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Convolution properties of some classes of meromorphic univalent functions

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Abstract. Convolution technique and subordination theorem are used to study certain class of meromorphic univalent functions in the punctured unit disc.

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1. Introduction

In this article, we study certain subclasses of functions of the form

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n \tag{1.1}$$

which are univalent and analytic in $0<|z|<1$. Thus, at $z=0$, these functions have a simple pole. Denote this class by Σ.

Let $\Sigma^*(A,B)$, $\Sigma_K(A,B)$ $(-1\leqslant B<1, B<A)$ be the subclasses of functions in Σ satisfying

$$-\frac{zg'(z)}{g(z)}\prec\frac{1+Az}{1+Bz},\quad z\in U \tag{1.2}$$

and

$$-\left(\frac{zg''(z)}{g'(z)}+1\right)\prec\frac{1+Az}{1+Bz},\quad z\in U \tag{1.3}$$

respectively. Here $\prec$ denotes subordination and $U=\{z:|z|<1\}$. For $0\leqslant\beta<1$, $\Sigma^*(1-2\beta,-1)$ and $\Sigma_K(1-2\beta,-1)$ are respectively the well-known subclasses of Σ consisting of functions meromorphic starlike and meromorphic convex of order β. Denote by $\Sigma^*(1-2\beta,-1)=\Sigma^*(\beta)$ and $\Sigma^*(0)=\Sigma^*$; $\Sigma_K(1-2\beta,-1)=\Sigma_K(\beta)$ and $\Sigma_K(0)=\Sigma_K$. For more details about these classes see Chapter 17 of [3].

We say that a function $g\in\Sigma$ is meromorphic λ-spiral-like of order β in $0<|z|<1$ if

$$-\operatorname{Re}\left\{e^{i\lambda}\frac{zg'(z)}{g(z)}\right\}>\beta\cos\lambda,\quad z\in U \tag{1.4}$$

for $0\leqslant\beta<1$ and $-\pi/2<\lambda<\pi/2$. We denote this class by $\Sigma^{*\lambda}(\beta)$.

Similarly we say that function ψ of the form (1.1) analytic in $0<|z|<1$ is meromorphic close-to-convex of order β and type $\lambda(\beta<-1/d_{-1},\ 0\leqslant\lambda<1)$ in $0<|z|<1$ if there exists a function ϕ of the form

$$\phi(z)=\frac{d_{-1}}{z}+\sum_{n=0}^{\infty}d_nz^n,\quad d_{-1}\neq 0, \tag{1.5}$$

which is meromorphic starlike of order λ (i.e. ϕ need not be normalized) such that

$$\operatorname{Re}\left\{\frac{z\psi'(z)}{\phi(z)}\right\}>\beta,\quad z\in U. \tag{1.6}$$

If we choose $\phi(z)=-1/z$, then the condition of meromorphic close-to-convexity reduces to

$$-\operatorname{Re}\{z^2\psi'(z)\}>\beta,\quad z\in U. \tag{1.7}$$

Now let g and ψ be two functions with series expansions

$$g(z)=\frac{d_{-1}}{z}+\sum_{n=0}^{\infty}d_nz^n,\quad (d_{-1}\neq 0);\quad \psi(z)=\frac{e_{-1}}{z}+\sum_{n=0}^{\infty}e_nz^n,\quad (e_{-1}\neq 0)$$

which are analytic and univalent in $0<|z|<1$. In [7], Robertson showed that the convolution or Hadamard product $g*\psi$ of such functions defined by

$$(g*\psi)(z)=\frac{d_{-1}e_{-1}}{z}+\sum_{n=0}^{\infty}d_ne_nz^n$$

is also analytic and univalent in $0<|z|<1$, for $d_{-1}=e_{-1}=1$. Furthermore he proved that $g*\psi$ is not only close-to-convex of order 0 in $0<|z|<1$ with respect to $\phi(z)=-1/z$ but also meromorphic starlike of order 0 in $0<|z|<1$.

Now we introduce the class $T_\delta(A,B)$ of functions having properties analogous to those of analytic univalent case considered by Ruscheweyh [8].

DEFINITION

Let δ, A, B be arbitrarily fixed real numbers such that $\delta>-1$, $-1\leqslant B<1$ with $B<A$. A function $g\in\Sigma$ is said to be in the class $T_\delta(A,B)$ if it satisfies

$$-\left[\frac{E^{\delta+1}g(z)}{E^\delta g(z)}-2\right]\prec\frac{1+Az}{1+Bz},\quad z\in U, \tag{1.8}$$

where

$$E^\delta g(z)=\frac{1}{z(1-z)^{\delta+1}}*g(z),\quad \delta>-1,\quad 0<|z|<1. \tag{1.9}$$

It is readily seen that for $\delta=n\in N\cup\{0\}$,

$$E^n g(z)=\frac{z^{-1}}{n!}\frac{d^n}{dz^n}(z^{n+1}g(z)).$$

With this notation the well-known condition for $g\in\Sigma$ to be in $\Sigma^*(A,B)$ can be written as

$$g\in\Sigma^*(A,B) \text{ if and only if } g\in T_0(A,B).$$

Note that $E^0 g(z)=g(z)$, $E^1 g(z)=zg'(z)+2g(z)$ and so

$$E^1 g(z)-2E^0 g(z)=zg'(z)=\frac{2z-1}{z(1-z)^2}*g(z).$$

In order to prove the main theorems of this article, we need the following lemmas.

Lemma 1. [4, *Corollary* 3.2]. *If A, B, β are reals with $\beta\neq 0$, $-1\leqslant B<1$ and $A\neq B$, and complex number γ satisfies*

$$\operatorname{Re}\gamma\geqslant -\frac{\beta(1-A)}{2} \quad \textit{whenever } B=-1 \textit{ and } A\neq -1$$

and

$$\operatorname{Re}\gamma\geqslant \max\left\{-\frac{\beta(1-A)}{1-B}, -\frac{\beta(1+A)}{1+B}\right\} \quad \textit{whenever } |B|<1 \textit{ and } A\neq B,$$

then the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz}$$

has a univalent solution given by

$$q(z)=\begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\mathrm{d}t}-\dfrac{\gamma}{\beta} & \textit{if } B\neq 0\\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)\mathrm{d}t}-\dfrac{\gamma}{\beta} & \textit{if } B=0. \end{cases} \tag{1.10}$$

If $p(z)$ is analytic in U and satisfies

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\frac{1+Az}{1+Bz}, \quad z\in U$$

then

$$p(z)\prec q(z)\prec\frac{1+Az}{1+Bz}$$

and $q(z)$ is the best dominant of the above differential subordination.

The method of proof of the following Lemma is based on the lines of proof of Wilken and Feng [10]. So we omit the details.

Lemma 2. *Let $\mu(t)$ be a positive measure on the unit interval $I=[0,1]$. Let $g(t,z)$ be a complex valued function defined on $[0,1]\times U$, analytic with respect to $z\in U$ and integrable in t for each $z\in U$ and for almost all $t\in[0,1]$, and suppose that* $\operatorname{Re}\{g(t,z)\}>0$

on U and $g(z) = \int_I g(t,z)\mathrm{d}\mu(t)$. *If, for fixed* $0 \leqslant \lambda < 2\pi$, $g(t, re^{i\lambda})$ *is real for r real and*

$$\mathrm{Re}\left\{\frac{1}{g(t,z)}\right\} \geqslant \frac{1}{g(t,re^{i\lambda})}, \quad \text{for } |z| \leqslant r \text{ and } t \in [0,1]$$

then

$$\mathrm{Re}\left\{\frac{1}{g(z)}\right\} \geqslant \frac{1}{g(re^{i\lambda})} \quad \text{for } |z| \leqslant r \text{ and } 0 \leqslant \lambda < 2\pi.$$

For a, b, c real numbers other than $0, -1, -2, \cdots$ the hypergeometric series

$$F(a,b;c;z) = 1 + \frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{1.2c(c+1)}z^2 + \cdots \tag{1.11}$$

represents an analytic function in U [9, p. 281]. The following identities are well known.

Lemma 3. [9, *Chapter* XIV]: *For a, b, c real numbers other than* $0, -1, -2, \cdots$ *and* $c > b > 0$ *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z) \tag{1.12}$$

$$F(a,b;c;z) = F(b,a;c;z) \tag{1.13}$$

$$F(a,b;c;z) = (1-z)^{-a}F(a,c-b;c;z/(1-z)) \tag{1.14}$$

$$F\left(a,b;\frac{a+b+1}{2};\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}.$$

To a certain extent the work on univalent meromorphic functions has paralleled that of univalent analytic functions since one is tempted to search for a class of functions in Σ which is analogous to that of analytic case. However, though a large number of papers have appeared dealing with some of the classes defined, through convolution of analytic functions, it is somewhat surprising that not much attempt appears to have been made in defining the classes, through convolution of meromorphic functions. In the present paper we make a modest attempt in this direction. Recently, Mogra [5] has made an easy attempt in a different context.

In §2, we give a necessary and sufficient condition for a function g in Σ to be in $\Sigma^*(A,B)$ and $\Sigma_K(A,B)$, $(-1 \leqslant B < A \leqslant 1)$ respectively in terms of convolution. In §3, we give containment relations for the classes $T_{\delta+1}(A,B)$. In §4, we study certain integral transforms in the classes $T_\delta(A,B)$ and in turn to the classes $\Sigma^*(A,B)$ and $\Sigma_K(A,B)$ which are much more general than the one considered by Goel and Sohi [2], Bajpai [1] and others. In the last section we study some sort of converse problem for functions in $T_\delta(1-2\beta,-1)$. In the analytic case converse problem of this type has been considered by many workers (see [11]).

2. Convolution theorems

Theorem 2.1. *A function g is in $\Sigma^*(A,B)$, $-1 \leqslant B < A \leqslant 1$, if and only if*

$$\left[g(z)*\left\{\frac{1+(1-(A-2B)x)(A-B)^{-1}x^{-1}z}{z(1-z)^2}\right\}\right] \neq 0 \tag{2.1}$$

for $0<|z|<1$, $|x|=1$.

Proof. The function g is in $\Sigma^*(A,B)$, $-1 \leqslant B < A \leqslant 1$ if and only if

$$-\frac{zg'(z)}{g(z)} \neq \frac{1+Ax}{1+Bx} \tag{2.2}$$

for $z \in U$ and $|x|=1$. Since $-(zg'(z)/g(z))=1$ at $z=0$, (2.2) is equivalent to

$$-zg'(z)(1+Bx)-g(z)(1+Ax) \neq 0, \quad 0<|z|<1. \tag{2.3}$$

Since

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n = g(z)*\left(\frac{1}{z(1-z)}\right)$$

we have

$$\begin{aligned} -zg'(z) &= \frac{1}{z}-\sum_{n=0}^{\infty} nb_n z^n \\ &= g(z)*\left(\frac{1}{z}-\frac{z}{(1-z)^2}\right), \quad 0<|z|<1 \\ &= g(z)*\left(\frac{1-2z}{z(1-z)^2}\right) \end{aligned}$$

so that the left hand side of (2.3) may be expressed as

$$g(z)*\left[\frac{1+(1-(A-2B)x)(A-B)^{-1}x^{-1}z}{z(1-z)^2}\right] \neq 0$$

for $0<|z|<1$, $|x|=1$, which is the desired convolution condition. This completes the proof. □

If we set

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n,$$

then

$$(zg'(z))' = (-zg'(z))*\left(\frac{1-2z}{z(1-z)^2}\right)$$

and so from Theorem 2.1, and the identity

$$zg'(z)*\psi(z)=g(z)*z\psi'(z), \quad g,\psi \in \Sigma$$

we obtain

Theorem 2.2. *A function g is in $\Sigma_K(A,B)$, $-1 \leqslant B < A \leqslant 1$, if and only if*

$$g(z)*\left[\frac{-1+(1-(A-2B)x)(A-B)^{-1}x^{-1}[3z+2z^2]}{z(1-z)^3}\right] \neq 0$$

for $0<|z|<1$, $|x|=1$.

Remark 1. For $A = 1-2\beta$, $B = -1$, Theorems 2.1 and 2.2 give necessary and sufficient convolution conditions for a function $g \in \Sigma$ to be in $\Sigma^*(\beta)$ and $\Sigma_K(\beta)$, $0 \leqslant \beta < 1$ respectively.

Theorem 2.3. *A function $g \in \Sigma$ is λ-spiral-like of order β, $0 \leqslant \beta < 1$, in $0<|z|<1$ if and only if*

$$g(z)*\left[\frac{1+\left(\dfrac{2\beta\cos\lambda e^{-i\lambda}-e^{-2i\lambda}-2-x}{1+e^{-2i\lambda}-2\beta\cos\lambda e^{-i\lambda}}\right)z}{z(1-z)^2}\right] \neq 0 \tag{2.4}$$

for $0<|z|<1$, $|x|=1$.

Proof. The function $g \in \Sigma$ is λ-spiral-like of order β if and only if

$$-\operatorname{Re}\left\{e^{i\lambda}\frac{zg'(z)}{g(z)}\right\} > \beta\cos\lambda, \quad z \in U, \quad |\lambda| < \frac{\pi}{2}. \tag{2.5}$$

Since

$$\left(-e^{i\lambda}\frac{zg'(z)}{g(z)} - i\sin\lambda\right)(\cos\lambda)^{-1} = 1 \text{ at } z=0,$$

(2.5) is equivalent to

$$\left(-e^{i\lambda}\frac{zg'(z)}{g(z)} - i\sin\lambda\right)(\cos\lambda)^{-1} \neq \frac{1+(2\beta-1)x}{1+x}, \quad z\in U, |x|=1, x \neq -1$$

which simplifies to

$$-(1+\bar{x})zg'(z) + (e^{-2i\lambda} - 2\beta\cos\lambda e^{-i\lambda} - \bar{x})g(z) \neq 0, \tag{2.6}$$

where $\bar{x}$ stands for the conjugate of the complex x. The remainder of the argument is the same as that of Theorem 2.1. □

3. Containment relations

For the proof of our next theorem we need the following lemma:

Lemma 3.1. *For $g \in \Sigma$ and $\delta > -1$ we have the following identity*

$$z\frac{\mathrm{d}}{\mathrm{d}z}(E^\delta g(z)) = (\delta+1)E^{\delta+1}g(z) - (\delta+2)E^\delta g(z), \quad 0<|z|<1 \tag{3.1}$$

where $E^\delta g(z)$ is given by (1.9).

Proof. If we set

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n,$$

then

$$E^{\delta}g(z)=\frac{1}{z(1-z)^{\delta+1}}*g(z)$$
$$\equiv\left[\frac{1}{z}+\sum_{n=0}^{\infty}\binom{\delta+n+1}{n+1}z^n\right]*\left[\frac{1}{z}+\sum_{n=0}^{\infty}b_n z^n\right]. \tag{3.2}$$

Now equating the coefficients of z^{-1}, constant term and z^n in the expansion of $(\delta+1)E^{\delta+1}g(z)-(\delta+2)E^{\delta}g(z)$ with the corresponding coefficients in the expansion of $z(E^{\delta}g(z))'$, the result follows. □

Theorem 3.1. *Let* $\delta>-1$.

(a) *Suppose that the constants A and B satisfy*

$$B<A\leqslant\frac{\delta+1+B}{\delta+2}\quad \textit{for}\ -1\leqslant B<1. \tag{3.3}$$

Then for $g\in T_{\delta+1}(A,B)$ *we have*

$$g\in T_{\delta}(A',B) \tag{3.4}$$

where

$$A'=A+(A-B)/(\delta+1). \tag{3.5}$$

Further

$$-\left(\frac{E^{\delta+1}g(z)}{E^{\delta}g(z)}-2\right)\prec-\frac{1}{\delta+1}\left[\frac{1}{Q(z)}\right]+2\equiv\tilde{q}(z)\prec\frac{1+A'z}{1+Bz},\quad z\in U \tag{3.6}$$

where

$$Q(z)=\begin{cases}\displaystyle\int_0^1\left(\frac{1+Btz}{1+Bz}\right)^{-(\delta+2)(A-B)/B}t^{\delta}\mathrm{d}t, & \textit{if } B\neq 0\\ \displaystyle\int_0^1 \exp\{(\delta+2)(1-t)Az\}t^{\delta}\mathrm{d}t, & \textit{if } B=0.\end{cases} \tag{3.7}$$

(b) *Suppose that the constants A and B satisfy*

$$B<A\leqslant\min\left\{\frac{\delta+1+B}{\delta+2},2B\right\}$$

with $0<B<1$, *then*

$$T_{\delta+1}(A,B)\subset T_{\delta}(1-2\rho',-1) \tag{3.8}$$

where

$$\rho'=2-[F(1,(\delta+2)(A-B)/B;\delta+2;B/(1+B))]^{-1}. \tag{3.9}$$

(c) *Suppose that*

$$B<A\leqslant\min\left\{\frac{\delta+1+B}{\delta+2},0\right\}$$

with $-1 \leqslant B < 0$, *then*

$$T_{\delta+1}(A,B) \subset T_\delta(1-2\rho'', -1) \tag{3.10}$$

where

$$\rho'' = 2 - [F(1,(\delta+2)(B-A)/B; \delta+2; -B/(1-B))]^{-1} \tag{3.11}$$

and $F(a,b;c;z)$ *is as defined by* (1.11). *The relations* (3.6), (3.8) *and* (3.10) *are all the best possible.*

Proof. Let $g \in T_{\delta+1}(A,B)$ where $\delta > -1$, $-1 \leqslant B < 1$ and $B < A$. Set

$$\phi(z) = z[zE^\delta g(z)]^{-1/(1+\delta)}$$

and

$$r_1 = \sup\{r: \phi(z) \neq 0, \quad 0 < |z| < r < 1\}.$$

Then ϕ is single valued in $0 < |z| < r_1$ and using (3.1) it follows that the function p defined by

$$p(z) = \frac{z\phi'(z)}{\phi(z)} = -\left(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2\right) \tag{3.12}$$

is analytic in $|z| < r_1$ and $p(0) = 1$. Since $g \in T_{\delta+1}(A,B)$, (1.8) coupled with (3.12) and (3.1) easily leads to

$$-\left(\frac{E^{\delta+2}g(z)}{E^{\delta+1}g(z)} - 2\right) = \frac{1}{\delta+2} + \frac{\delta+1}{\delta+2}\left(p(z) + \frac{zp'(z)}{(\delta+1)(2-p(z))}\right)$$
$$\prec \frac{1+Az}{1+Bz}, \quad |z| < r_1.$$

In other words,

$$P(z) + \frac{zP'(z)}{\beta P(z) + \gamma} \prec \frac{1+Az}{1+Bz} \tag{3.13}$$

where

$$P(z) = \left(1 - \frac{1}{\delta+2}\right)p(z) + \frac{1}{\delta+2}, \quad \beta = -(\delta+2), \quad \gamma = 2\delta+3. \tag{3.14}$$

It can be easily seen that for $-1 \leqslant B < 1$ and $A \neq B$,

$$\operatorname{Re}\left\{\beta\left(\frac{1+Az}{1+Bz}\right) + \gamma\right\} > 0 \text{ in } U$$

if and only if A and B satisfy the following inequalities:

$$\frac{-(\delta+1)+(2\delta+3)B}{\delta+2} \leqslant A \leqslant \frac{(\delta+1)+(2\delta+3)B}{\delta+2} \quad \text{for } -1 < B < 1$$

and

$$A \geqslant -\frac{(3\delta+4)}{\delta+2} \cdot \text{ for } B = -1.$$

It may be noted that (3.3) obviously satisfies the above inequalities and so it follows that

$$\operatorname{Re}\left\{\beta\left(\frac{1+Az}{1+Bz}\right)+\gamma\right\}>0 \text{ in } U$$

under the condition (3.3). Now using Lemma 1 we deduce that

$$P(z)\prec q(z)\prec\frac{1+Az}{1+Bz}, \quad |z|<r_1 \tag{3.15}$$

where q is the best dominant of (3.13) and is given by

$$q(z)=\frac{\delta+1}{\delta+2}\tilde{q}(z)+\frac{1}{\delta+2}.$$

Again by (3.14) and (3.15) we easily get

$$p(z)\prec\tilde{q}(z)\equiv\frac{-1}{(\delta+1)}\left[\frac{1}{Q(z)}\right]+2\prec\frac{1+((A(\delta+2)-B)/(\delta+1))z}{1+Bz}, \quad |z|<r_1 \tag{3.16}$$

where $Q(z)$ is given by (3.7). From (3.3) and (3.16) we see that Re $p(z)>0$ in $|z|<r_1$. This, by (3.12), shows that ϕ is starlike (univalent) in $|z|<r_1$. Thus it is not possible that ϕ vanishes in $|z|<r_1$ if $r_1<1$. So we conclude that $r_1=1$. Therefore p is analytic in U. Hence by (3.12) and (3.16),

$$g\in T_{\delta+1}(A,B) \text{ implies} -\left(\frac{E^{\delta+1}g(z)}{E^{\delta}g(z)}-2\right)\prec\tilde{q}(z), \quad z\in U$$

provided δ, A and B satisfy (3.3). This proves (3.4) and (3.6).

(b) Next we show that

$$\inf_{|z|<1}\{\operatorname{Re}\tilde{q}(z)\}=\tilde{q}(1), \quad \tilde{q}(z)=2-\frac{1}{\delta+1}\left[\frac{1}{Q(z)}\right] \tag{3.17}$$

provided $\delta>-1$, A and B satisfy

$$B<A<\min\left\{\frac{\delta+1+B}{\delta+2},2B\right\}. \tag{3.18}$$

If we set

$$a=-\beta\left(\frac{A-B}{B}\right), \quad b=\beta+\gamma, \quad c=\beta+\gamma+1(\beta=-(\delta+2),\gamma=2\delta+3)$$

then $c>b>0$. From (3.7) by using (1.12), (1.13) and (1.14) we see that for $0\neq B$

$$Q(z)=(1+Bz)^{a}\int_0^1(1+Btz)^{-a}t^{b-1}\,dt$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, c-b; c; Bz/(1+Bz))$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(1, a; c; Bz/(1+Bz)). \tag{3.19}$$

Again (3.7), by (3.19) for $0 < B < 1$, $B < A < \min\left\{\frac{\delta+1+B}{\delta+2}, 2B\right\}$, so that $c > a > 0$, can be rewritten as

$$Q(z) = \int_0^1 g(t,z)\mathrm{d}\mu(t)$$

with

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz}$$

and

$$\mathrm{d}\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\mathrm{d}t.$$

Using Lemma 2, (with $\lambda = 0$) we easily obtain

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\} \geqslant \frac{1}{Q(1)}, \quad z \in U.$$

This proves (3.17) and so by (3.6) we obtain (3.8). For the case $A = \min\left\{\frac{\delta+1+B}{\delta+2}, 2B\right\}$ we obtain (3.8) by letting $A \to \left[\min\left\{\frac{\delta+1+B}{\delta+2}, 2B\right\}\right]^+$.

(c) In a manner similar to that of part (b), using the Lemma 2 (with $\lambda = \pi$) we can easily show

$$\inf_{|z|<1} \mathrm{Re}\,\tilde{q}(z) = \tilde{q}(-1)$$

provided $\delta > -1$, A and B satisfy $-1 \leqslant B < 0$ with $B < A \leqslant \min\left\{0, \frac{(\delta+1)+B}{\delta+2}\right\}$. The proof of part (c) now follows on the same lines. Sharpness follows from the best dominant property. □

4. Integral transforms

We study in this section certain integral transforms of functions in the class $T_\delta(A,B)$. For a function $g \in \Sigma$, defined by $g(z) = z^{-1} + \Sigma_{n=0}^{\infty} b_n z^n$, Bajpai [2] defined the integral transform G_c by

$$G_c(z) = \begin{cases} c\displaystyle\int_0^1 u^c g(uz)\mathrm{d}u \\ z^{-1} + \displaystyle\sum_{n=0}^{\infty} \frac{c}{c+n+1} b_n z^n \end{cases} \tag{4.1}$$

where c is real with $c \geqslant 1$ and showed that

$$g \in \Sigma^*(\beta) \text{ implies } G_c \in \Sigma^*(\beta), \quad (0 \leqslant \beta < 1). \tag{4.2}$$

Reddy and Juneja [6] showed that the relation (4.2) continues to hold if c in (4.1) is taken to be a complex number satisfying $\operatorname{Re} c > 0$. They improved the relation (4.2) further by using a weaker condition on g. In the following theorem we give the sharp results in generalized form as follows.

Theorem 4.1. *Let* $\delta > -1$, *c be a complex number satisfying* $\operatorname{Re} c > 0$ *and the constants* A, B, δ *and* c *satisfy*

$$B - \frac{(1-B)\operatorname{Re} c}{\delta + 1} \leqslant A \leqslant B + \frac{(1+B)\operatorname{Re} c}{\delta + 1} \quad \text{for } -1 < B < 1 \tag{4.3}$$

and

$$A \geqslant -1 - \frac{2\operatorname{Re} c}{\delta + 1} \quad \text{for } B = -1. \tag{4.4}$$

(a) *If* $g \in T_\delta(A, B)$ *then the function* G_c *defined by* (4.1) *satisfies* $G_c \in T_\delta(A, B)$. *Furthermore*

$$-\left[\frac{E^{\delta+1} G_c(z)}{E^\delta G_c(z)} - 2\right] \prec -\frac{1}{\delta+1}\left[\frac{1}{Q(z)}\right] + \left(\frac{c+\delta+1}{\delta+1}\right) \equiv \tilde{q}(z), \quad z \in U \tag{4.5}$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{-(\delta+1)(A-B)/B} t^{c-1}\,\mathrm{d}t & \text{if } B \neq 0 \\ \displaystyle\int_0^1 \exp\{-(1+\delta)A(t-1)z\} t^{c-1}\,\mathrm{d}t & \text{if } B = 0. \end{cases} \tag{4.6}$$

(b) *If c is real with* $c > 0$, $0 < B < 1$ *and* $B < A \leqslant \min\left\{B + \frac{(c+1)B}{\delta+1}, B + \frac{(1+B)c}{\delta+1}\right\}$ *then for* $g \in T_\delta(A, B)$ *we have*

$$G_c \in T_\delta(1 - 2\rho', -1) \tag{4.7}$$

where

$$\rho' = \frac{1}{1+\delta}\{c + \delta + 1 - [F(1, (\delta+1)(A-B)/B; c+1; B/(1+B))]^{-1}. \tag{4.8}$$

(c) *If c is real with* $c > 0$,

$$B < A \leqslant \min\left\{-\frac{B(c-\delta)}{\delta+1}, B + \frac{(1+B)c}{\delta+1}\right\} \quad \text{for } -1 < B < 0$$

and

$$-1 < A \leqslant \frac{c-\delta}{\delta+1} \quad \text{for } B = -1$$

then for $g \in T_\delta(A, B)$ *we have*

$$G_c \in T_\delta(1 - 2\rho'', -1) \tag{4.9}$$

where

$$\rho'' = \frac{1}{\delta+1}\{c+\delta+1-[F(1,(\delta+1)(B-A)/B;c+1;-B/(1-B))]^{-1}\}. \tag{4.10}$$

The results are all the best possible ones.

Proof. Suppose that $g\in T_\delta(A,B)$ and A, B, δ and c satisfy (4.3) and (4.4). Since G_c is defined by

$$G(z)\equiv G_c(z)=\left(z^{-1}+\sum_{n=0}^{\infty}\frac{c}{c+n+1}z^n\right)*g(z)$$

and

$$E^\delta g(z)=z^{-1}+\sum_{n=0}^{\infty}\frac{\Gamma(n+\delta+2)}{\Gamma(\delta+1)}\frac{1}{(n+1)!}z^n$$

it can be easily seen that

$$z\frac{\mathrm{d}}{\mathrm{d}z}[E^\delta G(z)]=cE^\delta g(z)-(1+c)E^\delta G(z) \tag{4.11}$$

and as in Lemma 3.1,

$$z\frac{\mathrm{d}}{\mathrm{d}z}[E^\delta G(z)]=(\delta+1)E^{\delta+1}G(z)-(\delta+2)E^\delta G(z). \tag{4.12}$$

Equating (4.11) and (4.12) we get

$$cE^\delta g(z)=(c-\delta-1)E^\delta G(z)+(\delta+1)E^{\delta+1}G(z). \tag{4.13}$$

Let $g\in T_\delta(A,B)$. We put

$$\phi(z)=z[zE^\delta G(z)]^{-1/(\delta+1)}$$

and $r_1=\sup\{r:\phi(z)\neq 0, 0<|z|<r<1\}$. Then ϕ is single valued and analytic in $|z|<r_1$ and p defined by

$$p(z)=\frac{z\phi'(z)}{\phi(z)}=-\left(\frac{E^{\delta+1}G(z)}{E^\delta G(z)}-2\right) \tag{4.14}$$

is analytic in $|z|<r_1$, $p(0)=1$.

Using (4.11) and (4.12), (4.14) by differentiation reduces to

$$-\left(\frac{E^{\delta+1}g(z)}{E^\delta g(z)}-2\right)=p(z)+\frac{zp'(z)}{c+\delta+1-(\delta+1)p(z)},\quad |z|<r_1. \tag{4.15}$$

Since $g\in T_\delta(A,B)$, we have by (4.15) that

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\frac{1+Az}{1+Bz}, \tag{4.16}$$

where

$$\beta = -(\delta + 1), \quad \gamma = c + \delta + 1.$$

Using Lemma 1, we deduce that

$$p(z) \prec \tilde{q}(z) = \frac{1}{\beta}\left[\frac{1}{Q(z)}\right] - \frac{\gamma}{\beta} \prec \frac{1 + Az}{1 + Bz}, \quad |z| < r_1, \tag{4.17}$$

where $Q(z)$ is defined by (4.6). It may be noted that for $\beta = -(\delta + 1)$ and $\gamma = c + \delta + 1$ with $-1 \leqslant B < 1$ and $B < A$,

$$\mathrm{Re}\left\{\beta\left(\frac{1 + Az}{1 + Bz}\right) + \gamma\right\} > 0 \text{ in } U$$

if the conditions (4.3) and (4.4) are satisfied. Thus it is not possible that $\phi(z)$ vanishes in $|z| < r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore p is analytic in U and hence by (4.14) and (4.17) we obtain the first part of the theorem from Lemma 1.

(b) For the second part it is enough to show that

$$\inf_{|z| < 1} \tilde{q}(z) = \tilde{q}(1), \tag{4.18}$$

provided $\delta > -1$, c, A and B satisfy $0 < B < 1$ with

$$A < B \leqslant \min\left\{B + \frac{(c + 1)B}{\delta + 1}, B + \frac{(1 + B)c}{\delta + 1}\right\},$$

where

$$\tilde{q}(z) = \frac{-1}{\delta + 1}\left[\frac{1}{Q(z)}\right] + \frac{c + \delta + 1}{\delta + 1}$$

and $Q(z)$ is given by (4.6). If we set

$$a = -\beta(A - B)/B, b = \beta + \gamma, c' = \beta + \gamma + 1, \quad (\beta = -(\delta + 1), \gamma = c + \delta + 1)$$

then $c' > b > 0$. From (4.6) we as before see that for $B \neq 0$

$$\begin{aligned} Q(z) &= (1 + Bz)^a \int_0^1 (1 + Btz)^{-a} t^{b-1} \mathrm{d}t \\ &= \frac{\Gamma(b)\Gamma(c' - b)}{\Gamma(c')} F(1, a; c'; Bz/(1 + Bz)). \end{aligned}$$

For $0 < B < 1, B < A < \min\left\{B + \dfrac{(c + 1)B}{1 + \delta}, B + \dfrac{(1 + B)c}{1 + \delta}\right\}$ we see that $c' > a > 0$ and so (4.6) can be rewritten as

$$Q(z) = \int_0^1 g(t, z) \mathrm{d}\mu(t)$$

where

$$g(t, z) = \frac{1 + Bz}{1 + (1 - t)Bz},$$

and

$$\mathrm{d}\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c'-a)} t^{a-1}(1-t)^{c'-a-1}\,\mathrm{d}t.$$

Using Lemma 2, (with $\lambda = 0$), we easily obtain

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\} \geqslant \frac{1}{Q(1)}, \quad z \in U.$$

This proves (4.18) and hence by (4.17) and (4.14) we obtain (4.7). For the case

$$A = \min\left\{B + \frac{(c+1)B}{1+\delta}, B + \frac{(1+B)c}{1+\delta}\right\}.$$

we obtain (4.7) by taking limit as in Theorem 3.1(b).

Part (c) can be proved on similar lines using Lemma 2 with $\lambda = \pi$. □

Taking $\delta = 0$ in the above theorem we obtain

COROLLARY 4.1.

Let c be a complex number satisfying $\mathrm{Re}\, c > 0$, *and such that*

$$B - (1-B)\mathrm{Re}\, c < A \leqslant B + (1+B)\mathrm{Re}\, c \text{ when } -1 < B < 1,$$

and

$$A \geqslant -1 - 2\mathrm{Re}\, c \text{ when } B = -1.$$

(a) *If* $g \in \Sigma^*(A,B)$ *then the function* G_c *defined by* (4.1) *satisfies*

$$-\mathrm{Re}\left(\frac{zG_c'(z)}{G_c(z)}\right) > \inf \mathrm{Re}\left[-\left(\frac{1}{Q(z)}\right) + \frac{c+\delta+1}{\delta+2}\right]$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{(B-A)/B} t^{c-1}\,\mathrm{d}t & \text{if } B \neq 0 \\ \displaystyle\int_0^1 \exp\{A(1-t)z\} t^{c-1}\,\mathrm{d}t & \text{if } B = 0. \end{cases}$$

(b) *If c is real with* $c > 0, 0 < B < 1$ *and* $B < A \leqslant \min\{B + (1+c)B, B + (1+B)c\}$ *then*

$$g \in \Sigma^*(A,B) \text{ implies } G_c \in \Sigma^*(1 - 2\rho_1', -1)$$

where

$$\rho_1' = c + 1 - [F(1,(A-B)/B; c+1; B/(1+B))]^{-1}.$$

(c) *If c is real with* $c > 0$,

$$B < A \leqslant \min\{-Bc, B + (1+B)c\} \text{ when } -1 < B < 0$$

and

$$-1 < A \leqslant c \text{ when } B = -1$$

then

$$g \in \Sigma^*(A,B) \text{ implies } G_c \in \Sigma^*(1 - \rho_2'', -1)$$

where

$$\rho_2'' = c + 1 - [F(1,(B-A)/B; c+1; -B/(1-B))]^{-1}.$$

Remark 2. Since $g \in \Sigma_K(A,B)$ if and only if $-zg'(z) \in \Sigma^*(A,B)$ one can show that the Corollary 4.2 remains true on replacing $\Sigma^*(A,B)$ by $\Sigma_K(A,B)$ both in the hypothesis and the conclusion of the theorem.

Substituting $A = 1 - 2\rho$ with $B = -1$ in part (c) of the above theorem, we get the following sharp result.

COROLLARY 4.2.

Let $g \in \Sigma$, and for real $c > 0$, let

$$G_c(z) = \left(z^{-1} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} z^n \right) * g(z).$$

Then for $-(c-1)/2 \leqslant \rho < 1$ we have

$$g \in \Sigma^*(\rho) \text{ (or } g \in \Sigma^k(\rho) \text{ resp.) implies } G_c \in \Sigma^*(\rho_3'') \text{ (or } G_c \in \Sigma^k(\rho_3'') \text{ resp.)} \tag{4.19}$$

where

$$\rho_3'' = c + 1 - [F(1,(1-\rho)/2; c+1; 1/2)]^{-1}.$$

The result is sharp.

The above Corollary shows that the result obtained improves (and also extends) the earlier results of Bajpai [1], Goel and Sohi [2], Reddy and Juneja [6] and others.

5. Inverse problem

In the earlier section we considered the function G_c defined by (4.1). If we want to solve the equation for g in terms of G_c we must first see that

$$g(z) = \frac{1}{c}[(1+c)G_c(z) + zG_c'(z)]. \tag{5.1}$$

Given some property of $G_c(z)$, we can ask questions about the nature of $g(z)$. A problem of this type can be regarded as a sort of inverse problem.

For considering such an inverse problem of Theorem 4.1 for $A = 1 - 2\rho$ and $B = -1$ we need the following, proof of which can be had in [11]. However we only need the special case of it in this article.

Theorem 5.1. *Let p and q be regular in U and* $\mathrm{Re}\, p(z) > 0$, $q(0) = 1$ *and* $\mathrm{Re}\, q(z) > 0$ *for $z \in U$. Further let $C \neq 0$ and D be complex constants such that $C + D \neq 0$. Then*

$$\mathrm{Re}\left\{ p(z) + \frac{zp'(z)}{C + Dq(z)} \right\} > 0 \text{ in } |z| < \rho(C,D) \tag{5.2}$$

where

$$\rho(C,D) = \frac{[|C|^2 + 2 + 4|D| + |D|^2 - \sqrt{R}]^{1/2}}{|C-D|} \tag{5.3}$$

$$= \frac{|C+D|}{[|C|^2 + 2 + 4|D| + |D|^2 + \sqrt{R}]^{1/2}}$$

and

$$R = (|C|^2 + 2 + 4|D| + |D|^2)^2 - |C^2 - D^2|^2.$$

The result is sharp for C real and non-negative constant.

In [2], Goel and Sohi proved that if G_c belongs to Σ^* then g defined by (5.1) is meromorphically star-like in $0 < |z| < [c/(c+2)]^{1/2}$. However their claim is incorrect. This can be easily seen from the following example.

Consider

$$G(z) = \frac{(1+z)^2}{z} = \frac{1}{z} + 2 + z \text{ with } c = 2/3.$$

Then from (5.1) we have

$$g(z) = \frac{1}{z} + 5 + 4z,$$

and hence

$$-\frac{zg'(z)}{g(z)} = \frac{1-4z^2}{1+5z+4z^2} = \frac{1-4z^2}{(4z+1)(z+1)}.$$

Therefore according to Goel and Sohi [2] g is meromorphically star-like in $0 < |z| < 1/2$. But this is not so because

$$-\operatorname{Re}\frac{zg'(z)}{g(z)}\bigg|_{z=-1/3} = -\frac{5}{2} < 0.$$

The following theorem not only rectifies the result of Goel and Sohi [2] but also generalizes the corresponding results in this direction.

Theorem 5.2. *Let $G_c \in T_\delta(\beta)$, $(0 \leqslant \beta < 1)$ and c be a complex number such that* $\operatorname{Re} c > 0$. *Define g by*

$$g(z) = \frac{1}{c}[(1+c)G_c(z) + zG_c'(z)]. \tag{5.4}$$

Then g belongs to $T_\delta(\beta)$ in $|z| < \rho(C, D)$ for $C = c + (1+\delta)(1-\beta)$ and $D = -(1+\delta)(1-\beta)$ where $\rho(C, D)$ is given by (5.3).

Proof. Since $G_c \in T_\delta(\beta)$ we can write

$$p(z) = (1-\beta)^{-1}\left[-\left(\frac{E^{\delta+1}G_c(z)}{E^\delta G_c(z)} - 2\right) - \beta\right]. \tag{5.5}$$

By hypothesis p is analytic, $\operatorname{Re} p(z) > 0$ in U and $p(0) = 1$. As in the proof of Theorem 4.1 we, from (5.4) and (5.5), easily get the identity

$$(1-\beta)^{-1}\left[-\left(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2\right) - \beta\right] = p(z)$$
$$+ \frac{zp'(z)}{c + (\delta+1)(1-\beta) - (1+\delta)(1-\beta)p(z)}.$$

Set

$$C = c + (1+\delta)(1-\beta) \text{ and } D = -(1+\delta)(1-\beta) \tag{5.6}$$

so that $C + D \neq 0$. Hence by Theorem 5.1 we obtain that $g \in T_\delta(\beta)$ in $|z| < \rho(C, D)$ where $\rho(C, D)$ is obtained from (5.3) using (5.6). □

Substituting $\delta = 0$ in the above theorem and noting Remark 2 we obtain the rectified form of [2] in the following corollary.

COROLLARY 5.1.

If G_c belongs to $\Sigma^*(\beta)$ (or $\Sigma_K(\beta)$ resp.) for $0 \leqslant \beta < 1$ and

$$g(z) = \frac{1}{c}[(1+c)G_c(z) + zG_c'(z)], \quad \operatorname{Re} c > 0$$

then

$$g \in \Sigma^*(\beta) \text{ (or } \Sigma_K(\beta) \text{ resp.) in } |z| < \rho(C, D)$$

with

$$C = c + (1-\beta) \text{ and } D = -(1-\beta).$$

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