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Simultaneous operational calculus involving a product of a general class of polynomials, Fox's H-function and the multivariable H-function

V B L CHAURASIA and GIRDHARI LAL
Department of Mathematics, University of Rajasthan, Jaipur 302004, India

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Abstract. New operational relations between the original and the image for two-dimensional Laplace transforms involving a general class of polynomials, Fox's H-function and the multivariable H-function are obtained. The result provides a unification of the bivariate Laplace transforms for the H-functions given by Chaurasia [2, 3].

Keywords. Laplace transform; Fox's H-function; multivariable H-function; general class of polynomials; original and image functions; operational calculus; integral equation.

1. Introduction

The Laplace-Carson transform in two variables is defined and represented by the integral equation [4, p. 39]

$$F(p,q)=pq\int_0^\infty\int_0^\infty \exp(-px-qy)f(x,y)\,\mathrm{d}x\,\mathrm{d}y,\quad \mathrm{Re}(p,q)>0, \tag{1}$$

where $F(p,q)$ and $f(x,y)$ are said to be operationally related to each other. $F(p,q)$ is called the image and $f(x,y)$ the original. Symbolically, we can write

$$F(p,q)\doteqdot f(x,y) \text{ or vice versa} \tag{2}$$

and the symbol $\doteqdot$ is called operational.

Srivastava [7, p. 1, eqn. (1)] has introduced the general class of polynomials

$$S_\lambda^\mu[x]=\sum_{\alpha=0}^{[\lambda/\mu]}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}x^\alpha,\quad \lambda=0,1,2,\ldots, \tag{3}$$

where μ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda,\alpha\geqslant 0)$ are arbitrary constants real or complex. On suitably specializing the coefficients $L_{\lambda,\alpha}$, $S_\lambda^\mu[x]$ yields a number of known polynomials as its particular cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and several others [9, pp. 158–61].

The series representation of Fox's H-function [1, 5] is

$$H_{u,d}^{m,n}\left[z\left|\begin{matrix}(e_a,E_a)\\(f_d,F_d)\end{matrix}\right.\right]=\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,F_G}\phi(g_s)z^{g_s} \tag{4}$$

where

$$\phi(g_s)=\prod_{j=1,j\neq G}^{m}\Gamma(f_j-F_jg_s)\prod_{j=1}^{n}\Gamma(1-e_j+E_jg_s)$$
$$\times\left\{\prod_{j=1+m}^{d}\Gamma(1-f_g+F_jg_s)\prod_{j=1}^{u}\Gamma(a_j-E_jg_s)\right\}^{-1}\tag{5}$$

and $g_s=(f_G+s)/F_G$.

Srivastava and Panda [8] have introduced the multivariable H-function as

$$H^{0,u:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}]:[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(c):C',\ldots,C^{(r)}]:[(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}z_1,\ldots,z_r\right).\tag{6}$$

The defining integral of the above function, its various special cases and other details can be found in the paper referred to above.

For the sake of brevity

$$T_i=-\sum_{j=u+1}^{u}A_j^{(i)}+\sum_{j=1}^{N^{(i)}}B_j^{(i)}-\sum_{j=1+N^{(i)}}^{P^{(i)}}B_j^{(i)}$$
$$-\sum_{j=1}^{w}C_j^{(i)}+\sum_{j=1}^{M^{(i)}}D_j^{(i)}-\sum_{j=1+M^{(i)}}^{Q^{(i)}}D_j^{(i)}\tag{7}$$

$$\left.\begin{aligned}&\theta_i=d_j^{(i)}/D_j^{(i)},\quad j=1,\ldots,M^{(i)}\\&\phi_i=(1-b_j^{(i)})/B_j^{(i)},\quad j=1,\ldots,N^{(i)},\quad i=1,\ldots,r\end{aligned}\right\}\tag{8}$$

$$T=\sum_1^n E_i-\sum_{n+1}^{u}E_i+\sum_1^m F_i-\sum_{m+1}^{d}F_i\tag{9}$$

$$\left.\begin{aligned}&\theta=f_l/F_l,\quad l=1,\ldots,m\\&\phi=(e_{l'}-1)/E_{l'},\quad l'=1,\ldots,n\end{aligned}\right\}.\tag{10}$$

Also we use the notation

$$H^{0,u:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1,\ldots,z_r)\tag{11}$$

to denote (6).

The importance of our main result lies in the fact that it involves the product of a general class of polynomials, Fox's H-function and the multivariable H-function having general arguments. Thus this result serves as a key formula from which the bivariate Laplace transform for the product of a large number of polynomials which are special cases of $S_\lambda^\mu[x]$ [9, pp. 158–61] and simpler special functions which are particular cases of Fox's H-function [6, pp. 145–151] and of the multivariable H-function follow merely by specializing the parameters. In this paper we shall obtain correspondences, involving a product of a general class of polynomials, Fox's H-function and the multivariable H-function, between the original and the image in two variables.

In what follows we shall denote the original variables by x and y and the transformed variables by p and q. The notations employed are those of Ditkin and Prudnikov's operational calculus.

Theorem 1. *With T_i, θ_i, ϕ_i, T, θ and ϕ given by (7), (8), (9) and (10) respectively, let $T_i > 0$, $T > 0$, $|\arg(z_i)| < T_i\pi/2$, $|\arg(z)| < T\pi/2$, $h_i > 0$, $k > 0$, $\delta > 0$, $i = 1, \ldots, r$, μ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda,\alpha \geqslant 0)$ are arbitrary constants real or complex, and*

(i) $\mathrm{Re}\left(\rho - \sigma - k\phi - \sum_{i=1}^{r} h_i\phi_i\right) < 3/4$

(ii) $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}\left(\sigma + k\theta + \sum_{i=1}^{r} h_i\theta_i\right) > 0$,

Also, let $0 \leqslant n \leqslant u$, $0 \leqslant m \leqslant d$ and

(iii) $R(\rho) > 0$.

$$p^{-1/2}(pq)^{\sigma/2-\rho+1} \sum_{\alpha=0}^{[\lambda/\mu]} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!F_G}\phi(g_s)$$

$$\times z^{g_s}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}(pq)^{(kg_s+\delta\alpha/2} \cdot H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1(\sqrt{pq})^{h_1}, \ldots, z_r(\sqrt{pq})^{h_r})$$

$$\doteqdot \frac{(4xy)^{\rho-(\sigma/2)-(1/2)}}{\sqrt{\pi y}} \sum_{\alpha=0}^{[\lambda/\mu]} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!F_G}\phi(g_s)z^{g_s}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}$$

$$\times L_{\lambda,\alpha}(4xy)^{(-kg_s/2)-(\delta\alpha/2)} H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v+1,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}],\\ [(c):C',\ldots,C^{(r)}]:\end{matrix}\right.$$

$$\times [2\rho-\sigma-kg_s-\delta\alpha:h_1,\ldots,h_r]:\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}$$

$$\left.\times z_1(2\sqrt{xy})^{-h_1},\ldots,z_r(2\sqrt{xy})^{-h_r}\right) \tag{12}$$

Proof. The Laplace transform of the product of a general class of polynomials, Fox's H-function and the multivariable H-function is given by

$$L\left\{t^{\sigma-1}S^{\mu}_{\lambda}[t^{\delta}]H^{m,n}_{u,d}\left[zt^k\left|\begin{matrix}(e_u,E_u)\\(f_d,F_d)\end{matrix}\right.\right]H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w+1:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}]:\\ [(c):C',\ldots,C^{(r)}],\end{matrix}\right.\right.$$

$$\times [1-\sigma-kg_s-\delta\alpha:h_1,\ldots,h_r]:\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}$$

$$\left.\left.\times z_1t^{h_1},\ldots,z_rt^{h_r}\right)\right\}$$

$$= p^{-\sigma}\sum_{\alpha=0}^{[\lambda/\mu]} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!F_G}\phi(g_s)z^{g_s}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}\cdot p^{-kg_s-\delta\alpha}$$

$$\times H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1p^{-h_1},\ldots,z_rp^{-h_r}), \tag{13}$$

where

$$\mathrm{Re}(p) > 0, \quad \mathrm{Re}\left(\sigma + k\theta + \sum_{i=1}^{r} h_i\theta_i\right) > 0,$$

$$\operatorname{Re}\left(\sigma + k\phi + \sum_{i=1}^{r} h_i\phi_i\right) < 0, \quad h_i > 0, \quad k > 0, \quad \delta > 0,$$

$$|\arg(z_i)| < T_i\pi/2, \quad |\arg(z)| < T\pi/2, \quad T_i > 0, \quad T > 0,$$

μ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}$ $(\lambda,\alpha \geqslant 0)$ are arbitrary constants real or complex. The result in (13) can easily be established by making use of (3) and a result recently obtained by Chaurasia [3, eqn. (2.2), p. 22].

On writing $(pq)^{-1/2}$ for p and multiplying both sides of (13) by $p^{-1/2}(pq)^{1-\rho}$ and then interpreting it with the help of a known result [4, p. 144, eqn. (3.26)], we get

$$\frac{(4xy)^{(\rho/2)-(1/4)}}{\sqrt{\pi y}} \int_0^\infty t^{\sigma-\rho-(1/2)} J_{2\rho-1}[(64xyt^2)^{1/4}]$$

$$\times H_{u,d}^{m,n}\left[zt^k \begin{matrix}(e_u, E_u)\\(f_d, F_d)\end{matrix}\right] \cdot S_\lambda^\mu[t^\delta]\, H_{v,w+1:(P',Q');\ldots;(P^{(r)},Q^{(r)})}^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}$$

$$\times \left(\begin{matrix}[(a):A',\ldots,A^{(r)}]:\\ [(c):C',\ldots,C^{(r)}],\end{matrix}\ [1-\sigma-kg_s-\delta\alpha:h_1,\ldots,h_r]:\right.$$

$$\left.\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}\ z_1t^{h_1},\ldots,z_rt^{h_r}\right)\mathrm{d}t$$

$$\doteqdot p^{-1/2}(pq)^{(\sigma/2)-\rho+1} \sum_{\alpha=0}^{[\lambda/\mu]} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!F_G} \phi(g_s) z^{g_s}$$

$$\times \frac{(-\lambda)_{\mu\alpha}}{\alpha!} L_{\lambda,\alpha} (pq)^{(kg_s+\delta\alpha)/2} \cdot H_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}$$

$$\times (z_1(\sqrt{pq})^{h_1},\ldots,z_r(\sqrt{pq})^{h_r}). \tag{14}$$

Now, evaluating LHS of (14) by the process mentioned in (13) we obtain the described result and (21) is established.

2. Special cases

Letting $K \to 0$, from (12) we get (after a little simplification) the following bivariate Laplace transform for a general class of polynomials and the multivariable H-function in the elegant form.

Theorem 2. *With T_i, θ_i and ϕ_i given by (7) and (8) respectively, let $T_i > 0$, $|\arg(z_i)| < T_i\pi/2$, $h_i > 0$, $i = 1,\ldots,r$, $\delta > 0$, μ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda,\alpha \geqslant 0)$ are arbitrary constants real or complex, and*

(i) $\operatorname{Re}\left(\sigma + \sum_{i=1}^{r} h_i\theta_i\right) > 0$

(ii) $\operatorname{Re}\left(\rho - \sigma - \sum_{i=1}^{r} h_i\phi_i\right) \dot{<} 3/4.$

Also let
(iii) $\mathrm{Re}(p) > 0, \quad \mathrm{Re}(\rho) > 0$

$$p^{-1/2}(pq)^{(\sigma/2)-\rho+1}\sum_{\alpha=0}^{[\lambda/\mu]}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}(pq)^{\delta\alpha/2}$$
$$\times H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1(\sqrt{pq})^{h_1},\ldots,z_r(\sqrt{pq})^{h_r})$$
$$\doteqdot \frac{(4xy)^{\rho-(\sigma/2)-(1/2)}}{\sqrt{\pi y}}\sum_{\alpha=0}^{[\lambda/\mu]}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}(4xy)^{-\delta\alpha/2}.$$
$$\times H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v+1,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}],\\ [(c):C',\ldots,C^{(r)}]:\end{matrix}[2\rho-\sigma-\delta\alpha:h_1,\ldots,h_r]:\right.$$
$$\left.\times\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}z_1(2\sqrt{xy})^{-h_1},\ldots,z_r(2\sqrt{xy})^{-h_r}\right) \tag{15}$$

(ii) *Letting* $\lambda\to 0$, *the theorem* 1 *reduces to a known theorem recently obtained by Chaurasia* [3, *eqn.* (2.1), *p.* 21].
(iii) *Letting* $\lambda\to 0$, $K\to 0$ *and* $r=2$, *the theorem* 1 *reduces to a known result* [2, *eqn.* (2.1), *p.* 86].
(iv) *Putting* $\lambda = 0$ *in* (15), *we get a known result recently obtained by Chaurasia* [3, *eqn.* (3.1), *p.* 24].
(v) *Taking* $r=1$ *and* $\lambda\to 0$, *the theorem 2 reduces to a known theorem obtained in* [2, *theorem* (3b), *p.* 88].

The importance of our results lies in its manifold generality. In view of the generality of the polynomials $s^{\mu}_{\lambda}[x]$, on suitably specializing the coefficients $L_{\lambda,\alpha}$, and making a free use of the special cases of $S^{\mu}_{\lambda}[x]$ listed by Srivastava and Singh [9], our results can be reduced to a large number of bivariate Laplace transforms involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in Fox's H-function and in the multivariable H-function, from our results, several bivariate Laplace transforms involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H functions of one and several variables. Thus the results presented in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions occurring in the problems of mathematical analysis.

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