Simultaneous operational calculus involving a product of a general class of polynomials, Fox’s $H$-function and the multivariable $H$-function

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Abstract. New operational relations between the original and the image for two-dimensional Laplace transforms involving a general class of polynomials, Fox’s $H$-function and the multivariable $H$-function are obtained. The result provides a unification of the bivariate Laplace transforms for the $H$-functions given by Chaurasia [2, 3].

Keywords. Laplace transform; Fox’s $H$-function; multivariable $H$-function; general class of polynomials; original and image functions; operational calculus; integral equation.

1. Introduction

The Laplace-Carson transform in two variables is defined and represented by the integral equation [4, p. 39]

$$F(p, q) = pq \int_0^\infty \int_0^\infty \exp(-px - qy)f(x, y) \, dx \, dy, \quad \text{Re}(p, q) > 0,$$

where $F(p, q)$ and $f(x, y)$ are said to be operationally related to each other. $F(p, q)$ is called the image and $f(x, y)$ the original. Symbolically, we can write

$$F(p, q) \doteq f(x, y) \text{ or vice versa}$$

and the symbol $\doteq$ is called operational.

Srivastava [7, p. 1, eqn. (1)] has introduced the general class of polynomials

$$S_\mu^\nu[x] = \sum_{\alpha=0}^{t+\mu} \frac{(-\lambda)^\mu}{\alpha!} L_{\lambda,\alpha} x^\alpha, \quad \lambda = 0, 1, 2, \ldots,$$

where $\mu$ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda, \alpha \geq 0)$ are arbitrary constants real or complex. On suitably specializing the coefficients $L_{\lambda,\alpha}, S_\mu^\nu[x]$ yields a number of known polynomials as its particular cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and several others [9, pp. 158–61].

The series representation of Fox’s $H$-function [1, 5] is

$$H_{m,n}^{\mu,\nu}[z] = \sum_{\alpha=1}^{m} \sum_{\beta=0}^{\infty} (-1)^\beta \frac{(g_s)_{\nu}}{s!} \phi(g_s) z^{\alpha}$$

where $m, n, \mu, \nu$ are non-negative integers and $g_s, s = 0, 1, 2, \ldots$ are the parameters of the Fox’s $H$-function.

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where
\[ \phi(g_x) = \prod_{j=1}^{m} \Gamma(f_j - F_j g_x) \prod_{j=1}^{n} \Gamma(1 - e_j + E_j g_x) \]
\[ \times \left\{ \prod_{j=1}^{d} \Gamma(1 - f_j + F_j g_x) \prod_{j=1}^{u} \Gamma(a_j - E_j g_x) \right\}^{-1} \]  \hspace{1cm} (5)

and \( g_x = (F_g + s)/F_G \).

Srivastava and Panda [8] have introduced the multivariable \( H \)-function as
\[ H^{\nu_1, \ldots, \nu_r; \ldots; \nu_{s+1}, \ldots, \nu_s; \ldots; \nu_{t+1}, \ldots, \nu_t; \ldots; \nu_{u+1}, \ldots, \nu_u}_{v_1, \ldots, v_r; \ldots; v_{s+1}, \ldots, v_s; \ldots; v_{t+1}, \ldots, v_t; \ldots; v_{u+1}, \ldots, v_u} \left( \frac{[a]; \ldots, [b]; \ldots; [c]; \ldots, [d]; \ldots; [e]; \ldots, [f]; \ldots; [g]; \ldots, [h]; \ldots; [i]; \ldots, [j]; \ldots; [k]; \ldots, [l]; \ldots; [m]; \ldots, [n]; \ldots; [o]; \ldots, [p]; \ldots; [q]; \ldots, [r]; \ldots; [s]; \ldots, [t]; \ldots; [u]; \ldots, [v]; \ldots; [w]; \ldots, [x]; \ldots; [y]; \ldots, [z]; \ldots; [\nu_1]; \ldots, [\nu_r]; \ldots; [\nu_{s+1}]; \ldots, [\nu_s]; \ldots; [\nu_{t+1}]; \ldots, [\nu_t]; \ldots; [\nu_{u+1}]; \ldots, [\nu_u]}{[A]; \ldots, [B]; \ldots; [C]; \ldots, [D]; \ldots; [E]; \ldots, [F]; \ldots; [G]; \ldots, [H]; \ldots; [I]; \ldots, [J]; \ldots; [K]; \ldots, [L]; \ldots; [M]; \ldots, [N]; \ldots; [O]; \ldots, [P]; \ldots; [Q]; \ldots, [R]; \ldots; [S]; \ldots, [T]; \ldots; [U]; \ldots, [V]; \ldots; [W]; \ldots, [X]; \ldots, [Y]; \ldots, [Z]; \ldots; [\nu_1]; \ldots, [\nu_r]; \ldots; [\nu_{s+1}]; \ldots, [\nu_s]; \ldots; [\nu_{t+1}]; \ldots, [\nu_t]; \ldots; [\nu_{u+1}]; \ldots, [\nu_u]} \right) \]  \hspace{1cm} (6)

The defining integral of the above function, its various special cases and other details can be found in the paper referred to above.

For the sake of brevity
\[ T_i = - \sum_{j=1}^{u+1} A_j^{(i)} + \sum_{j=1}^{v+1} B_j^{(i)} - \sum_{j=1}^{w+1} C_j^{(i)} + \sum_{j=1}^{x+1} D_j^{(i)} \]
\[ \theta_i = (d_j^{(i)}/D_j^{(i)}), \quad j = 1, \ldots, M^{(i)} \]
\[ \phi_i = (1 - b_j^{(i)})/B_j^{(i)}, \quad j = 1, \ldots, N^{(i)}, \quad i = 1, \ldots, r \]  \hspace{1cm} (7)
\[ T = \sum_{i=1}^{n} E_i - \sum_{i=1}^{m} E_i + \sum_{i=1}^{d} F_i - \sum_{i=1}^{d+1} F_i \]
\[ \theta = f_i/F_i, \quad i = 1, \ldots, m \]
\[ \phi = (e_{r'} - 1)/E_{r'}, \quad i' = 1, \ldots, n \]  \hspace{1cm} (8)
\[ \theta_{r'} = f_{r'}/F_{r'}, \quad i' = 1, \ldots, n \]
\[ \phi = (e_{r'} - 1)/E_{r'}, \quad i' = 1, \ldots, n \]
\[ \phi = (e_{r'} - 1)/E_{r'}, \quad i' = 1, \ldots, n \]  \hspace{1cm} (9)

Also we use the notation
\[ H^{\nu_1, \ldots, \nu_r; \ldots; \nu_{s+1}, \ldots, \nu_s; \ldots; \nu_{t+1}, \ldots, \nu_t; \ldots; \nu_{u+1}, \ldots, \nu_u}_{v_1, \ldots, v_r; \ldots; v_{s+1}, \ldots, v_s; \ldots; v_{t+1}, \ldots, v_t; \ldots; v_{u+1}, \ldots, v_u} \left( \frac{[a]; \ldots, [b]; \ldots; [c]; \ldots, [d]; \ldots; [e]; \ldots, [f]; \ldots; [g]; \ldots, [h]; \ldots; [i]; \ldots, [j]; \ldots; [k]; \ldots, [l]; \ldots; [m]; \ldots, [n]; \ldots; [o]; \ldots, [p]; \ldots; [q]; \ldots, [r]; \ldots; [s]; \ldots, [t]; \ldots; [u]; \ldots, [v]; \ldots; [w]; \ldots, [x]; \ldots, [y]; \ldots, [z]; \ldots; [\nu_1]; \ldots, [\nu_r]; \ldots; [\nu_{s+1}]; \ldots, [\nu_s]; \ldots; [\nu_{t+1}]; \ldots, [\nu_t]; \ldots; [\nu_{u+1}]; \ldots, [\nu_u]}{[A]; \ldots, [B]; \ldots; [C]; \ldots, [D]; \ldots; [E]; \ldots, [F]; \ldots; [G]; \ldots, [H]; \ldots; [I]; \ldots, [J]; \ldots; [K]; \ldots, [L]; \ldots; [M]; \ldots, [N]; \ldots; [O]; \ldots, [P]; \ldots; [Q]; \ldots, [R]; \ldots; [S]; \ldots, [T]; \ldots; [U]; \ldots, [V]; \ldots; [W]; \ldots, [X]; \ldots, [Y]; \ldots, [Z]; \ldots; [\nu_1]; \ldots, [\nu_r]; \ldots; [\nu_{s+1}]; \ldots, [\nu_s]; \ldots; [\nu_{t+1}]; \ldots, [\nu_t]; \ldots; [\nu_{u+1}]; \ldots, [\nu_u]} \right) \]  \hspace{1cm} (10)

The importance of our main result lies in the fact that it involves the product of a general class of polynomials, Fox's \( H \)-function and the multivariable \( H \)-function having general arguments. Thus this result serves as a key formula from which the bivariate Laplace transform for the product of a large number of polynomials which are special cases of \( S_n^p[x] \) [9, pp. 158–61] and simpler special functions which are particular cases of Fox's \( H \)-function [6, pp. 145–151] and of the multivariable \( H \)-function follow merely by specializing the parameters. In this paper we shall obtain correspondences, involving a product of a general class of polynomials, Fox's \( H \)-function and the multivariable \( H \)-function, between the original and the image in two variables.

In what follows we shall denote the original variables by \( x \) and \( y \) and the transformed variables by \( p \) and \( q \). The notations employed are those of Ditkin and Prudnikov's operational calculus.
Simultaneous operational calculus

Theorem 1. With \( T_i, \theta_i, \phi_i, T, \theta \) and \( \phi \) given by (7), (8), (9) and (10) respectively, let \( T_i > 0, T > 0 \), \( |\arg(z_i)| < T_i \pi /2, |\arg(z)| < T \pi /2, h_i > 0, k > 0, \delta > 0, i = 1, \ldots, r, \mu \) is an arbitrary positive integer and the coefficients \( L_{x,a}(\lambda, \alpha > 0) \) are arbitrary constants real or complex, and

(i) \( \text{Re} \left( \rho - \sigma - k \phi - \sum_{i=1}^{r} h_i \phi_i \right) < 3/4 \)

(ii) \( \text{Re} (\rho) > 0, \quad \text{Re} \left( \sigma + k \theta + \sum_{i=1}^{r} h_i \theta_i \right) > 0. \)

Also, let \( 0 \leq n \leq u, 0 \leq m \leq d \) and

(iii) \( \text{Re}(p) > 0. \)

\[
P^{-1/2}(pq)^{\sigma/2} - \rho + \sum_{a=0}^{[2/\pi]} \sum_{s=0}^{m} \frac{(-1)^s}{s! F_G} \phi(g_s) \\
\times z^{a} \left( -\frac{\lambda}{\alpha} \right)^{m} L_{x,a}(p)(\frac{\alpha^{a} + \beta \alpha /2}{(\frac{\alpha}{2})^{a/2}}) F_{v+1, w+1}(\frac{\alpha}{2}; P, Q_{\alpha} \cdots P_{\alpha}) (z_{1}(\sqrt{pq})^{h_{1}}, \ldots, z_{r}(\sqrt{pq})^{h_{r}}) \\
\times (4xy)^{-1/2} \frac{(\sigma/2) - (1/2)}{\sqrt{\pi y}} \sum_{a=0}^{[2/\pi]} \sum_{s=0}^{m} \frac{(-1)^s}{s! F_G} \phi(g_s) z^{a} \left( -\frac{\lambda}{\alpha} \right)^{m} \\
\times L_{x,a}(4xy)^{-k \alpha /2} - (\alpha/2) H_{0,0;1}^{0,0}((M_{\alpha}, N_{\alpha}) \cdots (M_{\alpha}, N_{\alpha})) \left[ ([a]; A', \ldots, A^{(r)}), \\
\quad ([c]; C', \ldots, C^{(r)}); \\
\quad [2 \rho - \sigma - k \alpha; \delta \alpha; h_{1}, \ldots, h_{r}; ([b']; B'); \ldots; ([b^{(r)}]; B^{(r)}); \\
\quad ([d']; D'); \ldots; ([d^{(r)}]; D^{(r)}); \\
\quad z_{1}(2\sqrt{xy})^{-h_{1}}, \ldots, z_{r}(2\sqrt{xy})^{-h_{r}} \right) \tag{12}
\]

Proof. The Laplace transform of the product of a general class of polynomials, Fox's \( H \)-function and the multivariable \( H \)-function is given by

\[
L \left\{ t^{\sigma-1} S_{\alpha}^{\alpha}(t^{\beta}) H_{\alpha}^{\beta \gamma \delta} \left[ \begin{array}{c}
(x_1, E_1) \\
(y_1, F_1)
\end{array} \right] H_{p, w + 1}^{1, q \pi}(P, Q_{\alpha} \cdots P_{\alpha}) \left[ ([a]; A', \ldots, A^{(r)}); \\
\quad ([c]; C', \ldots, C^{(r)}); \\
\quad [1 - \sigma - k \alpha; \delta \alpha; h_{1}, \ldots, h_{r}; ([b']; B'); \ldots; ([b^{(r)}]; B^{(r)}); \\
\quad ([d']; D'); \ldots; ([d^{(r)}]; D^{(r)}); \\
\quad z_{1} t^{h_{1}}, \ldots, z_{r} t^{h_{r}} \right) \right\} \}
\]

\[
= p^{-\sigma} \sum_{a=0}^{[2/\pi]} \sum_{s=0}^{m} \frac{(-1)^s}{s! F_G} \phi(g_s) z^{a} \left( -\frac{\lambda}{\alpha} \right)^{m} L_{x,a}(p)^{-k \alpha - \delta \alpha} \\
\times H_{p, w + 1}^{1, q \pi}(P, Q_{\alpha} \cdots P_{\alpha}) (z_{1} p^{-h_{1}}, \ldots, z_{r} p^{-h_{r}}), \tag{13}
\]

where

\[
\text{Re}(p) > 0, \quad \text{Re} \left( \sigma + k \theta + \sum_{i=1}^{r} h_i \theta_i \right) > 0,
\]
\[
\Re \left( \sigma + k\phi + \sum_{i=1}^{r} h_i \phi_i \right) < 0, \quad h_i > 0, \quad k > 0, \quad \delta > 0, \\
|\arg(z_i)| < T_i \pi/2, \quad |\arg(z)| < T \pi/2, \quad T_i > 0, \quad T > 0,
\]

\(\mu\) is an arbitrary positive integer and the coefficients \(L_{\lambda,\alpha} (\lambda, \alpha \geq 0)\) are arbitrary constants real or complex. The result in (13) can easily be established by making use of (3) and a result recently obtained by Chaurasia [3, eqn. (2.2), p. 22].

On writing \((pq)^{-1/2}\) for \(p\) and multiplying both sides of (13) by \(p^{-1/2}(pq)^{1-\rho}\) and then interpreting it with the help of a known result [4, p. 144, eqn. (3.26)], we get

\[
\frac{(4xy)^{(\rho/2)-(1/4)}}{\sqrt{\pi y}} \int_0^\infty t^{\sigma-\rho-\alpha} J_{2\rho - 1} [(64xyt^2)^{1/4}] \\
\times H_{u,d}^{m,n} \left[ \sum_{i=1}^{r} (E_u, E_d) \right] S_{u,d} \left[ t^d \right] H_{u,w+1}^{0,0} (M'_1, N'_1; \ldots; (M'_m, N'_m)) \\
\times \left[ (a) : (A'_1), \ldots, A'_{r} : \left[ 1 - \sigma - kG - \delta : h_1, \ldots, h_r \right] : \\
\left( (c) : C'_1, \ldots, C'_{r} : \right) \right] \\
\times \left[ (b) : B'_1 : \ldots : \left( (b') : B'(1) : \ldots : \left( (d) : D'_1 : \ldots : \left( (d') : D'(1) : \\ z_1 t^{h_1}, \ldots, z_r t^{h_r} \right) \right) \right] dt \\
\pm p^{-1/2}(pq)^{\rho/2 - 1/4} \sum_{\alpha=0}^{\rho-1} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s! F_G} \phi(g_s) z^{2s} \\
\times \left( -\frac{\lambda}{\alpha} \right)^\mu L_{\lambda,\alpha} (pq)^{(k+\delta)/2} H_{u,w}^{0,0} (M'_1, N'_1; \ldots; (M'_m, N'_m)) \\
\times (z_1 (\sqrt{pq})^{h_1}, \ldots, z_r (\sqrt{pq})^{h_r}). \quad (14)
\]

Now, evaluating LHS of (14) by the process mentioned in (13) we obtain the described result and (21) is established.

2. Special cases

Letting \(K \to 0\), from (12) we get (after a little simplification) the following bivariate Laplace transform for a general class of polynomials and the multivariable \(H\)-function in the elegant form.

**Theorem 2.** With \( T_i, \theta_i \) and \( \phi_i \) given by (7) and (8) respectively, let \( T_i > 0, \) \( |\arg(z_i)| < T_i \pi/2, h_i > 0, i = 1, \ldots, r, \delta > 0, \mu \) is an arbitrary positive integer and the coefficients \( L_{\lambda,\alpha} (\lambda, \alpha \geq 0) \) are arbitrary constants real or complex, and

(i) \( \Re \left( \sigma + \sum_{i=1}^{r} h_i \theta_i \right) > 0 \)

(ii) \( \Re \left( \rho - \sigma - \sum_{i=1}^{r} h_i \phi_i \right) < 3/4. \)
Also let

(iii) $\text{Re}(p) > 0$, $\text{Re}(\rho) > 0$

\[
p^{-1/2}(pq)^{(o/2) - \rho + 1} \sum_{a=0}^{\{2a\}} \frac{(-\lambda)^{a}}{a!} L_{a,a}(pq)^{a/2} \times H_{\nu+1,\nu+1}(\nu+1) \left( (a): A', \ldots, A^{(o)}; \ldots; (c): C', \ldots, C^{(o)}; \left[ 2\rho - \sigma - \deltax: h_1, \ldots, h_r \right] \right)
\]

\[
\times \left[ (b'): B'; \ldots; [(b^{(o)}): B^{(o)}]; \right] \]

\[
\times \left[ (d'): D'; \ldots; [(d^{(o)}): D^{(o)}]; z_1(2\sqrt{xy})^{-h_1}, \ldots, z_r(2\sqrt{xy})^{-h_r} \right]
\]

(ii) Letting $\lambda \to 0$, the theorem 1 reduces to a known theorem recently obtained by Chaurasia [3, eqn. (2.1), p. 21].

(iii) Letting $\lambda \to 0$, $K \to 0$ and $r = 2$, the theorem 1 reduces to a known result [2, eqn. (2.1), p. 86].

(iv) Putting $\lambda = 0$ in (15), we get a known result recently obtained by Chaurasia [3, eqn. (3.1), p. 24].

(v) Taking $r = 1$ and $\lambda \to 0$, the theorem 2 reduces to a known theorem obtained in [2, theorem (3b), p. 88].

The importance of our results lies in its manifold generality. In view of the generality of the polynomials $s^a[x]$, on suitably specializing the coefficients $L_{a,a}$, and making a free use of the special cases of $S^a[x]$ listed by Srivastava and Singh [9], our results can be reduced to a large number of bivariate Laplace transforms involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in Fox's $H$-function and in the multivariable $H$-function, from our results, several bivariate Laplace transforms involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of $E$, $F$, $G$ and $H$ functions of one and several variables. Thus the results presented in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions occurring in the problems of mathematical analysis.

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