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Dirichlet L-function and power series for Hurwitz zeta function

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Abstract. For $0<\alpha<1$, let $\zeta(s,\alpha)$ be the Hurwitz zeta function and let $\zeta_1(s,\alpha)=\zeta(s,\alpha)-\alpha^{-s}$. For a fixed s, we develop $\zeta_1(s,\alpha)$ as a power series in α in the complex circle $|\alpha|<1$. If

$$\sum_{\chi(\bmod q)} L(s,\chi)\,L(s',\bar{\chi})=\frac{\phi(q)}{q^{s+s'}}\sum_{k/q}\mu\left(\frac{q}{k}\right)\left(\sum_{a=1}^{k}\left(\frac{k}{a}\right)^{\operatorname{Re}s+\operatorname{Re}s'}+Q(s,s',k)\right),$$

we obtain an asymptotic expansion for $Q(k)=Q(s,s',k)$ using the power series for $\zeta_1(s,\alpha)$

Keywords. Hurwitz zeta function; Dirichlet L-function; power series.

For integer $q\geqslant 1$, let $\chi(\bmod q)$ be a Dirichlet character and let $L(s,\chi)$ be the corresponding Dirichlet L-series, where $s=\sigma+it$. For $0<\alpha\leqslant 1$, let $\zeta(s,\alpha)$ be Hurwitz zeta function, defined by

$$\zeta(s,\alpha)=\sum_{n\geqslant 0}(n+\alpha)^{-s}\text{ for }\operatorname{Re}s>1;$$

and its analytic continuation. For $0\leqslant\alpha\leqslant 1$, let

$$\zeta_1(s,\alpha)=\sum_{n\geqslant 1}(n+\alpha)^{-s}\text{ for }\operatorname{Re}(s)>1;$$

and its analytic continuation.

Our aim is to prove the following two theorems. Theorem 1 deals with $\zeta_1(s,\alpha)$ as a function of complex variable α for fixed complex number s. Consequently, we shall get some identities involving Riemann zeta function. Using Theorem 1, we shall prove Theorem 2, which deals with

$$\sum_{\chi(\bmod q)} L(s,\chi)\,L(s',\bar{\chi})$$

for fixed complex numbers $s=\sigma+it$ and $s'=\sigma'+it'$ with $0<\sigma,\ \sigma'<1$. We state our theorems now.

Theorem 1. *For a fixed complex number $s\neq 1$, $\zeta_1(s,\alpha)$ is an analytic function of α in the region $|\alpha|<1$ of the complex plane, where it admits of the power series expansion*

$$\zeta_1(s,\alpha)=\sum_{n\geqslant 0}b_n(s)\alpha^n,\ \textit{where } b_0(s)=\zeta(s).\ \textit{For } n\geqslant 1,$$

$$b_n(s)=\frac{(-1)^n}{n!}s(s+1)(s+2)\ldots(s+n-1)\zeta(s+n).$$

Here $\zeta(s)$ is Riemann zeta function. The above expansion is valid also for $\alpha=1$, when $0<\operatorname{Re} s<1$.

Theorem 2. *We have, for $s=\sigma+it$ and $s'=\sigma'+it'$ with $0<\sigma,\sigma'<1$; $\sigma_0=\max(\sigma,\sigma')$; t,t' real; and for integer $q\geqslant 1$,*

$$\frac{1}{\phi(q)}\sum_{\chi(\bmod q)}L(s,\chi)\,L(s',\bar{\chi})=\sum_{\substack{a=1\\(a,q)=1}}^{q}a^{-s-s'}+q^{-s-s'}\sum_{k/q}\mu(q/k)Q(s,s',k),$$

where

$$Q(s,s',k)=\sum_{a=1}^{k}\left(\frac{k^s}{a^s}\zeta_1\left(s',\frac{a}{k}\right)+\frac{k^{s'}}{a^{s'}}\zeta_1\left(s,\frac{a}{k}\right)+\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)\right)$$

$$=A(s,s')k+\sum_{n=0}^{N-1}(a_n(s,s')k^{s-n}+b_n(s,s')k^{s'-n}+c_n(s,s')k^{-n})+O(k^{\sigma_0-N})$$

for any $N\geqslant 1$. Here ϕ denotes Euler's function, and $A(s,s')$, $a_n(s,s')$, $b_n(s,s')$ and $c_n(s,s')$ are functions of s and s' alone (being independent of k); and are explicitly computable. The O-constant, which is dependent on s and s' alone is also explicitly computable.

Corollaries of Theorem 2.

COROLLARY 1.

For $s'=\bar{s}$, and $s=\sigma+it$ with $0<\sigma<1$,

$$\frac{1}{\phi(q)}\sum_{\chi(\bmod q)}|L(s,\chi)|^2=\sum_{\substack{a=1\\(a,q)=1}}^{q}a^{-2\sigma}+q^{-2\sigma}\sum_{k/q}\mu(q/k)Q(s,\bar{s},k)$$

and hence for $s=\frac{1}{2}+it$,

$$\frac{q}{\phi(q)}\sum_{\chi(\bmod q)}|L(\tfrac{1}{2}+it,\chi)|^2=q\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{1}{a}$$

$$+\sum_{k/q}\mu(q/k)\left(A(s,\bar{s})k+\sum_{n=0}^{N-1}a_n(s,\bar{s})k^{s-n}+\sum_{n=0}^{N-1}b_n(s,\bar{s})k^{\bar{s}-n}\right.$$

$$\left.+\sum_{n=0}^{N-1}c_n(s,\bar{s})k^{-n}+O(k^{-N})\right).$$

COROLLARY 2.

We have for prime p,

$$\sum_{\chi(\bmod p)}L(s,\chi)\,L(s',\bar{\chi})+(p-1)\cdot p^{-s-s'}\cdot\zeta(s)\zeta(s')$$

$$= (p-1)\sum_{a=1}^{p} a^{-s-s'} + (p-1)p^{-s-s'}\left(A(s,s')p + \sum_{n=0}^{N-1} a_n(s,s')p^{s-n}\right.$$
$$\left. + \sum_{n=0}^{N-1} b_n(s,s')p^{s'-n} + \sum_{n=0}^{N-1} c_n(s,s')p^{-n} + O(p^{-N})\right)$$

Notation: In what follows, any function of s and s' will be termed 'scalar'. Thus, a scalar will be independent of q. Also $a_n = a_n(s,s')$, $b_n = b_n(s,s')$, $c_n = c_n(s,s'), \ldots$ shall denote sequence of scalars. Similarly $A = A(s,s')$, $B = B(s,s')$, $C = C(s,s'), \ldots$ shall denote scalars. The $\ll$ and O-constants shall depend on scalars only. p shall denote a prime number. $\phi(n)$, $\mu(n)$ will denote Euler's function and Moebius function respectively. $\zeta(s,1) = \zeta(s)$ shall denote Riemann zeta function. $\sum_{a=1}^{L}{}'$ shall denote summation $\sum_{a=1}^{L}$ with restriction $(a,q) = 1$, for any L. We shall write

$$\psi_0(a) = a - [a] - \tfrac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2\pi na}{\pi n}.$$

ψ_1 shall denote the indefinite integral of $\psi_0(a)$. Thus

$$\psi_1(a) = \int^{a} \psi_0(t)\mathrm{d}t = 2\sum_{n=1}^{\infty} \frac{\cos 2\pi na}{(2\pi n)^2}.$$

By induction $\psi_r(a) =$ indefinite integral of $\psi_{r-1}(a)$ i.e. $\int^{a} \psi_{r-1}(t)\mathrm{d}t$, for $r \geqslant 1$. Thus, for fixed r, $\psi_r(n) = C_r$ is a constant for integer n.

Paley [4], first showed that

$$\sum_{\substack{\chi(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} L(s,\chi)\, L(s',\bar{\chi}) = \phi(q)\, L(s+s',\chi_0) + A(s,s')\phi^2(q)q^{-s-s'}$$
$$+ O(q^{1-\sigma+G} + q^{1-\sigma'+G'}),$$

where χ_0 denotes principal character $(\mathrm{mod}\, q)$, for $s + s' \neq 1$; and a corresponding expression, for $s + s' = 1$. Here $\varepsilon, \varepsilon' > 0$ are arbitrary and $A(s,s')$ is some function of s, s'. Heath–Brown [1] obtained our corollary 1 of Theorem 2 for $s = s' = \frac{1}{2}$. Motohashi [3] obtained Paley's result mentioned above for

$$\sum_{\chi(\mathrm{mod}\, p)} |L(\tfrac{1}{2} + it, \chi)|^2$$

with vastly improved error term viz. $O(p^{-1/2})$ for prime p. While, very recently Katsurada and Matsumoto [2] have obtained our corollary 1 of Theorem 2 for

$$\sum_{\chi(\mathrm{mod}\, q)} |L(s,\chi)|^2$$

in a slightly different form.

The method of our proof consists in the use of Euler summation formula (along with power series expansion of $\zeta_1(s,\alpha)$), which makes the statement of Theorem 2 almost obvious. We have used the fact that the expansion $\zeta_1(s,\alpha)=\sum_{n\geqslant 0} b_n(s)\alpha^n$ naturally separates 's part' and 'α part' of $\zeta_1(s,\alpha)$.

Next, we prove our Theorem 1.

Corollaries of Theorem 1

First, we state corollaries of Theorem 1.

COROLLARY

Putting $\alpha = 1, -\frac{1}{2}$ *and* $\frac{1}{2}$ *respectively in the statement of Theorem* 1, *we get*

$$\text{I)}\quad 1 - s\zeta(s+1) + \frac{s(s+1)}{2!}\zeta(s+2) - \frac{s(s+1)(s+2)}{3!}\zeta(s+3) + \cdots = 0, \text{ for } 0 < \operatorname{Re} s < 1.$$

$$\text{II)}\quad (2^s - 2)\zeta(s) = \sum_{n\geqslant 1} \frac{s(s+1)(s+2)\ldots(s+n-1)\zeta(s+n)}{n!\cdot 2^n}$$

$$\text{III)}\quad (2^s - 2)\zeta(s) = 2^s + \sum_{n\geqslant 1} \frac{(-1)^n s(s+1)\ldots(s+n-1)\zeta(s+n)}{n!\cdot 2^n}$$

Remarks on Theorem 1: 1) As $\sum_{n\geqslant 0} b_n(s)$ is convergent for $0<\operatorname{Re} s<1$, we see that the series $\zeta_1(s,\alpha)=\sum_{n\geqslant 0} b_n(s)\alpha^n$ is uniformly convergent on the closed interval $[0,1]$ of α. Hence multiplying $\zeta_1(s,\alpha)$ by a bounded Riemann-integrable function, we can integrate the consequent series term-by-term on $[0,1]$. 2) For positive integer k, we have,

$$\frac{\partial^k}{\partial\alpha^k}\zeta_1(s,\alpha) = (-1)^k s(s+1)(s+2)\ldots(s+k-1)\zeta_1(s+k,\alpha)$$

for any complex number s. 3) Considering real and imaginary parts of $\zeta_1(s,\alpha)$ separately as function of real variable α on the interval $[0,1]$, we get for any positive integer k, the following:

$$\zeta_1(s,\alpha) = \zeta(s) - \alpha\cdot s\zeta(s+1) + \frac{\alpha^2}{2!}s(s+1)\zeta(s+2) + \cdots + \frac{(-1)^{k-1}}{(k-1)!}\alpha^{k-1}\cdot s(s+1)(s+2)\cdots(s+k-2)\zeta(s+k-1) + O\left(\frac{s(s+1)(s+2)\cdots(s+k-1)}{k!}\zeta(\sigma+k)\cdot\alpha^k\right) \quad \text{for } \operatorname{Re} s = \sigma > 0,$$

where the O-constant is absolute. 4) From

$$\zeta_1(s,\alpha) = \sum_{n\geqslant 0} b_n(s)\alpha^n,$$

by multiplying it with power series for

$$\zeta_1(s',\alpha) = \sum_{n\geqslant 0} b_n(s')\alpha^n,$$

we get

$$\zeta_1(s,\alpha)\zeta_1(s',\alpha)=\sum_{n\geqslant 0} d_n(s,s')\alpha^n,$$

where

$$d_n=d_n(s,s')=\sum_{i=0}^{n} b_i(s)b_{n-i}(s')$$

and the new series $\sum_{n\geqslant 0} d_n(s,s')\alpha^n$ also is uniformly convergent on the closed interval $[0,1]$ of α.

Proof of Theorem 1: Let $s=\sigma+it$ with $\sigma>1$ and α be real with $|\alpha|<1$. Then

$$\begin{aligned}\zeta_1(s,\alpha)&=\sum_{n\geqslant 1}(n+\alpha)^{-s}=\sum_{n\geqslant 1} n^{-s}\left(1+\frac{\alpha}{n}\right)^{-s}\\ &=\sum_{n\geqslant 1} n^{-s}\left(1-s\cdot\frac{\alpha}{n}+\frac{s(s+1)\alpha^2}{2!n^2}-\frac{s(s+1)(s+2)}{3!}\frac{\alpha^3}{n^3}+\cdots\right)\\ &=\sum_{n\geqslant 1} n^{-s}-\alpha s\sum_{n\geqslant 1} n^{-s-1}+\frac{s(s+1)\alpha^2}{2}\sum_{n\geqslant 1} n^{-s-2}-\cdots\\ &=\zeta(s)-\alpha s\zeta(s+1)+\frac{s(s+1)}{2}\alpha^2\zeta(s+2)-\cdots\\ &=\sum_{n\geqslant 0} b_n(s)\alpha^n,\text{ say.}\end{aligned}$$

Thus for $s=\sigma+it$, with $\sigma>1$; and real α, with $0<\alpha<1$, we have

$$\zeta_1(s,\alpha)=\sum_{n\geqslant 0} b_n(s)\alpha^n.$$

Next, we show the series on the right hand converges uniformly and absolutely on every compact subset of the complex plane with $|\alpha|<1$, for any complex number $s\neq 1$. For convenience, we take $s(\neq 1)$ to be a complex number such that $\mathrm{Re}(s)>0$. Now, the coefficients $b_n(s)$ are well-defined functions of s and for $n\geqslant 1$, if $\mathrm{Re}(s)>0$, then

$$\begin{aligned}|b_n(s)|&\leqslant\frac{|s|(|s|+1)(|s|+2)\cdots(|s|+n-1)\zeta(\sigma+n)}{n!}\\ &\leqslant\frac{|s|(|s|+1)(|s|+2)\cdots(|s|+n-1)\zeta(\sigma+1)}{n!}.\end{aligned}$$

Let $|\alpha|\leqslant\beta<1$. Then, $|\sum_{n\geqq 1} b_n(s)\alpha^n|$

$$\leqslant\zeta(\sigma+1)\sum_{n\geqq 1}\frac{|s|(|s|+1)\cdots(|s|+n-1)\beta^n}{n!}\leqslant\zeta(\sigma+1)\{(1-\beta)^{-|s|}-1\}.$$

Thus by analytic continuation, for any fixed s with $\mathrm{Re}(s)>0$, $s\neq 1$, and for complex

α with $|\alpha| < 1$, $\zeta_1(s, \alpha)$ is an analytic function of α with the above power series expansion. This result can be extended for all complex s.

Next, we prove the validity of the above series for $\alpha = 1$. Let s be a complex number with $\mathrm{Re}(s) > 0$. Then we have to prove

$$\zeta_1(s, 1) = \sum_{n \geqslant 0} b_n(s).$$

i.e. to prove

$$\zeta(s) - 1 = \zeta(s) + \sum_{n \geqslant 1} b_n(s).$$

This amounts to proving that

$$1 + \sum_{n=1}^{\infty} b_n(s) = 0.$$

We define

$$\zeta_0(s) = \zeta(s) - 1.$$

Consider

$$\begin{aligned} 1 + \sum_{n \geqslant 1} b_n(s) &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1)(1 + \zeta_0(s+i)) \\ &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1) \\ &\quad + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1)\zeta_0(s+i) \\ &= 1 + A + B, \text{ say, provided both the sums exist.} \end{aligned}$$

Now

$$\begin{aligned} B &= \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1)\zeta_0(s+i) \\ &= -s \sum_{n \geqslant 2} \frac{1}{n^{s+1}} + \frac{s(s+1)}{2!} \sum_{n \geqslant 2} \frac{1}{n^{s+2}} - \frac{s(s+1)(s+2)}{3!} \sum_{n \geqslant 2} \frac{1}{n^{s+3}} + \cdots \\ &= \sum_{n \geqslant 2} n^{-s} \left\{ -\frac{s}{n} + \frac{s(s+1)}{2!} \frac{1}{n^2} - \frac{s(s+1)(s+2)}{3!} \frac{1}{n^3} + \cdots \right\} \\ &= \sum_{n \geqslant 2} n^{-s} \left\{ \left(1 + \frac{1}{n}\right)^{-s} - 1 \right\} = \sum_{n \geqslant 2} ((n+1)^{-s} - n^{-s}) = -2^{-s}. \end{aligned}$$

The inversion of the order of summation is justified for $\sigma = \mathrm{Re}(s) > 0$ by the convergence of

$$\sum_{n \geqslant 2} n^{-\sigma} \sum_{k=0}^{\infty} \frac{|s|(|s|+1)\cdots(|s|+k)}{(k+1)!n^{k+1}} = \sum_{n \geqslant 2} n^{-\sigma} \left(\left(1 - \frac{1}{n}\right)^{-|s|} - 1 \right)$$

Next,

$$1+A=1-s+\frac{s(s+1)}{2!}-\frac{s(s+1)(s+2)}{3!}+\cdots$$
$$=(1+1)^{-s}=2^{-s},\quad \text{for } 0<\operatorname{Re}s<1.$$

Thus we have proved

$$1+\sum_{n\geqslant 1} b_n(s)=2^{-s}-2^{-s}=0 \text{ for } 0<\operatorname{Re}(s)<1.$$

This completes the proof of our Theorem 1.

Proof of Theorem 2. For $s=\sigma+it$ and $s'=\sigma'+it'$ with $0<\sigma,\ \sigma'<1$, we have

$$\sum_{\chi(\bmod q)} L(s,\chi)\,L(s',\bar{\chi})$$
$$=\sum_{\chi(\bmod q)}\Bigg(q^{-s}\sum_{\substack{a=1\\(a,q)=1}}^{q}\chi(a)\zeta\Big(s,\frac{a}{q}\Big)\Bigg)\Bigg(q^{-s'}\sum_{\substack{b=1\\(b,q)=1}}^{q}\bar{\chi}(b)\zeta\Big(s',\frac{b}{q}\Big)\Bigg)$$
$$=q^{-s-s'}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,q)=1}}^{q}\zeta\Big(s,\frac{a}{q}\Big)\zeta\Big(s',\frac{b}{q}\Big)\sum_{\chi(\bmod q)}\chi(a)\overline{\chi(b)}$$
$$=\phi(q)q^{-s-s'}\sideset{}{'}\sum_{a=1}^{q}\zeta\Big(s,\frac{a}{q}\Big)\zeta\Big(s',\frac{a}{q}\Big).$$

Next,

$$\sideset{}{'}\sum_{a=1}^{q}\zeta\Big(s,\frac{a}{q}\Big)\zeta\Big(s',\frac{a}{q}\Big)$$
$$=\sideset{}{'}\sum_{a=1}^{q}\Big(\frac{q^s}{a^s}+\zeta_1\Big(s,\frac{a}{q}\Big)\Big)\Big(\frac{q^{s'}}{a^{s'}}+\zeta_1\Big(s',\frac{a}{q}\Big)\Big)$$
$$=q^{s+s'}\sideset{}{'}\sum_{a=1}^{q}\frac{1}{a^{s+s'}}+q^{s}\sideset{}{'}\sum_{a=1}^{q}\frac{\zeta_1\Big(s',\frac{a}{q}\Big)}{a^{s}}+q^{s'}\sideset{}{'}\sum_{a=1}^{q}\frac{\zeta_1\Big(s,\frac{a}{q}\Big)}{a^{s'}}$$
$$+\sideset{}{'}\sum_{a=1}^{q}\zeta_1\Big(s,\frac{a}{q}\Big)\zeta_1\Big(s',\frac{a}{q}\Big).$$

Thus

$$\sum_{\chi(\bmod q)} L(s,\chi)\,L(s',\bar{\chi})=\phi(q)\sideset{}{'}\sum_{a=1}^{q}a^{-s-s'}+\phi(q)q^{-s-s'}\cdot\sum_{k/q}\mu(q/k)Q(s,s',k),$$

where

$$Q(s,s',k)=\sum_{a=1}^{k}\zeta_1\Big(s,\frac{a}{k}\Big)\zeta_1\Big(s',\frac{a}{k}\Big)+\sum_{a=1}^{k}k^{s}\cdot\frac{\zeta_1\Big(s',\frac{a}{k}\Big)}{a^{s}}$$
$$+\sum_{a=1}^{k}\frac{k^{s'}\zeta_1\Big(s,\frac{a}{k}\Big)}{a^{s'}}$$

$$= \sum_{a=1}^{k} f_1\left(\frac{a}{k}\right) + \sum_{a=1}^{k} f_2\left(\frac{a}{k}\right) + \sum_{a=1}^{k} f_3\left(\frac{a}{k}\right), \quad \text{say.}$$

By Euler summation formula, for $1 \leqslant i \leqslant 3$,

$$\sum_{a=1}^{k} f_i\left(\frac{a}{k}\right) = \int_1^k f_i\left(\frac{a}{k}\right) da + \int_1^k (a - [a] - \tfrac{1}{2})\left(\frac{d}{da} f_i\left(\frac{a}{k}\right)\right) da$$
$$+ \left(\frac{f_i\left(\frac{1}{k}\right) + f_i(1)}{2}\right).$$

In view of finite power series expression for $\zeta_1\left(s, \frac{1}{k}\right)$ for any complex s, we find that $\dfrac{f_1\left(\frac{1}{k}\right) + f_1(1)}{2}$ has the required form. Thus

$$\frac{f_1\left(\frac{1}{k}\right) + f_1(1)}{2} = \sum_{n=0}^{N-1} a_n k^{-n} + O(k^{-N}) \quad (\text{for } N \geqslant 1),$$

where a_n's are some scalars. Similarly $\dfrac{f_2\left(\frac{1}{k}\right) + f_2(1)}{2}$ has the expression of the form

$$\sum_{n=0}^{N-1} a_n k^{s-n} + \sum_{n=0}^{N-1} b_n k^{-n} + O(k^{\sigma - N})$$

for some scalars a_n and b_n. Similarly, $\dfrac{f_3\left(\frac{1}{k}\right) + f_3(1)}{2}$ has the expression of the form

$$\sum_{n=0}^{N-1} a_n k^{s'-n} + \sum_{n=0}^{N-1} b_n k^{-n} + O(k^{\sigma' - N}).$$

Next, we evaluate

$$\int_1^k f_i\left(\frac{a}{k}\right) da \quad \text{for } 1 \leqslant i \leqslant 3.$$

We have

$$\int_1^k \zeta_1\left(s, \frac{a}{k}\right) \zeta_1\left(s', \frac{a}{k}\right) da = \int_1^k f_1\left(\frac{a}{k}\right) da.$$

Noting

$$\zeta_1\left(s, \frac{a}{k}\right) = \sum_{n \geqslant 0} b_n(s) \left(\frac{a}{k}\right)^n; \quad \zeta_1\left(s', \frac{a}{k}\right) = \sum_{n \geqslant 0} b_n(s') \left(\frac{a}{k}\right)^n,$$

we have

$$\zeta_1\left(s, \frac{a}{k}\right)\zeta_1\left(s', \frac{a}{k}\right) = \sum_{n\geqslant 0} d_n \left(\frac{a}{k}\right)^n,$$

for sequence of scalars (d_n).

Thus

$$\begin{aligned}\int_1^k \zeta_1\left(s, \frac{a}{k}\right)\zeta_1\left(s', \frac{a}{k}\right) da &= \sum_{n\geqslant 0} d_n \int_1^k \left(\frac{a}{k}\right)^n da \\ &= \sum_{n\geqslant 0} \frac{d_n k^{-n}}{n+1}[a^{n+1}]_{a=1}^k = k\sum_{n\geqslant 0}\frac{d_n}{n+1} - \sum_{n\geqslant 0}\frac{d_n k^{-n}}{n+1} \\ &= \left(\sum_{n\geqslant 0}\frac{d_n}{n+1}\right)k - \sum_{n\geqslant 0}^{N-1}\frac{d_n}{n+1}k^{-n} + O(k^{-N}).\end{aligned}$$

Next,

$$\begin{aligned}\int_1^k f_2\left(\frac{a}{k}\right) da &= \int_1^k \frac{k^s}{a^s}\zeta_1\left(s', \frac{a}{k}\right) da \\ &= \sum_{n\geqslant 0} k^s \int_1^k a^{-s}\left(\sum_{n\geqslant 0} b_n(s')\left(\frac{a}{k}\right)^n\right) da \\ &= \sum_{n\geqslant 0} b_n(s')k^{s-n}\cdot \int_1^k a^{n-s} da = \sum_{n\geqslant 0} b_n(s')k^{s-n}\left[\frac{a^{n-s+1}}{n-s+1}\right]_{a=1}^k \\ &= k\sum_{n\geqslant 0}\frac{b_n(s')}{n-s-1} - \sum_{n\geqslant 0}\frac{b_n(s')}{n-s+1}k^{s-n} \\ &= k\left(\sum_{n\geqslant 0}\frac{b_n(s')}{n-s-1}\right) - \sum_{n\geqslant 0}^{N-1}\frac{b_n(s')}{n-s+1}k^{s-n} + O(k^{\sigma-N}).\end{aligned}$$

Similarly

$$\begin{aligned}&\int_1^k \frac{k^{s'}}{a^{s'}}\zeta_1\left(s, \frac{a}{k}\right) da \\ &= k\left(\sum_{n\geqslant 0}\frac{b_n(s)}{n-s'-1}\right) - \sum_{n\geqslant 0}^{N-1}\frac{b_n(s)}{n-s'-1}k^{s'-n} + O(k^{\sigma'-N}).\end{aligned}$$

Thus

$$\int_1^k \left(\sum_{i=1}^3 f_i\left(\frac{a}{k}\right)\right) da$$

has the form

$$Ak + \sum_{n=0}^{N-1} b_n k^{s-n} + \sum_{n=0}^{N-1} c_n k^{s'-n} + O(k^{\sigma-N} + k^{\sigma'-N})$$

for some scalars A, b_n and c_n.

Next, we evaluate

$$\int_1^k \psi_0(a)\frac{d}{da}\left(f_i\left(\frac{a}{k}\right)\right) da, \quad \text{for } 1 \leqslant i \leqslant 3.$$

Now

$$\int_1^k \psi_0(a)\frac{d}{da}f_1\left(\frac{a}{k}\right)da = \int_1^k \psi_0(a)\left(\frac{d}{da}\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)\right)da.$$

On integration by parts N times, we get

$$\int_1^k \psi_0(a)\frac{d}{da}f_1\left(\frac{a}{k}\right)da = \sum_{i=1}^{N}\left[(-1)^{i-1}\psi_i(a)\frac{d^i}{da^i}\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)\right]_{a=1}^{a=k}$$
$$+\int_1^k (-1)^N\psi_N(a)\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)\right)da,$$

where $[f(a)]_{a=1}^{a=k}$ denotes $f(k)-f(1)$ for any function f. Noting, $\frac{d^i}{da^i}\left(\zeta_1\left(s,\frac{a}{k}\right)\right)$ is a scalar multiple of $k^{-i}\zeta_1\left(s+i,\frac{a}{k}\right)$ for any complex s; and also noting $\psi_i(n)=\psi_i=\text{constant}$ for any integer n; and expanding $\zeta_1\left(s,\frac{1}{k}\right)$ as a finite power series in $\frac{1}{k}$ for any s, we find that the integrated terms have the form

$$\sum_{n=1}^{N-1} b_n k^{-n} + O(k^{-N})$$

and the integral

$$\int_1^k da\,\psi_N(a)\frac{d^{N+1}}{da^{N+1}}\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)$$
$$\ll k^{-N-1}\int_1^k |\psi_N(a)|da \ll k^{-N-1}\int_1^k da \ll k^{-N}.$$

Next, we consider

$$\int_1^k \psi_0(a)\frac{d}{da}f_2\left(\frac{a}{k}\right)da.$$

We can treat

$$\int_1^k \psi_0(a)\frac{d}{da}f_3\left(\frac{a}{k}\right)da$$

similarly.

Now

$$\int_1^k \psi_0(a)\frac{d}{da}f_2\left(\frac{a}{k}\right)da = k^{s'}\cdot\int_1^k da\,\psi_0(a)\frac{d}{da}\left(\zeta_1\left(s,\frac{a}{k}\right)a^{-s'}\right).$$

On integration by parts N times, we get

$$\int_1^k \psi_0(a)\frac{d}{da}\left(f_2\left(\frac{a}{k}\right)\right)da = k^{s'}\cdot\sum_{i=1}^{N}(-1)^{i-1}\left[\psi_i(a)\frac{d^i}{da^i}\left(\zeta_1\left(s,\frac{a}{k}\right)a^{-s'}\right)\right]_{a=1}^{k}$$
$$-k^{s'}\cdot\int_1^k(-1)^{N-1}\psi_N(a)\cdot\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)\cdot a^{-s'}\right)da.$$

Considering finite power series for $\zeta_1\left(s,\frac{1}{k}\right)$ for any s, as before, we find the integrated terms have the form

$$\sum_{n=0}^{N-1} a_n k^{s'-n} + \sum_{n=1}^{N-1} b_n k^{-n} + O(k^{-N})$$

for some scalars (a_n) and (b_n).

Now

$$k^{s'}\int_1^k \psi_N(a)\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)\cdot a^{-s'}\right)da = k^{s'}\int_1^k \psi_N(a)\left(\frac{d^{N+1}}{da^{N+1}}\zeta_1\left(s,\frac{a}{k}\right)\right)a^{-s'}\,da$$
$$+\sum_{j=0}^{N} a(j)k^{s'-j}\int_1^k \psi_N(a)\zeta_1\left(s+j,\frac{a}{k}\right)\cdot a^{-s'-1-r}\,da, \text{ where } r+j=N.$$

Here $a(j)$ are some scalars. Note that

$$k^{s'}\int_1^k \psi_N(a)\cdot\left(\frac{d^{N+1}}{da^{N+1}}\zeta_1\left(s,\frac{a}{k}\right)\right)a^{-s'}\,da$$
$$\ll k^{\sigma'}\int_1^k |\psi_N(a)|\cdot k^{-(N+1)}\left|\zeta_1\left(s+N+1,\frac{a}{k}\right)\right|a^{-\sigma'}\,da$$
$$\ll k^{\sigma'-(N+1)}\int_1^k a^{-\sigma'}\,da \ll k^{-N}.$$

On substitution

$$\zeta_1\left(s+j,\frac{a}{k}\right) = \sum_{i=0}^{r-1} b_i(s+j)\left(\frac{a}{k}\right)^i + O\left(\left(\frac{a}{k}\right)^r\right),$$

we find that

$$k^{s'-j}\int_1^k \psi_N(a)\zeta_1\left(s+j,\frac{a}{k}\right)a^{-s'-1-r}\,da = k^{s'-j}\int_1^k \frac{da\cdot\psi_N(a)}{a^{s'+1+r}}$$
$$\times\left(b_0(s+j)+b_1(s+j)\frac{a}{k}+b_2(s+j)\frac{a^2}{k^2}\right.$$
$$\left.+\cdots+b_{r-1}(s+j)\left(\frac{a}{k}\right)^{r-1}+O\left(\frac{a}{k}\right)^r\right).$$

The

$$O-\text{term}=k^{s'-j-r}\int_1^k \frac{\psi_N(a)}{a^{s'+1}}da \ll k^{\sigma'-N},$$

as $r+j=N$. For $1\leqslant i\leqslant r-1$, the ith term $=k^{s'-i-j}\cdot\displaystyle\int_1^k \frac{\psi_N(a)b_i(s+j)}{a^{s'+1+r-i}}da.$

Hence,

$$\begin{aligned}
&k^{s'-i-j}\int_1^k \frac{\psi_N(a)}{a^{s'+1+r-i}}da\\
&=k^{s'-i-j}\left(\int_1^\infty \frac{\psi_N(a)da}{a^{s'+1+(r-i)}}-\int_k^\infty \frac{\psi_N(a)da}{a^{s'+1+(r-i)}}\right)\\
&=c_{ij}k^{s'-i-j}-k^{s'-i-j}\int_k^\infty \frac{\psi_N(a)da}{a^{s'+1+(r-i)}},\quad \text{say.}\\
&=c_{ij}k^{s'-i-j}+O(k^{-r-j})=c_{ij}k^{s'-i-j}+O(k^{-N}),\quad \text{as } r+j=N
\end{aligned}$$

Thus,

$$\begin{aligned}
&k^{s'-j}\int_1^k \zeta_1\left(s+j,\frac{a}{k}\right)\cdot a^{-s'-1-r}\psi_N(a)da\\
&=\sum_{i=0}^{r-1}\alpha_{ij}k^{s'-i-j}+O(k^{\sigma'-N}),
\end{aligned}$$

where α_{ij} are some scalars, $0\leqslant j\leqslant N+1$ and $r+j=N$. Taking together the terms corresponding to all j, we find that

$$k^{s'}\cdot\int_1^k \psi_N(a)\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)a^{-s'}\right)da$$

has the form

$$\sum_{n=0}^{N-1}a_nk^{s'-n}+O(k^{\sigma'-N})$$

and hence consequently,

$$k^{s'}\cdot\int_1^k \psi_0(a)\frac{d}{da}\left(a^{-s'}\cdot\zeta_1\left(s,\frac{a}{k}\right)\right)da$$

has the form $\sum_{n=0}^{N-1}a_nk^{s'-n}+\sum_{n=0}^{N-1}b_nk^{-n}+O(k^{\sigma'-N})$ for some scalars a_n,b_n. Thus, $\sum_{a=1}^{k}f_2\left(\frac{a}{k}\right)$ has the form

$$Ak+\sum_{n=0}^{N-1}a_nk^{s'-n}+\sum_{n=0}^{N-1}b_nk^{-n}+O(k^{\sigma'-N}).$$

Similarly, we find that $\sum_{a=1}^{k} f_3\left(\frac{a}{k}\right)$ has the form

$$Ak + \sum_{n=0}^{N-1} a_n k^{s-n} + \sum_{n=0}^{N-1} b_n k^{-n} + O(k^{\sigma-n}).$$

This completes the proof of Theorem 2.

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