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Some applications of Briot–Bouquet differential subordination

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Abstract. Some applications of Briot–Bouquet differential subordination are obtained which improve or extend a number of classical results in the univalent function theory.

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1. Introduction

Let V denote the class of functions f of the form $f(z)=z+\Sigma_{n=2}^{\infty}a_nz^n$ which are analytic in the unit disc $E=\{z:|z|<1\}$. For given arbitrary numbers A and B satisfying $-1\leqslant B<A\leqslant 1$, we denote by $P(A,B)$, the class of functions of the form $p(z)=1+p_1z+\dots$ which are analytic in E and satisfy the condition

$$p(z)\ll\frac{1+Az}{1+Bz},\quad z\in E,$$

where "$\ll$" stands for subordination. The class $P(A,B)$ was investigated by Janowski [4].

By $S^*(A,B)$, we mean the class of functions $f\in V$ such that $zf'(z)/f(z)\in P(A,B)$. Similarly, $K(A,B)$ is meant as the class of functions $f\in V$ satisfying $(zf'(z))'/f'(z)\in P(A,B)$. It is clear that $S^*(1-2\rho,-1)\equiv S^*(\rho)$ and $K(1-2\rho,-1)\equiv K(\rho)$, $0\leqslant\rho<1$, $S^*(\rho)$ and $K(\rho)$ denote the subclasses of V which are respectively star-like of order ρ and convex of order ρ in E.

We now define the class $KS^*_{\delta,\alpha}(A,B)$ as below and obtain some interesting results by an application of Briot–Bouquet differential subordination.

DEFINITION 1

Let A, B, α and δ be arbitrary fixed real numbers such that $-1\leqslant B<A\leqslant 1$, $\alpha\geqslant 0$ and $\delta\geqslant -1$. A function $f\in V$ is said to be in the class $KS^*_{\delta,\alpha}(A,B)$ if it satisfies

$$J_\delta(\alpha;f(z))\ll\frac{1+Az}{1+Bz},\quad z\in E,\tag{1}$$

where

$$J_\delta(\alpha;f(z))=(1-\alpha)\frac{z(D^\delta f(z))'}{D^\delta f(z)}+\alpha\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}$$

and

$$D^{\delta}f(z) = (z/(1-z)^{\delta+1}) * f(z)$$

(Here $*$ stands for the Hadamard product of two analytic functions).

Clearly $KS^*_{0,0}(A,B) \equiv S^*(A,B)$, $KS^*_{0,1}((A+B)/2, B) \equiv K(A,B)$. It is readily seen that $KS^*_{n,\alpha}(1-2\rho, -1)$, $\rho < 1$, $n \in N_0 = \{0, 1, 2, \ldots\}$ is the class defined by Bulboaca [3], whereas the class

$$KS^*_{n,0}\left(\frac{b^2 - a^2 + a}{b}, \frac{1-a}{b}\right), \quad |a-1| < b \leqslant a,$$

is the class considered by us [2]. Further taking $\delta = 0$, $\alpha = 2\mu/(\mu+1)(\mu \geqslant 0)$, $A = 1 - 2\mu/(\mu+1)$, $B = -1$, it is seen that the class $KS^*_{\delta,\alpha}(A,B)$ reduces to the well-known class of μ-convex functions ([5] and [6]).

For a, b, c real numbers with c other than $0, -1, -2, \ldots$ the hypergeometric series

$$F(a,b,c;z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \ldots$$

represents an analytic function in E[11]. The following identities are well known [11, Chapter 15].

$$F(a,b,c;z) = F(b,a,c;z) \tag{2}$$

$$F(a,b,c;z) = (1-z)^{-a} F(a, c-b, c; z/(z-1)), \quad z \in C - \{1, \infty\} \tag{3}$$

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b,c;z) \tag{4}$$

with $c > b > 0$.

2. Main results

Theorem 2.1 *Let* $-1 \leqslant B < A \leqslant 1$, $\delta > -1$ *and* $0 < \alpha < (\delta+2)/(\delta+1)$ *satisfy*

$$(\delta+2)(1-A) + [\delta(\delta+2)(1-\alpha) - \alpha](1-B) \geqslant 0. \tag{5}$$

(a) *Then*

$$KS^*_{\delta,\alpha}(A,B) \subset KS^*_{\delta,0}(A',B) \tag{6}$$

where

$$A' = 1 - \frac{(\delta+2)(1-A) - \alpha(\delta+1)(1-B)}{(\delta+2) - \alpha(\delta+1)}.$$

Further for $f \in KS^*_{\delta,\alpha}(A,B)$ *we also have*

$$\frac{z(D^{\delta}f(z))'}{D^{\delta}f(z)} \ll \frac{\alpha}{[(\delta+2) - \alpha(\delta+1)]}\left\{\frac{1}{Q(z)}\right\} - \delta \equiv \tilde{q}(z), \quad z \in E \tag{7}$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{((\delta+2)/\alpha)((A-B)/B)} t^{((\delta+2)/\alpha)(\delta(1-\alpha)+1)-2}\,\mathrm{d}t & \text{if } B \neq 0 \\ \displaystyle\int_0^1 \exp\left\{\frac{\delta+2}{\alpha}(t-1)Az\right\} t^{((\delta+2)/\alpha)(\delta(1-\alpha)+1)-2}\,\mathrm{d}t & \text{if } B = 0. \end{cases} \tag{8}$$

(b) *If in addition to* (5) *one has* $A/B > -\delta(1-\alpha)$ *with* $B<0$, *then*

$$KS^*_{\delta,\alpha}(A,B) \subset KS^*_{\delta,0}(1-2\rho', -1) \tag{9}$$

where

$$\rho' = \frac{\delta+1}{F\left(1, \left(\frac{\delta+2}{\alpha}\right)\left(\frac{B-A}{B}\right), \left(\frac{\delta+2}{\alpha}\right)(\delta(1-\alpha)+1); \frac{-B}{1-B}\right)} - \delta.$$

The result is sharp.

Proof. (a) We follow the method similar to that of Mocanu *et al* [8]. It can be easily verified that

$$z(D^\delta f(z))' = (\delta+1)D^{\delta+1}f(z) - \delta D^\delta f(z). \tag{10}$$

Let $f \in KS^*_{\delta,\alpha}(A,B)$ where $\delta > -1$, $\alpha > 0$ and $-1 \leqslant B < A \leqslant 1$. Set

$$\tilde{p}(z) = \frac{z(D^\delta f(z))'}{D^\delta f(z)}. \tag{11}$$

Clearly $\tilde{p}(z)$ is analytic in E and $\tilde{p}(o) = 1$. From (10) and (11) we obtain

$$\frac{D^{\delta+1}f(z)}{D^\delta f(z)} = \frac{\tilde{p}(z)+\delta}{1+\delta} = p(z); \quad p(o) = 1.$$

Since $f \in KS^*_{\delta,\alpha}(A,B)$, (1) coupled with (10) easily leads to

$$P(z) + \frac{zP'(z)}{\beta P(z)+\gamma} \ll \frac{1+Az}{1+Bz}, \quad z \in E \tag{12}$$

where

$$P(z) = (\delta+1)\left(\frac{1}{\beta} + 1 - \alpha\right)p(z) + \frac{1}{\beta} - \delta(1-\alpha) \tag{13}$$

with $\beta = (\delta+2)/\alpha$ and $\gamma = \delta\beta(1-\alpha) - 1$. Considering the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz}, \quad z \in E$$

and using Corollary 3.2 of Miller and Mocanu [7] we deduce that

$$P(z) \ll q(z) \ll \frac{1+Az}{1+Bz}, \quad z \in E \tag{14}$$

provided δ, α, A and B satisfy (5), where $q(z)$ is the best dominant of (12) and is given by (23) in [7, Corollary 3.2]. (6) and (7) now follow from (13) and (14).

(b) Next we show that

$$\operatorname*{Inf}_{z \in E}\{\tilde{q}(z)\} = \tilde{q}(-1). \tag{15}$$

If we set $a = \beta(B-A)/B$, $b = \beta+\gamma$, $c = \beta+\gamma+1$, then $c > b > 0$. From (8) by using

(2), (3) and (4) we see that for $B \neq 0$

$$Q(z) = (1+Bz)^a \int_0^1 (1+Btz)^{-a} t^{b-1}\, dt$$
$$= (1+Bz)^a \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} (1+Bz)^{-a} F\left(a, c-b, c; \frac{Bz}{Bz+1}\right)$$
$$= \frac{1}{b} F\left(1, a, c; \frac{Bz}{Bz+1}\right).$$

Supposing $-1 \leqslant B < A < 0$ we have $c > a > 0$ and by using (4) we obtain

$$Q(z) = \int_0^1 \left[\frac{1+Bz}{1+(1-t)Bz}\right] d\mu(t)$$

where

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\, dt$$

is a positive measure. Denoting $g(z,t) = (1+Bz)/(1+(1-t)Bz)$, it may be seen that $\mathrm{Re}\{g(z,t)\} > 0$, $g(-r,t)$ is real for $0 \leqslant r < 1$, $t \in [0,1]$ and

$$\mathrm{Re}\left\{\frac{1}{g(z,t)}\right\} = \mathrm{Re}\left\{\frac{1+(1-t)Bz}{1+Bz}\right\} \geqslant \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(-r,t)}$$

for $|z| \leqslant r < 1$ and $t \in [0,1]$. Therefore by using Lemma 2 [12] we deduce that $\mathrm{Re}(1/Q(z)) \geqslant 1/(Q(-r))$, $|z| \leqslant r < 1$ and letting $r \to 1^-$ we obtain $\mathrm{Re}(1/Q(z)) \geqslant 1/(Q(-1))$, $z \in E$. This by (15) and (7) leads to (9). Hence the theorem.

We write $KS^*_{\delta,0}(A,B) \equiv S^*_\delta(A,B)$ and $KS^*_{\delta,1}(A,B) = K_{\delta+1}(A,B)$. Then it can be readily seen, using the identity (10), that

$$S^*_\delta(A,B) \equiv K_\delta\left(\frac{A+\delta B}{1+\delta}, B\right) \text{ or } K_\delta(A,B) \equiv S^*_\delta((\delta+1)A - \delta B, B) \tag{16}$$

provided $\delta > 0$. Further we have $S^*_\delta(1-2\rho, -1) \equiv S^*_\delta(\rho)$ and $K_\delta(1-2\rho, -1) \equiv K_\delta(\rho)$ for $\rho < 1$.

Taking $\alpha = 1$ in Theorem 2.1 (a) and using (16) we obtain the following result of Ramareddy [9].

COROLLARY 2.1

Let $-1 \leqslant B < A \leqslant 1$ *and* $\delta > -(1-A)/(1-B)$. *Then* $S^*_{\delta+1}(A,B) \subset S^*_\delta(A,B)$.

Taking $A = 1-2\rho$ and $B = -1$ in Theorem 2.1 we get

COROLLARY 2.2

Let $\delta > -1$, $0 < \alpha < (\delta+2)/(\delta+1)$ *and* $\dfrac{\alpha - \delta(\delta+2)(1-\alpha)}{\delta+2} \leqslant \rho < 1$. *Then*

$$KS^*_{\delta,\alpha}(1-2\rho, -1) \subset S^*_\delta\left(1 - 2\frac{(\delta+2)\rho - \alpha(\delta+1)}{(\delta+2) - \alpha(\delta+1)}, -1\right). \tag{17}$$

Further if $f \in KS^*_{\delta,\alpha}(1-2\rho,-1)$, then

$$\frac{z(D^\delta f(z))'}{D^\delta f(z)} \ll \frac{\alpha}{[(\delta+2)-\alpha(\delta+1)]}\left[\int_0^1 \left(\frac{1-tz}{1-z}\right)^{[-2(\delta+2)(1-\rho)]/\alpha} \times t^{(\delta+2)/\alpha)(\delta(1-\alpha)+1)-2}\mathrm{d}t\right]^{-1} - \delta.$$

Furthermore if $\max\left\{\frac{\alpha-\delta(\delta+2)(1-\alpha)}{\delta+2}, \frac{1-\delta(1-\alpha)}{2}\right\} \leqslant \rho < 1$,

then

$$KS^*_{\delta,\alpha}(1-2\rho,-1) \subset S^*_\delta(1-2\rho'',-1) \tag{18}$$

where

$$\rho'' = (\delta+1)\left[F\left\{1, \frac{2(\delta+2)(1-\rho)}{\alpha}, \frac{(\delta+2)}{\alpha}(\delta(1-\alpha)+1); \frac{1}{2}\right\}\right]^{-1} - \delta.$$

The result is sharp.

Remarks i) For $\delta = n \in N_0$, (17) and (18) were obtained by Bulboaca [3].

ii) If we take $\alpha = 1$, $\rho = (\delta+1)/(\delta+2)$ in Corollary 2.2, it follows, from (18), that for $f \in V$ and $\delta \geqslant 0$ we have the sharp result $\mathrm{Re}\{D^{\delta+2}f(z)/D^{\delta+1}f(z)\} > (\delta+1)/(\delta+2)$ in E implies $\mathrm{Re}\{z(D^\delta f(z))'/D^\delta f(z)\} > ((\delta+1)/F(1,2,\delta+2;1/2)) - \delta$ in E which in turn implies $\mathrm{Re}\{D^{\delta+1}f(z)/D^\delta f(z)\} > 1/F(1,2,\delta+2;1/2) > \delta/(\delta+1)$, $z \in E$. This improves Singh and Singh's results [10] obtained for $\delta = n \in N_0$.

(iii) Letting $\alpha = 1$ in Corollary 2.2, we obtain that, for

$$\rho \in \left(\max\left\{\frac{1}{\delta+2}, \frac{1}{2}\right\}, 1\right),\ K_{\delta+1}(\rho) \subset S^*_\delta(\rho''),$$

where ρ'' is as stated in Corollary 2.2 with $\alpha = 1$. Using $K_\delta(\rho) = S^*_\delta((\delta+1)\rho - \delta)$, this result takes the form $S^*_{\delta+1}(\rho) \subset S^*_\delta(\rho^*)$ for $-(\delta/2) \leqslant \rho < 1$, where $\rho^* = (\delta+1)[F(1, 2(1-\rho), \delta+2; 1/2]^{-1} - \delta$, which represents an improvement and extension of Ahuja's result [1] obtained for $\delta = n \in N_0$ and $0 \leqslant \rho < 1$.

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