1-q^{k+m-1})} = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \Bigg/ \begin{bmatrix} k+m-1 \\ k \end{bmatrix}. \tag{7.3}$$

Putting $m = 1$ in (7.2) and (7.3), we get

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{(1-q^k)} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k=1}^{n} \frac{q^k}{1-q^k}, \tag{7.4}$$

where $\left[{n \atop k}\right]$ is the Gaussian polynomial defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})}, \quad \text{if } 0 \leqslant k \leqslant n,$$
$$= 0, \quad \text{otherwise,}$$

and (7.1) respectively.

The above result (7.4) was proposed by L. Van Hamme [27] as an advanced problem.

Andrews [6] proved the identity (7.4) by using an identity of hypergeometric functions (7.2) is a finite version of (7.1).

As a final application we consider a result of Ramanujan [21] in connection with the development of obtaining formulae for $r_{2s}(n)$, the number of representation of n as sums of $2s$ squares. Ramanujan introduced the function $\phi_{r,s}(x) = \sum_m \sum_n m^r n^s \cdot x^{mn}$, symmetric in r and s and expressed $\phi_{r,s}(x)$, when $(r+s)$ is odd, as polynomials in the transcedentals in P, Q and R where

$$P = 1 - 24\phi_{0,1}(x) = 1 - 24\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}$$

$$Q = 1 + 240\phi_{0,3}(x) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3x^n}{1-x^n}$$

$$R = 1 - 504\phi_{0,5}(x) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5x^n}{1-x^n}.$$

V Ramamani [22] in 1987 defined two allied functions, viz

$$\Psi_{r,s}(x) = \sum_m \sum_n (-1)^{n-1} m^r n^s x^{mn}$$

$$F_{r,s}(x) = \sum_m \sum_n (2m-1)^r n^s x^{(2m-1)n/2}, \quad r, s > 0$$

which are not symmetric in r and s and considered the problem of obtaining the polynomial expressions for them when $(r+s)$ is odd. She succeeded in expressing $\Psi_{r,s}(x)$ (when $(r+s)$ is odd) as polynomial in e_1, P_1 and Q_1, where $e_1 = \wp(\pi)$ (where $\wp$ is Weierstrassian elliptic function)

$$P_1 = 1 + 8\sum_{n=1}^{\infty} \frac{(-1)^{n-1} nx^n}{1-x^n} = 1 + 8\Psi_{0,1}(x)$$

$$Q_1 = 1 - 16\sum_{1}^{\infty} \frac{(-1)^{n-1} n^3 x^n}{1-x^n} = 1 - 16\Psi_{0,3}(x).$$

The behaviour of the function $F_{r,s}(x)$ is similar to that of $\Psi_{r,s}(x)$ and can be expressed, when $r+s$ is odd, as polynomials in e_3, P_2 and Q_2 where

$$e_3 = \wp(\pi\tau), \ \operatorname{Im} \tau > 0,$$

$$P_2 = -8\sum_{m=1}^{\infty} \frac{mx^{m/2}}{1-x^m} = -8F_{0,1}(x)$$

$$Q_2 = 16 \sum_{m=1}^{\infty} \frac{m^3 x^{m/2}}{1 - x^m} = 16F_{0,3}(x).$$

Further by noticing that $\Psi_{r,s}(x)$ and $F_{r,s}(x)$ are functions of $\phi_{r,s}(x)$, she expressed $\Psi_{r,s}(x)$ as polynomials in e_1, P_1 and Q_1 and $F_{r,s}(x)$ in e_3, P_2 and Q_2.

8. Lambert series and mock theta functions of order 'six'

Very recently, Andrews and Dean Hickerson [10a] proved five identities involving seven new mock theta functions found in the 'Lost' notebook. Andrews and Hickerson 'formally' labelled these functions as mock theta functions of order 'six'. The seven mock theta functions are defined as below;

$$\phi(q) = \sum_{n \geq 0} \frac{(-)^n q^{n^2}(q; q^2)_n}{(-q)_{2n}} \tag{8.1}$$

$$\Psi(q) = \sum_{n \geq 0} \frac{(-)^n q^{(n+1)^2}(q; q^2)_n}{(-q)_{2n+1}} \tag{8.2}$$

$$\rho(q) = \sum_{n \geq 0} \frac{q^{1/2n(n+1)}(-q)_n}{(q; q^2)_{n+1}} \tag{8.3}$$

$$\sigma(q) = \sum_{n \geq 0} \frac{q^{1/2n(n+2)}(-q)_n}{(q; q^2)_{n+1}} \tag{8.4}$$

$$\lambda(q) = \sum_{n \geq 0} \frac{(-)^n q^n (q; q^2)_n}{(-q)_n} \tag{8.5}$$

$$\mu(q) = \sum_{n \geq 0} \frac{(-)^n (q; q^2)_n}{(-q)_n} \tag{8.6}$$

$$\gamma(q) = \sum_{n \geq 0} \frac{q^{n^2}(q)_n}{(q^3; q^3)_n}. \tag{8.7}$$

It should be noted that (8.6) does not converge. However, we give it a meaning by taking the average of sequence of odd and sequence of even partial sums, each of which converges.

Andrews and Hickerson derived the following simple Lambert series representations for five of these functions:-

$$J_{1,3}\phi(q) = 2 \sum_{r=-\infty}^{\infty} \frac{q^{r(3r+1)/2}}{1 + q^{3r}} \tag{8.8}$$

$$J_{1,3}\Psi(q) = 2 \sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(3r+2)/2}}{1 + q^{3r+1}} \tag{8.9}$$

$$J_{1,6}\rho(q) = \sum_{r=-\infty}^{\infty} \frac{(-)^r q^{r(3r+4)}}{1 - q^{6r+1}} \tag{8.10}$$

$$J_{1,6}\sigma(q) = \sum_{r=-\infty}^{\infty} \frac{(-)^r q^{(r+1)(3r+1)}}{1-q^{6r+3}} \tag{8.11}$$

$$(q)_\infty \gamma(q) = 3 \sum_{r=-\infty}^{\infty} \frac{(-)^r q^{r(3r+1)/2}}{1+q^r+q^{2r}}, \tag{8.12}$$

where $J_{a,m} = (q^a, q^{m-a}, q^m; q^m)_\infty$.

Similar representations for $\lambda(q)$ and $\mu(q)$ have also been given by them but are not so elegant (see [10a]; (4.19) and (4.20)).

They used these alternative representations to prove the various 'unproven' identities connecting them, as given by Ramanujan.

It may be remarked that (8.12) is also found in earlier works of N J Fine (for details see Agarwal ([3a]).

The rationality of the 'adhoc' labelling of these seven mock theta functions as of order 'six' has, very recently, aroused much interest. This has motivated Agarwal [3a] to give a formal definition of the order of a mock theta function, in an attempt to settle a seventy-year old 'adhocism' in naming such functions as mock theta functions of orders three, five, seven and six. Agarwal's discussion also leaves a question mark on the existence of 'mock theta functions' of orders other than those of orders three, five and seven given by Ramanujan.

I think I have said enough to justify the effectiveness and elegance with which Lambert series can be used to partially lift the curtain of ignorance regarding Ramanujan's creative and ever vibrant genius.

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