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Transformation formula for exponential sums involving Fourier coefficients of modular forms

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Abstract. In 1984 Jutila [5] obtained a transformation formula for certain exponential sums involving the Fourier coefficients of a holomorphic cusp form for the full modular group $SL(2, \mathbb{Z})$. With the help of the transformation formula he obtained good estimates for the distance between consecutive zeros on the critical line of the Dirichlet series associated with the cusp form and for the order of the Dirichlet series on the critical line, [7]. In this paper we follow Jutila to obtain a transformation formula for exponential sums involving the Fourier coefficients of either holomorphic cusp forms or certain Maass forms for congruence subgroups of $SL(2, \mathbb{Z})$ and prove similar estimates for the corresponding Dirichlet series.

Keywords. Exponential sums; summation formula; cusp forms and Maass forms; transformation formula.

1. Introduction

By an exponential sum we mean a sum of the form

$$\sum_{M \leq n \leq M'} a(n)e(f(n)),$$

where $a(n)$ is an arithmetical function and f is a real valued function on $[M, M']$. Many a problem in number theory reduces in the ultimate analysis to estimating an exponential sum. The Waring's problem, the Goldbach's conjecture, the Dirichlet divisor problem and the order of the Dirichlet series in the critical strip are very good examples of this phenomenon.

The most commonly employed method to estimate such sums is due to Van der Corput and Vornoi. The basic idea here is to transform an exponential sum into a new shape by first converting the sum into an integral—Van der Corput's lemma and Vornoi summation formula—and then evaluating the integral by the 'Saddle-point method'. Since then various summation formulae of the Voronoi type have been found; a very good survey is to be found in [2]. In 1984, Jutila [5] discovered that by replacing $f(n)$ by $f(n) + rn$ where r is an integer (which does not affect the sum) before transforming the sum one was led to much better transformed sums. Another important observation made by Jutila was the flexibility of this method which, he showed, works in the case when $a(n)$'s are Fourier coefficients of a cusp form of weight k for the full modular group, $SL(2, \mathbb{Z})$. With the help of this transformation formula he was able to obtain for the Dirichlet series associated to cusp forms for $SL(2,\mathbb{Z})$ analogues of many results known in the case of the Riemann zeta function, $\zeta(s)$, like the distance between consecutive zeros on the critical line, the order on the critical line, mean square estimates and higher power moments [5, 6, 7]. In this paper we show that the transformation formula and the above mentioned applications carry over to the case when $a(n)$'s are Fourier coefficients of either holomorphic cusp forms or certain Maass forms (Maass forms f with $\Delta f = 1/4f$) of higher levels. While some of the above mentioned applications were already known in the case of cusp forms for $SL(2,\mathbb{Z})$ due to Good [4] they seem to be new in the case of cusp forms of higher levels.

Mention must also be made of the work of Meurman [8, 9] who has extended some of the results of Jutila to the case of L-functions associated to Maass wave forms for $SL(2,\mathbb{Z})$. Presumably his work also extends to higher level Maass forms. The class of Maass forms we consider in this thesis does not occur at level one.

2. Functional equations and summation formulae

In this section we are concerned with functional equations for Dirichlet series (and their twists by additive characters) associated to certain modular forms for congruence subgroups of the full modular group $SL(2,\mathbb{Z})$. The class of modular forms we consider consists of all holomorphic forms and the subclass of Maass forms with $1\backslash 4$ as the eigenvalue of the Laplacian on the upper half plane. We consider the holomorphic case (Theorem 2.1) and the case of Maass forms (Theorem 2.2) separately. The summation formulae these functional equations lead to are written down at the end of the section.

If f is a function on the upper half-plane $\mathbb{H}$, k an integer and A is in $GL^+(2,\mathbb{R})$ then $f|^k_{[A]}(\tau)$ will denote the following function

$$(\det A)^{k/2}(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Holomorphic case

For k, $N \geqslant 1$ integers and ε a character mod N let $M(N,k,\varepsilon)$ denote the space of modular forms of level N, weight k and character ε. Thus if $f \in M(N,k,\varepsilon)$ and $A \in \Gamma_0(N)$ we have

$$f|^k_{[A]}(\tau) = \varepsilon(d) f(\tau), \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that $M(N,k,\varepsilon) = \{0\}$ unless $\varepsilon(-1) = (-1)^k$, and that $M(N,k,1)$ is $M(N,k)$, the space of modular forms of weight k for $\Gamma_0(N)$. If $H(N)$ denotes the matrix $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ then $f \to f|^k_{[H(N)]}$ defines an isomorphism from $M(N,k,\varepsilon)$ onto $M(N,k,\bar{\varepsilon})$ where $\bar{\varepsilon}$ is the (complex) conjugate character.

Let $f \in M(N,k,\varepsilon)$ and $f(\tau) = \sum_{n=0}^{\infty} a(n) \exp(2\pi i n\tau)$ be its Fourier expansion at the cusp $i\infty$. We are interested in functional equations for the Dirichlet series

$$\phi_f(s, h/m) = \sum_{n=1}^{\infty} \frac{a(n)\exp(2\pi i n h/m)}{n^s}.$$

Theorem 2.1. (a) *Case when* $(m,N)=N$. *Let*

$$\Phi(s,h/m)=(m/2\pi)^s\Gamma(s)\phi_f(s,h/m).$$

Then

$$\Phi(s,h/m)+\frac{a(0)}{s}+\frac{i^k\varepsilon(\bar{h})a(0)}{k-s},$$

is EBV (entire and bounded in every vertical strip) and we have the functional equation:

$\Phi(s,h/m)=i^k\varepsilon(\bar{h})\Phi(k-s,-\bar{h}/m)$, *where* $\bar{h}$ *is defined by* $h\bar{h}\equiv 1\ (\mathrm{mod}\ m)$.

(b) *Case when* $(m,N)=1$. *Let* $f|^k_{[H(N)]}(\tau)=g(\tau)\in M(N,k,\bar{\varepsilon})$ *and* $g(\tau)=\sum_{n=0}^{\infty}b(n)\exp(2\pi in\tau)$ *be it's Fourier expansion. Further let*

$$\Phi(s,h/m)=(m\sqrt{N}/2\pi)^s\Gamma(s)\phi_f(s,h/m) \text{ and}$$

$$\Psi(s,h/m)=(m\sqrt{N}/2\pi)^s\Gamma(s)\phi_g(s,h/m).$$

Then

$$\Phi(s,h/m)+(m^2N)^{-s/2}\left(\frac{a(0)}{s}+\frac{\varepsilon(m)b(0)}{k-s}\right)$$

is EBV and we have the functional equation:

$$\Phi(s,h/m)=\varepsilon(m)\Psi(k-s,-\overline{Nh}/m).$$ □

Proof. (a) Let $t>0$ be a real number and put $\tau=h/m+i/mt$ and $\tau'=-\bar{h}/m+it/m$. Then τ and τ' are in the upper half-plane and are equivalent under $\Gamma_0(N)$ by the matrix (remember $m\equiv 0\ (\mathrm{mod}\ N)$)

$$A=\begin{bmatrix} h & (\bar{h}h-1)/m \\ m & \bar{h}\end{bmatrix}$$

i.e. $A(\tau')=\tau$. Therefore we have $f(\tau)=\varepsilon(\bar{h})(it)^kf(\tau')$. If $\mathrm{Re}(s)$ is sufficiently large we have

$$\begin{aligned}\Phi(s,h/m)&=\sum_{n=1}^{\infty}a(n)\exp(2\pi inh/m)\int_0^{\infty}t^{s-1}\exp(-2\pi nt/m)\mathrm{d}t.\\ &=\int_0^{\infty}t^{s-1}\{f(h/m+it/m)-a(0)\}\,\mathrm{d}t.\\ &=\int_1^{\infty}t^{s-1}\{f(h/m+it/m)-a(0)\}\,\mathrm{d}t-\int_0^1 a(0)t^{s-1}\\ &\quad+\int_0^1 t^{s-1}f(h/m+it/m)\mathrm{d}t.\end{aligned}$$

Now consider

$$\begin{aligned}\int_0^1 t^{s-1}f(h/m+it/m)\mathrm{d}t&=\int_1^{\infty}t^{-s-1}f(h/m+i/mt)\mathrm{d}t.\\ &=\int_1^{\infty}t^{-s-1}f(\tau)\mathrm{d}t.\\ &=\varepsilon(\bar{h})i^k\int_1^{\infty}t^{k-s-1}f(\tau')\mathrm{d}t.\end{aligned}$$

Thus we have

$$\Phi(s,h/m)=\int\{[f(h/m+it/m)-a(0)]t^{s-1}$$
$$+\varepsilon(\bar{h})i^k[f(-\bar{h}/m+it/m)-a(0)]t^{k-s-1}\}\mathrm{d}t$$
$$-\frac{a(0)}{s}-\frac{\varepsilon(\bar{h})i^k a(0)}{k-s}.$$

This integral representation proves the claims made in (a).

(b). For $x\in\mathbb{R}$ let $\alpha(x)$ denote the matrix $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$. Let $t>0$ be a real number and set $\tau=h/m+i/m^2Nt$ and $\tau'=-\bar{h}/m+it$. We need to know $f(\tau)$ in terms of $g(\tau')$. For that first observe that

$$\alpha(h/m)H(m^2N)=H(N)\begin{pmatrix} m & -b\\ -Nh & n\end{pmatrix}\alpha(b/m)\begin{pmatrix} m & 0\\ 0 & m\end{pmatrix}$$

where b is defined via $Nhb\equiv -1\pmod m$ (which is possible because $(m,N)=(m,h)=1$.) and n is chosen such that $\begin{pmatrix} m & -b\\ -Nh & n\end{pmatrix}$ is in $\Gamma_0(N)$. Therefore we have

$$\begin{aligned}f(\tau)&=(m^2N)^{k/2}(it)^k f|_{[\alpha(h/m)][H(Nm^2)]}(\mathrm{it}).\\ &=(m^2N)^{k/2}(it)^k f|_{[H(N)][(\begin{smallmatrix} m & -b\\ -Nh & n\end{smallmatrix})][\alpha(b/m)]}(it).\\ &=(m^2N)^{k/2}(it)^k\bar{\varepsilon}(n)g|_{[\alpha(b/m)]}(it).\\ &=(m^2N)^{k/2}(it)^k\varepsilon(m)g(-\bar{N}\bar{h}/m+it).\end{aligned}$$

Note that we have made use of

(i) $\bar{\varepsilon}(n)=\varepsilon(m)$ as $mn\equiv 1\pmod N$,

and

(ii) $\exp(2\pi ib/m)=\exp(-2\pi i\bar{N}\bar{h}/m)$

since

$Nhb\equiv -1\pmod m$.

Consider now the following integral representation:

$$\Phi(s,h/m)=(m^2N)^{s/2}\int_0^\infty[f(h/m+it)-a(0)]t^{s-1}\mathrm{d}t.$$
$$=(m^2N)^{s/2}\left[\int_{1/m\sqrt{N}}^\infty[f(h/m+it)-a(0)]t^{s-1}\mathrm{d}t-\int_0^{1/m\sqrt{N}}a(0)t^{s-1}\mathrm{d}t\right.$$
$$\left.+\int_0^{1/m\sqrt{N}}f(h/m+it)t^{s-1}\mathrm{d}t\right].$$

Now

$$\int_0^{1/m\sqrt{N}}f(h/m+it)t^{s-1}\mathrm{d}t=(m^2N)^{-s}\int_{1/m\sqrt{N}}^\infty f(h/m+i/m^2Nt)t^{-s-1}\mathrm{d}t,$$
$$=(m^2N)^{(k/2)-s}\varepsilon(m)\int_{1/m\sqrt{N}}^\infty g(-\bar{N}\bar{h}/m+it)t^{k-s-1}\mathrm{d}t.$$

Thus we get

$$\Phi(s, h/m) = (m^2 N)^{s/2} \int_{1/m\sqrt{N}}^{\infty} \{[f(h/m + it) - a(0)]t^{s-1} + \varepsilon(m)(m^2 N)^{(k/2)-s}[g(-\bar{N}\bar{h}/m + it) - b(0)]t^{k-s-1}\} dt - (m^2 N)^{-s/2}\left(\frac{a(0)}{s} + \frac{\varepsilon(m)b(0)}{k-s}\right).$$

This integral representation verifies the claims made in (b). Thus the proof of the Theorem is complete.

Remark. As these functional equations characterise modular forms of level N (see [12] and [13]) we cannot in general hope to get similar functional equations when $1 < (m, N) < N$. For instance if f is a new form of level N then existence of functional equations for $1 < (m, N) < N$ would mean that f is a form of lower level which it is not.

Non-holomorphic case (see [11])

Let $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ denote the Laplacian on the upper half-plane $\mathbb{H}$ associated with the hyperbolic metric. Let $N \geqslant 1$ be an integer, ε a character mod N and λ a complex number. Let f be an even Maass form of level N, character ε and eigenvalue (for Δ) λ. This means:

(i) $f \in L^2(\Gamma_1(N)\backslash\mathbb{H})$;
(ii) $f(\gamma\tau) = \varepsilon(d)f(\tau)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$;
(iii) $\Delta f = \lambda f$, $\lambda = 1/4 + r^2$;
(iv) f is a simultaneous eigenfunction of the Hecke operators T_n,

$$(n, N) = 1 \text{ and } T_{-1}f(\tau) = f(-\bar{\tau}) = f(\tau).$$

Such an f has a Fourier expansion of the following type:

$$f(\tau) = \sum a(n) y^{1/2} K_{ir}(2\pi n y)\cos 2\pi n x, \quad \tau = x + iy.$$

Let f be an even Maass form with $\lambda = 1/4$ i.e. $r = 0$ with Fourier coefficients $a(n)$. Here again we are interested in functional equations for the following Dirichlet series associated with f:

$$\phi_f(s, h/m) = \sum_{n=1}^{\infty} \frac{a(n)\cos 2\pi n h/m}{n^s}$$

$$\phi'_f(s, h/m) = \sum_{n=1}^{\infty} \frac{a(n)\sin 2\pi n h/m}{n^s}$$

and in this direction we have:

Theorem 2.2. (a) $(m, N) = N$. *The functional equations are*

$$\Phi_f(s, h/m) = \varepsilon(h)\Phi_f(1 - s, -\bar{h}/m)$$

$$\Phi'_f(s, h/m) = -\varepsilon(h)\Phi'_f(1 - s, -\bar{h}/m),$$

where $h\bar{h} \equiv 1 \pmod m$

$$\Phi_f(s, h/m) = (m/\pi)^s \Gamma^2(s/2)\phi_f(s, h/m),$$

and

$$\Phi'_f(s, h/m) = (m/\pi)^s \Gamma^2\left(\frac{s+1}{2}\right)\phi'_f(s, h/m)$$

(b) $(m, N) = 1$. *Let* $f|^0_{[H(N)]}(\tau) = g(\tau)$; *then* g *is an even Maass form of level* N *and character* $\bar{\varepsilon}$. *Let* $b(n)$ *be the Fourier coefficients of* g. *Then the functional equations are*:

$$\Phi_f(s, h/m) = \varepsilon(m)\Phi_g(1 - s, -\bar{N}\bar{h}/m),$$

$$\Phi'_f(s, h/m) = \varepsilon(m)\Phi'_g(1 - s, -\bar{N}\bar{h}/m)$$

where

$$\Phi_f(s, h/m) = (\pi/m\sqrt{N})^{-s}\Gamma^2(s/2)\phi_f(s, h/m)$$

and

$$\Phi'_f(s, h/m) = (\pi/m\sqrt{N})^{-s}\Gamma^2\left(\frac{s+1}{2}\right)\phi'_f(s, h/m). \qquad \square$$

Proof. Let $f_x(\tau) = 1/2\pi i\,\partial/\partial x\, f(\tau)$; $(\tau = x + iy)$. Then we have

$$f_x(\tau) = i\sum a(n)n\sqrt{y}K_0(2\pi ny)\sin(2\pi nx).$$

We need to know how $f_x(\tau)$ transforms under the transformations A and $H(N)$. For this let $f_y = 1/2\pi i\,\partial/\partial y\, f(\tau)$. If $U \in \Gamma_0(N)$ then $f(U\tau) = \varepsilon(d)f(\tau)$, $U = \begin{pmatrix} a\,b \\ d\,c \end{pmatrix}$. So we get

$$\begin{aligned} f_x(\tau) &= 1/2\pi i\,\partial/\partial x\, f(\tau) = \varepsilon(d)/2\pi i\,\partial/\partial x\, f(Ut) \\ &= \varepsilon(d)[f_x(U\tau)\mathrm{Re}(\partial U\tau/\partial x) + f_y(U\tau)\mathrm{Im}(\partial U\tau/\partial x)] \\ &= \varepsilon(d)[f_x(U\tau)\mathrm{Re}((c\tau + d)^{-2}) + f_y(U\tau)\mathrm{Im}((c\tau + d)^{-2}] \end{aligned}$$

Now taking $\tau = h/m + i/mt$ and A as in the above proof we see that $f_x(h/m + i/mt) = \varepsilon(\bar{h})(-t)^{-2}f_x(-\bar{h}/m + it/m)$. We get a similar transformation formula under H(N).

The rest of the proof is along the same lines as that of theorem 2.1 (and so we will not reproduce it here) but will use the following formula to get an integral representation for $\Phi(s, h/m)$:

$$1/4(m/\pi n)^s\Gamma^2(s/2) = \int_0^\infty K_0(2\pi ny/m)y^{s-1}\,\mathrm{d}y.$$

Summation formulae

We begin by stating a theorem of Berndt [3]. Let $\{\lambda_n\}$ and $\{\mu_n\}$-be two sequences of positive numbers strictly increasing to infinity. Let $\{a(n)\}$ and $\{b(n)\}$ be two sequences of complex numbers, not identically zero such that the Dirichlet series:

$$\phi(s) = \sum a(n)\lambda_n^{-s} \quad \text{and} \quad \psi(s) = \sum b(n)\mu_n^{-s}$$

converge in some half-plane. Suppose further that they satisfy the functional equation:

$\chi(s)\ \phi(s) = \chi(r-s)\psi(r-s)$ where $\chi(s)$ is one of the following three gamma factors:

(i) $\Gamma(s)$ and r arbitrary real;
(ii) $\Gamma(s/2)\Gamma(s-p/2)$ where p is an integer and $r = p+1$;
(iii) $\Gamma^2(s+1/2)$ and $r=1$.

Also further, suppose that the poles of $\chi(s)\phi(s)$ are confined to some compact set. Define for $x>0$, $Q_q(x)$ and $I_q(x)$ as follows:

$$Q_q(x) = \frac{1}{2\pi i}\int_{C_q}\frac{\Gamma(s)\phi(s)}{\Gamma(s+q+1)}x^{s+q}ds$$

where C_q is a cycle enclosing all of the integrand's poles; and respectively as $\chi(s)$ is as in (i), (ii) and (iii):

$$\begin{aligned} I_q(x) &= x^{(r+q)/2}J_{r+q}(2\sqrt{x}) \\ &= x^{(p+q+1)/2}\Big\{\cos(\pi(p+1)/2)J_{p+q+1}(4\sqrt{x}) \\ &\qquad - \sin(\pi(p+1)/2)[Y_{p+q+1}(4\sqrt{x}) + \left(\frac{2(-1)^{p+q}}{\pi}\right)K_{p+q+1}(4\sqrt{x})\Big\} \\ &= x^{(q+1)/2}\left\{Y_{q+1}(4\sqrt{x}) + \left(\frac{2(-1)^{q+1}}{\pi}\right)K_{q+1}(4\sqrt{x})\right\}. \end{aligned}$$

Theorem 2.3. (Berndt [3]): *Let $f\in C^{(1)}(0,\infty)$. Then*

$$\sum_{a<\lambda_n<b}' a(n)f(\lambda_n) = \int_a^b Q_0'(t)f(t)dt + \sum\frac{b(n)}{\mu_n^{r-1}}\int_a^b I_{-1}(\mu_n t)f(t)dt \qquad \square$$

Summation formulae for the holomorphic case

For the sake of simplicity we write down the summation formulae only when f is a cusp form. In this case only the second term on the right hand side of the general summation formula in theorem 2.3 will survive for then $\phi(s)$ is entire. Accordingly let f be a cusp form of level N and character ε. The functional equations of theorem 2.1 give rise to the following summation formulae:

(a). *Case when $(m,N)=N$.*

$$\sum_{a\leqslant n\leqslant b} a(n)\exp(2\pi inh/m)f(n) = i^k\varepsilon(\bar{h})\frac{2\pi}{m}\sum a(n)\exp(-2\pi in\bar{h}/m)n^{-(k-1)/2}\times$$
$$\int_a^b x^{(k-1)/2}J_{k-1}\left(\frac{4\pi(nx)^{1/2}}{m}\right)f(x)dx$$

(b). *Case when $(m,N)=1$.*

$$\sum_{a\leqslant n\leqslant b} a(n)\exp(2\pi inh/m)f(n) = \bar{\varepsilon}(m)\left(\frac{2\pi}{m\sqrt{N}}\right)\sum b(n)\exp(-2\pi i\bar{N}\bar{h}/m)n^{-(k-1)/2}$$
$$\int_a^b x^{(k-1)/2}J_{k-1}\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right)f(x)dx.$$

Summation formulae for the non-holomorphic case

Let f be an even Maass form with $\lambda = 1/4$ with Fourier coefficients $a(n)$. Then the functional equations of theorem 2.2 imply the following summation formulae (note that $p = 0$):

(a). $(m, N) = N$.

$$\sum_{a \leqslant n \leqslant b} a(n)\cos(2\pi nh/m) f(n) = \frac{\varepsilon(\bar{h})\pi}{2m} \sum a(n)\cos(-2\pi n\bar{h}/m) \times \int_a^b \left[2/\pi K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) - Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) \right] f(x)\mathrm{d}x,$$

$$\sum_{a \leqslant n \leqslant b} a(n)\sin(2\pi nh/m) f(n) = \frac{\varepsilon(\bar{h})\pi}{2m} \sum a(n)\sin(-2\pi n\bar{h}/m) \times \int_a^b \left[Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) + 2/\pi K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) \right] f(x)\mathrm{d}x,$$

(b). $(m, N) = 1$.

$$\sum_{a \leqslant n \leqslant b} a(n)\cos(2\pi nh/m) f(n) = \frac{\varepsilon(m)\pi}{2m\sqrt{N}} \sum b(n)\cos(-2\pi n\bar{N}\bar{h}/m) \times \int_a^b \left[2/\pi K_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right) - Y_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right) \right] f(x)\mathrm{d}x,$$

$$\sum_{a \leqslant n \leqslant b} a(n)\sin(2\pi nh/m) f(n) = \frac{\varepsilon(m)\pi}{2m\sqrt{N}} \sum b(n)\sin(-2\pi\bar{N}\bar{h}/m) \times \int_a^b \left[Y_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right) + 2/\pi K_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right) \right] f(x)\mathrm{d}x.$$

3. Transformation formulae

In this section we obtain, following Jutila [6], a transformation formula for exponential sums of the type

$$\sum_{a \leqslant n \leqslant b} a(n) g(n) \exp(2\pi i f(n))$$

where $a(n)$'s are Fourier coefficients of either a holomorphic cusp form or a Maass form with $\lambda = 1/4$ for a congruence subgroup of $SL(2, \mathbb{Z})$ and f and g are functions on $[a, b]$. To begin with we recall results on exponential integrals due to Atkinson and Jutila. This is essentially chapter 2 of [6] without proofs. Then we obtain transformation formulae for exponential sums involving Fourier coefficients of cusp forms considered in § 1. We conclude this section with some special cases of the transformation formula for Dirichlet polynomials associated with these cusp forms.

Exponential integrals

Let

$$I = \int_a^b g(x)\exp(2\pi i(f(x) + \alpha x))\mathrm{d}x = \int_a^b h(x)\mathrm{d}x.$$

For a positive integer J and a positive real number U define a smoothed version I_J of I by:

$$U^{-1}\int_0^U \mathrm{d}u_1 \int_0^U \mathrm{d}u_J \int_{a+u}^{b-u} h(x)\mathrm{d}x = \int_a^b \eta_J(x)h(x)\mathrm{d}x, \quad u = u_1 + \cdots + u_J.$$

Also let $I_0 = I$. Note that $0 < \eta(x) \leqslant 1$ for $x \in (a, b)$ and $\eta(x) = 1$, for $a + JU \leqslant x \leqslant b - JU$.

We quote three theorems below first of which gives an approximate value of the integral I in terms of saddle points (Atkinson), the second theorem its generalization to I_J due to Jutila and the third gives an estimate of I_J when f has no saddle points in (a, b). For proofs of these theorems see [6].

Let f and g be functions on $[a, b]$ satisfying the following conditions:

(i) f is real for $a \leqslant x \leqslant b$;
(ii) f and g are holomorphic in the domain
$D = \{z \,|\, |z - x| < \mu$ for some $x \in (a, b)\}$ where μ is a positive real number;
(iii) there are positive numbers F and G such that:
$|g(z)| \ll G$ and $|f'(z)| \ll F\mu^{-1}$ for $z \in D$;
(iv) $f''(x) > 0$ and $f''(x) \gg F\mu^{-2}$.

Since $f''(x) > 0$, $f'(x) + \alpha$ is monotonically increasing and hence has at most one zero in (a, b), say x_0. Further let

$$E_J(x) = G(|f'(x) + \alpha| + f''(x)^{1/2})^{-J-1}$$

Theorem 3.1. *Let f and g be as above. Then*

$$\begin{aligned} I &= g(x_0)f''(x_0)^{-1/2}\exp[(2\pi i(f(x_0) + \alpha x_0 + 1/8)] \\ &\quad + O(G\exp[-A|\alpha|\mu - AF](b-a)) + O(G\mu F^{-3/2}) + O(E_0(a)) + O(E_0(b)). \end{aligned}$$

□

Theorem 3.2. *Let $U > 0, J \geqslant 0$ be a fixed integer, $JU < (b-a)/2$ and f and g be as above with the additional condition that $F \gg 1$. Suppose, also that $U \gg \mu F^{-1/2}$. Then with I_J as above we have*:

$$\begin{aligned} I_J &= \xi(x_0)g(x_0)f''(x_0)^{-1/2}\exp(f(x_0) + \alpha x_0 + 1/8) \\ &\quad + O((1 + (\mu/U)^J)G\exp(-A|\alpha|\mu - AF)(b-a)) \\ &\quad + O((1 + F^{1/2})G\mu F^{-3/2}) \\ &\quad + O\left(U^{-J}\sum_{j=0}^{J}(E_J(a + jU) + E_J(b - jU)\right), \end{aligned}$$

where $\xi(x_0) = 1$ *for* $a + JU < x_0 < b - JU$,

$$\xi(x_0) = (J!U^J)^{-1} \sum_{j=0}^{J1} \binom{J}{j} (-1)^j \sum_{0 \leqslant v \leqslant J/2} c_v f''(x_0)^{-v} (x_0 - a - jU)^{J-2v}$$

for $a < x_0 \leqslant a + JU$ *with* j_1 *the largest integer such that* $a + j_1 U < x_0$

$$\xi(x_0) = (J!U^J)^{-1} \sum_{j=0}^{J2} \binom{J}{j} (-1)^j \sum_{0 \leqslant v \leqslant J/2} c_v f''(x_0)^{-v} (b - x_0 - jU)^{J-2v}$$

for $b - JU \leqslant x_0 < b$ *with* j_2 *the largest integer such that* $b - j_2 U > x_0$. *The* c_v *are numerical constants.* □

Theorem 3.3. *Suppose* f *and* g *are functions satisfying* (*i*) *and* (*ii*) *above. Assume further that* $|g(z)| \ll G$, $|f'(x)| M$, *and* $|f'(z)| \ll M$ *for* $z \in D$ *and* $x \in (a, b)$. *Let* I_J *denote the smoothed version of* I *with* $\alpha = 0$ *and* $0 < JU < (b - a)/2$. *Then*

$$I_J \ll U^{-1} G M^{-J-1} + (\mu^J U^{1-J} + (b - a)) G \exp(-AM\mu). \quad \square$$

Transformation formulae

Before we proceed to the theorem we quote a Lemma (without proof) which summarizes the properties of Hankel functions we need to use.

Lemma 3.4 *Let* $\delta_1 < \pi$ *and* δ_2 *be fixed positive numbers. Then in the sector* $|\arg z| \leqslant \pi - \delta_1$, $|z| \geqslant \delta_2$ *we have*

$$H_n^{(j)}(z) = (2/\pi z)^{1/2} \exp((-1)^{j-1} i(z - n\pi/2 - \pi/4))(1 + g_j(z)),$$

where the functions $g_j(z)$ *are holomorphic in the slit complex plane* $z \neq 0$, $|\arg z| < \pi$, *and satisfy* $|g_j(z)| \ll |z|^{-1}$ *in the above sector. Further we have*

$$J_n(z) = 1/2(H_n^{(1)}(z) + H_n^{(2)}(z))$$

and

$$Y_n(z) = 1/2i(H_n^{(1)}(z) - H_n^{(2)}(z)).$$

We also have

$$K_n(x) = (\pi/2x)^{1/2} \exp(-x)(1 + O(x^{-1})). \quad \square$$

In what follows, δ denotes an arbitrary small positive constant not necessarily the same in each occurrence. Put $L = \log M$.

We have the following theorem which gives a transformation formula in the case of *holomorphic cusp forms*. Accordingly let

$$f(\tau) = \sum_{n=1}^{\infty} a(n) \exp(2\pi i n \tau)$$

be a cusp form of level N, weight k and character ε. Further let $f|^k_{[H(N)]}(\tau) = g(\tau) = \Sigma b(n) \exp(2\pi i n \tau)$.

Theorem 3.5. *Let $2 \leqslant M_1 < M_2 \leqslant 2M_1$ and let f and g be holomorphic functions in the domain*

$$D = \{z \mid |z - x| < cM_1 \text{ for some } x \in [M_1, M_2]\},$$

where c is a positive constant. Suppose that $f(x)$ is real for x in $[M_1, M_2]$. Suppose also that, for some positive numbers F and G

$$|g(z)| \ll G,$$

$$|f'(z)| \ll FM^{-1}, \text{ for } z \in D, \text{ and that } (0 <) f''(x) \gg FM_1^{-2} \text{ for } x \in [M_1, M_2].$$

Let $r = h/m$ be a rational number such that

$$1 \leqslant m \ll M_1^{1/2 - \delta},$$

$$|r| \asymp FM_1^{-1},$$

and

$$f'(M(r)) = r$$

for a certain number $M(r)$ in $[M_1, M_2]$. Write $M_j = M(r) + (-1)^j m_j$, $j = 1, 2$.

Suppose that $m_1 \asymp m_2$, and that

$$M_1^{\delta} \max(M_1 F^{-1/2}, |hm|) \ll m_1 \ll M_1^{1-\delta}.$$

Define for $j = 1, 2$

$$p_{j,n}(x) = \begin{cases} f(x) - rx + (-1)^{j-1}\left(\dfrac{2\sqrt{(nx)}}{m} - \dfrac{(k-1)}{4} - \dfrac{1}{8}\right), & \text{if } (m, N) = N; \text{ and} \\ f(x) - \mathrm{r}x + (-1)^{j-1}\left(\dfrac{2\sqrt{(nx)}}{m\sqrt{N}} - \dfrac{(k-1)}{4} - \dfrac{1}{8}\right), & \text{if } (m, N) = 1. \end{cases}$$

and

$$n_j = \begin{cases} (r - f'(M_j))^2 m^2 M_j, & \text{if } (m, N) = N; \\ (r - f'(M_j))^2 (m\sqrt{N}))^2 M_j & \text{if } (m, N) = 1. \end{cases}$$

and for $n < n$ let $x_{j,n}$ be the (unique) zero of $p'_{j,n}(x)$ in the interval (M_1, M_2). Set

$$A = \begin{cases} i^k \varepsilon(\bar{h})(2m)^{-1/2}, & \text{if } (m, N) = N \text{ and} \\ \varepsilon(m)(2m\sqrt{N})^{-1/2}, & \text{if } (m, N) = 1. \end{cases}$$

Then we have

$$\sum_{M_1 \leqslant n \leqslant M_2} a(n) g(n) e(f(n)) =$$

$$A \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} a'(n) e(nh'/m) n^{-(k/2) + 1/4} x_{j,n}^{(k/2) - (3/4)} g(x_{j,n}) \times$$

$$p''_{j,n}(x_{j,n})^{-1/2} e(p_{j,n}(x_{j,n}) + 1/8) + O(G(|h|m)^{1/2} M_1^{(k-1)/2} m_1^{1/2} L^2) +$$

$$O(F^{1/2} G |h|^{-3/4} m^{5/4} M_1^{(k-1)/2} m_1^{-1/4} L).$$

where $a'(n) = a(n)$, $h' = -\bar{h}$ if $(m, N) = N$ and $a(n) = b(n)$, $h' = -\bar{N}\bar{h}$ if $(m, N) = 1$. □

Proof. Without loss of generality suppose that $r(=h/m)$ is positive. Assume that $(m,N)=N$; the case $(m,N)=1$ is entirely similar. The transformation formula should be understood as an asymptotic result wherein M_1 and M_2 are large. Before we start on the proof, we shall note various estimates that are needed; like, for instance, the order of n_j. First note that $f''(x)\asymp FM^{-2}$ (for, by assumption, $f''(x)\gg FM^{-2}$ and the reverse inequality follows from the estimate for f' and holomorphy of f), and $F\gg M_1^{1/2+\delta}$ (for $F\gg M_1 r\geqslant m^{-1}M_1\gg M_1^{1/2+\delta}$). Thus we have

$$|r-f'(M_j)|\asymp m_j FM_1^{-2} \text{ (for } f'(M(r))=r).$$

This gives us the estimate: $n\asymp F^2m^2M_1^{-3}m_j^2$.

The n_j's are determined by the condition $p'_{j,n}(M_j)=0$. This implies that for $n<n_j, p'_{j,n}(x)$ has an unique zero in (M_1,M_2). For clearly $(-1)^j p'_{j,n}(M_j)>0$ and $(-1)^j p'_{j,n}(M(r))<0$, if $n<n_j$. Note also that $x_{1,n}<x_{2,n}$ and that $p'_{j,n}(x)$ has no zero in (M_1,M_2) if $n>n_j$. Uniqueness of $x_{j,n}$ follows from that $p''_{j,n}(x)$ has the same order as $f''(x)$ if M_1 is sufficiently large and hence is positive for $f''(x)$ is positive.

Let

$$S=S(M_1,M_2)=\sum_{M_1\leqslant n\leqslant M_2} a(n)g(n)e(f(n)).$$

We first replace S by its smoothed version S':

$$S'=U^{-1}\int_0^U S(u)\mathrm{d}u,$$

where

$$S(u)=\sum_{M_1+u\leqslant n\leqslant M_2-u} a(n)g(n)e(f(n))$$

and U is a parameter to be chosen later. For now we only assume

$$M_1^\delta \ll U\leqslant 1/2\min(m_1,m_2).$$

The estimate $a(n)\ll n^{(k-1)/2+\varepsilon}$ implies that $S-S'\ll GUM_1^{(k-1)/2}L$. The choice of the parameter U later will show that this error has been accounted for in the statement of the transformation formula.

The idea is to apply the summation formula to $S(u)$ and evaluate S' by using saddle-point theorems. But instead of applying the summation formula to $S(u)$ as it stands it has been observed by Jutila that we get better results if we introduce an exponential factor without disturbing the sum. Accordingly before applying the summation formula we modify the sum $S(u)$ as:

$$S(u)=\sum_{a\leqslant n\leqslant b} a(n)e(nr)g(n)e(f(n)-nr),\quad a=M_1+u,\quad b=M_2-u.$$

Applying the summation formula of § 2 we get:

$$S(u)=i^k\varepsilon(\bar{h})\frac{2\pi}{m}\sum a(n)e(-n\bar{h}/m)n^{-(k-1)/2}$$
$$\times\int_a^b x^{(k-1)/2}J_{k-1}\left(\frac{4\pi\sqrt{(nx)}}{m}\right)g(x)e(f(x)-rx)\mathrm{d}x.$$

Now write $J_{k-1}(\,)$ in terms of the Hankel functions to get:

$$S(u) = i^k \varepsilon(\bar{h}) \sum a(n) e(-n\bar{h}/m) n^{-(k-1)/2} I_n,$$

where

$$I_n = \frac{\pi}{m} \int_a^b x^{(k-1)/2} \left[H_{k-1}^{(1)}\left(\frac{4\pi\sqrt{(nx)}}{m}\right) + H_{k-1}^{(2)}\left(\frac{4\pi\sqrt{(nx)}}{m}\right) \right] g(x) e(f(x) - rx) \mathrm{d}x$$

whence by lemma 3.4 we get $I_n = I_n^{(1)} + I_n^{(2)}$
with

$$I_n^{(j)} = (2m\sqrt{n})^{-1/2} \int_a^b x^{(k/2)-(3/4)} g(x) \left[1 + g_j\left(\frac{4\pi\sqrt{(nx)}}{m}\right) \right] e(p_{j,n}(x)) \mathrm{d}x.$$

It can checked that the conditions of the theorems 3.1 and 3.3 are satisfied with $-r$ in place of r and $f(x)$ replaced by:

$$f(x) + (-1)^{j-1}(2\sqrt{(nx)}/m - (k-1)/4 - 1/8), \quad \text{and } \mu = M_1.$$

The number $x_{j,n}$ is by definition the saddle point for $I_n^{(j)}$ and it lies in the interval $[M_1, M_2]$ if and only if $n < n_j$. However, in $I_n^{(j)}$ the interval of integration is $[a,b] = [M_1 + u,\ M_2 - u]$, and $x_{j,n} \in [a,b]$ if and only if $n < n_j(u)$ where

$$n_j(u) = (r - f'(M_j + (-1)^{j-1}u)^2 m^2 (M_j + (-1)^{j-1}u).$$

But for simplicity we count the saddle point terms for all $n < n_j$ for this frees the saddle point terms from depending on u and thus we will have the same saddle point terms for all $S(u)$ and hence for S' as well. The number of extra terms counted will be

$$\ll 1 + n_j - n_j(u) \ll 1 + F^2 m^2 M_1^{-3} m_1 U.$$

The saddle point term for $I_n^{(j)}$, for $n < n_j$ is:

$$(2m)^{-1/2} n^{-1/4} x_{j,n}^{(k/2)-(3/4)} g(x_{j,n}) p''_{j,n}(x_{j,n})^{-1/2}$$
$$e(p_{j,n}(x_{j,n}) + 1/8)(1 + g_j(2(nx_{j,n})^{1/2}/m).$$

Thus up to $g_j(\,)$ we have the explicit terms claimed in the theorem. The effect of the omission of $g_j(\,)$ is:

$$\ll F^{-1/2} G m^{1/2} M_1^{(k/2)-(1/4)} \sum_{n < n_j} a(n) n^{-(k/2)-(1/4)}$$
$$\ll F^{-1/2} G m^{1/2} M_1^{(k/2)-(1/4)} n_j^{1/4} \ll G m_1^{1/2} M_1^{(k-1)/2} L.$$

This error can be absorbed into the first O-term in the formula given in the theorem.

The extra saddle-points counted while replacing $n_j(u)$ by n_j contribute

$$\ll (1 + F^2 m^2 M_1^{-3} m_1 U) F^{-1/2} G m^{-1/2} M_1^{(k/2)+(1/4)+\varepsilon} n_1^{-1/4}$$
$$\ll F^{1/2} G h^{-3/2} m^{1/2} m_1^{-1/2} M_1^{(k-1)/2+\varepsilon} + F^{-1/2} G h^{3/2} m^{-1/2} m_1^{1/2} M_1^{(k-1)/2+\varepsilon} U.$$

Here the first term is absorbed into the second O-term in the transformation formula and later U will be chosen so that the second term above also goes into the second O-term.

We shall now consider the error terms of theorem 3.1 which was applied to $I_n^{(j)}$ for $n < n_j$. The first error term is clearly negligible. The contribution of the second O-term is:

$$\ll F^{-3/2} G m^{-1/2} M_1^{(k/2)+(1/4)} \sum_{n \ll n_1} a(n) n^{-(k-1)/2} n^{-1/4}$$
$$\ll G m m_1^{3/2} M_1^{(k-1)/2-(3/2)} L \ll G m_1^{1/2} M_1^{(k-1)/2} L$$

which again goes into the first O-term of the theorem. The terms $O(E_0(a))$ and $O(E_0(b))$ are similar and so it is enough to consider one of them, say $O(E_0(a))$. This error term is

$$\ll G m^{-1/2} M_1^{(k/2)-(3/4)} n^{-1/4} (|p'_{j,n}(a)| + p''_{j,n}(a)^{1/2})^{-1}.$$

Consider the case $j = 1$; the case $j = 2$ is even simpler for $p'_{2,n}(b)$ cannot be very small. $p'_{1,n(u)}(a) = 0$ and $p''_{1,n}(a) = F^{-1} r^2$. Therefore we have

$$(|p'_{1,n}(a)| + p''_{1,n}(a)^{1/2})^{-1} \ll \begin{cases} F^{1/2} r^{-1} \text{ for } |n - n_1(u)| \ll F^{-1/2} h^2 m_1, \\ m M_1^{1/2} n_1^{1/2} |n - n(u)|_1^{-1} \text{ otherwise.} \end{cases}$$

Thus we get that the contribution to $S(u)$ of these error terms is $\ll G(hm)^{1/2} m_1^{1/2} M_1^{(k-1)/2} L^2$, which goes into the first error term of the formula.

We are now left with showing that the tail part in the summation formula, that is terms for $n > n_j$, are accounted for in the theorem. Here we make use of theorem 3.2 for the estimation of the exponential integral since for $n > n_j$ the integral has no saddle-points in the interval of integration. Here U will be the smoothing parameter with $J = 1$. The contribution of $I_n^{(j)}$ to S' equals

$$(2m)^{-1/2} \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} a(n) n^{-(k-1)/2} e(-n\bar{h}/m) n^{-1/4} \times$$
$$\int_a^b \eta_1(x) x^{(k/2)-(3/4)} g(x) \left[1 + g_j\left(\frac{4\pi \surd(nx)}{m} \right) \right] e(p_{j,n}(x)) \mathrm{d}x$$

where $\eta_1(x)$ is the weight function. Apply theorem 3.2 with $p_{j,n}(z)$ in place of $f(z)$ and $\mu \asymp m_1$. Note that the conditions of the theorem 3.3 are met if we choose $M = m^{-1} M_1^{-1/2} n^{1/2}$. The second term on the right hand side of the estimate given in theorem 3.3 is exponentially smalll and hence can be neglected because

$$M\mu \gg m^{-1} M_1^{-1/2} n^{1/2} m_1 \gg (n/n_1)^{1/2} F m_1^2 M_1^{-2} \gg (n/n_1)^{1/2} M_1^{\delta}.$$

The term corresponding to $U^{-1} G M^{-2}$ therein is

$$\ll G m^{3/2} M_1^{(k/2)+(1/4)} U^{-1} \sum_{n \gg n_1} a(n) n^{-(k-1)/2-(5/4)}$$
$$\ll G m^{3/2} M_1^{(k/2)+(1/4)} U^{-1} n_1^{-1/4} L$$
$$\ll \mathrm{G} \mathrm{F}^{-1/2} m M_1^{(k/2)+1} m_1^{-1/2} U^{-1} L$$
$$\ll G F h^{-3/2} m^{5/2} M_1^{(k-1)/2} m_1^{-1/2} U^{-1} L.$$

Thus proof of the theorem is complete up to the following error terms:

$$GUM_1^{(k-1)/2}L + F^{-1/2}Gh^{3/2}m^{-1/2}m_1^{1/2}M^{(k-1)/2+\varepsilon}U$$
$$+ GFh^{-3/2}m^{5/2}M_1^{(k-1)/2}m_1^{-1/2}U^{-1}L$$

The first and the last terms above coincide with the last term in the transformation formula if we choose $U = F^{1/2}h^{-3/4}m^{5/4}m_1^{-1/4}$. Then the second term above is

$$\ll Gh^{3/4}m^{3/4}m_1^{1/4}M_1 \ll G(hm)^{1/2}m_1^{1/2}M_1^{\delta}$$

which can be seen to go into the first O-term of the transformation formula. It only remains to be shown that the above choice of U satisfies our requirement: $M_1^{\delta} \ll U \leqslant 1/2\min(m_1, m_2)$. We have

$$Um_1^{-1} \ll U(M_1^{1+\delta}F^{-1/2})^{-1} \ll (hm)^{1/4}m_1^{-1/2}M_1^{-\delta} \ll M_1^{-\delta}.$$

For the other inequality

$$U \gg F^{1/2}h^{-3/4}m^{5/4}M_1^{-(1/4)+\delta} \gg M_1^{1/4+\delta}h^{-1/4}m^{3/4} \gg M_1^{\delta}.$$

This completes the proof of theorem.

In the case when $f(\tau) = \Sigma a(n)\sqrt{y}K_0(2\pi ny)\cos(2\pi nx)$ is an even Maass form of level N and character ε as in § 2, we have the following transformation formula.

Theorem 3.6. *Under the notations and assumptions of theorem* 3.5 *with* $k = 1$ *we have*

$$\sum_{M_1 \leqslant n \leqslant M_2} a(n)g(n)e(f(n)) = A\sum_{j=1}(-1)^{j-1}\sum_{n<n_j} b(n)e(nh'/m) \times$$
$$n^{-1/4}x_{j,n}^{-1/4}g(x_{j,n})p''_{j,n}(x_{j,n})^{-1/2}e(p_{j,n}(x_{j,n}) + 1/8) +$$
$$O(G(|h|m)^{1/2}m_1^{1/2}L^2) +$$
$$O(F^{1/2}G|h|^{-3/4}m^{5/4}M_1^{1/10}m_1^{-1/4}L). \qquad \square$$

Proof. Note that the second error term above is slightly worse than the corresponding error term in theorem 3.5. This is because the Deligne's estimate which was used in theorem 3.5 has not been proved for non-holomorphic forms and the best known estimate is $a(n) \ll n^{1/5+\varepsilon}$. Let S, S' and $S(u)$ be as in the proof of theorem 3.5; further assume that $(m, N) = N$, the other case is similar. Thus

$$S(u) = \sum_{a \leqslant n \leqslant b} a(n)g(n)e(f(n))$$
$$= \sum_{a \leqslant n \leqslant b} a(n)[\cos(2\pi nr) + i\sin(2\pi nr)]g(n)e(f(n) - nr)$$
$$= S_1(u) + iS_2(u), \text{ say.}$$

We now apply summation formulae of § 2 to $S_1(u)$ and $S_2(u)$ and proceed to evaluate the integrals as before.

$$S_1(u) = \sum_{a \leqslant n \leqslant b} a(n)\cos(2\pi nr)g(n)e(f(n) - nr)$$

$$= \frac{\varepsilon(\bar{h})\pi}{m} \sum_{n=1}^{\infty} a(n)\cos(-2\pi n\bar{h}/m) \times$$

$$\int_a^b \left[2/\pi K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) - Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) g(x)e(f(x) - rx)\mathrm{d}x\right.$$

$$= \frac{\varepsilon(\bar{h})\pi}{m} \sum_{n=1}^{\infty} a(n)\cos(-2\pi n\bar{h}/m)[i_n + I_n],$$

where

$$i_n = \pi/2 \int_a^b K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) g(x)e(f(x) - rx)\mathrm{d}x$$

and

$$I_n = -\int_a^b Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) g(x)e(f(x) - rx)\mathrm{d}x.$$

We first observe that the contribution from the integrals i_n is negligible. We have $\sqrt{(nM_1)}/m \gg \sqrt{n}M_1^{\delta}$ so that

$$m^{-1} \sum_{n=1}^{\infty} a(n)|i_n| \ll m^{-1} G \sum_{n=1}^{\infty} a(n)\exp(-A\sqrt{n}M_1^{\delta})$$

$$\ll G\exp(-AM_1^{\delta}).$$

Write the integrals I_n in terms of the Hankel functions to get:

$$I_n = i\int_a^b \left[H_0^{(1)}\left(\frac{4\pi\sqrt{(nx)}}{m}\right) - H_0^{(2)}\left(\frac{4\pi\sqrt{(nx)}}{m}\right)\right] g(x)e(f(x) - rx)\mathrm{d}x$$

$$= I_n^{(1)} - I_n^{(2)},$$

where

$$I_n^{(j)} = i\pi^{-1}m^{1/2}n^{-1/4}\int_a^b x^{-1/4}g(x)\left[1 + g_j\left(\frac{4\pi\sqrt{(nx)}}{m}\right)\right] e(p_{j,n}(x))\mathrm{d}x.$$

Notice that this is same as the integral '$I_n^{(j)}$' in the proof of theorem 3.5 with $k = 1$. Similarly for the sum $S_2(u)$ and putting these two terms together we get the transformation formula claimed in the theorem. Note also that the 'Rankin's trick' has been extended to the case of Maass forms to get the mean value estimate:

$$\sum_{n \leqslant X} |a(n)|^2 = CX + O(X^{3/5+\varepsilon}).$$

We now proceed to give analogs of the above transformation formulae for smoothed exponential sums provided with weights of the type $\eta_J(n)$ of pp. 9. We get much better error terms but we will have to allow for certain weights to appear in the transformed sum as well.

Theorem 3.7. *Suppose that the assumptions of the theorem 3.5 are satisfied. Let* $U \gg F^{-1} M_1^{1+\delta} F^{1/2} r^{-1} M_1^{\delta}$, *and J be a fixed positive integer exceeding a certain bound. Write for* $j = 1, 2$

$$M'_j = M_j + (-1)^{j-1} JU = M(r) + (-1)^j m'_j,$$

and suppose that $m'_j m_j$. L et n_j *be as before and*

$$n'_j = (r - f'(M'_j))^2 m^2 M'_j.$$

Then defining the weights $\eta_J(x)$ *in the interval* $[M_1, M_2]$ *as in pp. 9 we have*

$$\begin{aligned}&\sum_{M_1 \leqslant n \leqslant M_2} \eta_J(n) a(n) g(n) e(f(n)) \\ &= A \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} w_j(n) a'(n) e(nh'/m) \times \\ &n^{-(k/2)+1/4} x_{j,n}^{(k/2)-(3/4)} g(x_{j,n}) \times \\ &p''_{j,n}(x_{j,n})^{-1/2} e(p_{j,n}(x_{j,n}) + 1/8) + \\ &O(F^{-1} G |h|^{3/2} m^{-1/2} M_1^{(k-1)/2} m_1^{1/2} UL).\end{aligned}$$

where $w_j(n) = 1$ *for* $n < n_j$, *and* $w_j(n) \ll 1$ *for* $n < n'_j$; *further* $w_j(y)$ *and* $w'_j(y)$ *are piecewise continuous functions in* (n'_j, n_j) *with at most* $J - 1$ *discontinuities and* $w'_j(y) \ll (n_j - n'_j)^{-1}$ *for* $y \in (n'_j, n_j)$ *whenever* $w'_j(y)$ *exists.* □

The proof of this theorem is the same as that of theorem 3.5 but uses theorem 3.2 in place of theorem 3.1 for details see [6]. A similar theorem holds for the nonholomorphic case.

A particular case

We now want to specialise the transformation formula to the case of Dirichlet polynomials, that is to say, to

$$S(M_1, M_2) = \sum_{M_1 \leqslant n \leqslant M_2} a(n)^{-(k/2)-it}$$

when $M_1 < t/2\pi r < M_2$ with r satisfying the conditions of theorem 3.5 and where $a(n)$'s are Fourier coefficients of a cusp form of weight k. Such sums occur, for instance, while estimating the Dirichlet series (associated to cusp forms) on the critical line and studying their zeros on the critical line.

Here $g(z) = z^{-k/2}$, $f(z) = -(t/2\pi)\log z$ and $M(-r) = t/2\pi r$. The assumptions of the theorems 3.5 are satisfied (with $-r$ in place of r) if we choose $F = t$ and $G = M_1^{-k/2} r^{k/2} t^{-k/2}$. Then $n_j = h^2 m_j^2 M_j^{-1}$, $M_j = (t/2\pi r) + (-1)^j m_j$ and the function $p_{j,n}(x)$ takes the form

$$p_{j,n}(x) = -(t/2\pi)\log x + rx + (-1)^{j-1}(2\sqrt{(nx)}/\alpha - (k-1)/4 - 1/8)$$

where $\alpha = m$ if $(m, N) = N$ and $\alpha = m\sqrt{N}$ if $(m, N) = 1$. Assume for sake of simplicity that $(m,N)=N$; the other case is entirely similar. Thus $x_{j,n}$'s are the roots of the equation

$$p'_{j,n}(x) = -t/2\pi x + r + (-1)^{j-1}\sqrt{n}(m\sqrt{x})^{-1} = 0$$

or equivalently of the quadratic equation

$$x^2 - ((t/\pi r) + (n/h^2))x + (t/2\pi r)^2 = 0.$$

Therefore, since $x_{1,n} < x_{2,n}$, we have

$$x_{j,n} = \frac{t}{2\pi r} + \frac{n}{2h^2} + \frac{(-1)^j}{h^2}\left(\frac{n^2}{4} + \frac{hknt}{2\pi}\right)^{1/2}$$

and

$$(t/2\pi r)^2 x_{j,n}^{-1} = \frac{t}{2\pi r} + \frac{n}{2h^2} - \frac{(-1)^j}{h^2}\left(\frac{n^2}{4} + \frac{hknt}{2\pi}\right)^{1/2}.$$

To wirte the transformation formula here we need to calculate $2^{-1/2}m^{-1/2}x_{j,n}^{-3/4}p''_{j,n}(x_{j,n})^{-1/2}$ and $p_{j,n}(x_{j,n})$.

We have

$$p''_{j,n}(x_{j,n}) = t/2\pi x_{j,n}^2 + (-1)^j 2^{-1} n^{1/2} m^{-1} x_{j,n}^{-3/2}$$

So

$$\begin{aligned} 2mx_{j,n}^{3/2}p''_{j,n}(x_{j,n}) &= \pi^{-1}mtx_{j,n}^{-1/2} + (-1)^j n^{1/2} \\ &= (-1)^{j-1}h^2 n^{-1/2}(2(t/2\pi r)^2 x_{j,n}^{-1} - t/\pi r) + (-1)^j n^{1/2} \\ &= \pi^{-1/2}(2hkt)^{1/2}\left(1 + \frac{\pi n}{2hkt}\right)^{1/2}. \end{aligned}$$

Thus

$$2^{-1/2}m^{-1/2}x_{j,n}^{-3/4}p''_{j,n}(x_{j,n})^{-1/2} = \pi^{1/4}(2hkt)^{-1/4}\left(1 + \frac{\pi n}{2hkt}\right)^{-1/4}.$$

Calculation of $p_{j,n}(x_{j,n})$ is more delicate. We have

$$\begin{aligned} (2\pi r t^{-1}x_{j,n})^{(-1)^j} &= 1 + \frac{\pi n}{hkt} + \left(\left(\frac{\pi n}{hkt}\right)^2 + \frac{2\pi n}{hkt}\right)^{1/2} \\ &= \left(\left(\frac{\pi n}{2hkt}\right)^{1/2} + \left(1 + \frac{\pi n}{2hkt}\right)^{1/2}\right)^2 \end{aligned}$$

whence $\log(2\pi r t^{-1}x_{j,n}) = (-1)^j 2\,\operatorname{arcsinh}((\pi n/2hkt)^{1/2})$. We also have, by $p'_{j,n}(x_{j,n})=0$,

$$\begin{aligned} 2\pi r x_{j,n} + 4\pi(-1)^{j-1}n^{1/2}x_{j,n}^{1/2}m^{-1} &= 2t - 2\pi r x_{j,n} \\ &= t - \frac{\pi n}{hk} + (-1)^{j-1}2t\left(\frac{\pi n}{2hkt} + \left(\frac{\pi n}{2hkt}\right)^2\right)^{1/2}. \end{aligned}$$

Thus

$$2\pi p_{j,n}(x_{j,n}) = (-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right) - t\log\left(\frac{t}{2\pi}\right) + \\ + t\log r + t - \frac{\pi n}{hk},$$

where we have put $\phi(x) = \operatorname{arcsinh}(x^{1/2}) + (x + x^2)^{1/2}$. Thus we have

$$e(p_{j,n}(x_{j,n} + 1/8) = e(-n/2hk)\exp(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right) \times r^{it}\exp(i(t + \pi/4))(2\pi/t)^{it}$$

Thus finally we have:

$$\begin{aligned} S(M_1, M_2) &= \sum_{M_1 \leqslant n \leqslant M_2} a(n) n^{-(k/2)-it} = \\ &= \pi^{1/4}(2hkt)^{-1/4}\left(\frac{2\pi r}{t}\right)^{it} \exp(i(t + \pi/4)) \\ &\times \sum_{j=1}^{2} \sum_{n < n_j} a(n) e\left(n\left(\frac{\bar{h}}{m} - \frac{1}{2hk}\right)\right) \\ &n^{(1/4)-(k/2)}\left(1 + \frac{\pi n}{2hkt}\right)^{-1/} \\ &\times \exp(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt} - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right)\right) \\ &+ O(hm_1^{1/2}t^{-1/2}L^2) + O(h^{-1/4}m^{3/4}m_1^{-1/4}L). \end{aligned}$$

The smoothed version in this case reads:

$$\begin{aligned} S(M_1, M_2) &= \sum_{M_1 \leqslant n \leqslant M_2} \eta_j(n) a(n) n^{-(k/2)-it} = \\ &= \pi^{1/4}(2hkt)^{-1/4}\left(\frac{2\pi r}{t}\right)^{it} \exp(i(t + \pi/4)) \\ &\times \sum_{j=1}^{2} \sum_{n < n_j} a(n) e\left(n\left(\frac{\bar{h}}{m} - \frac{1}{2hk}\right)\right) \\ &\times n^{(1/4)-(k/2)}\left(1 + \frac{\pi n}{2hkt}\right)^{-1/4} \exp(i(-1)^{j-1} \\ &\times \left(2t\phi\left(\frac{\pi n}{2hkt} - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right)\right) + O(h^2 m^{-1} m_1^{1/2} t^{-3/2} UL). \end{aligned}$$

It is advantageous to choose U as small as the condition

$$U \gg F^{-1/2} M_1^{1+\delta} \asymp F^{1/2} M_1^{1+\delta} r^{-1}, \quad \text{i.e. } U \asymp F^{1/2+\varepsilon} r^{-1}.$$

With this choice the above error term becomes $O(F^{-1/2+\varepsilon} G(|h|m)^{1/2} M_1^{(k-1)/2} m_1^{1/2})$.
As usual we have a similar formula for the non-holomorphic case.

Remark In the case of Dirichlet series coming from cusp-forms of higher level, $N \geqslant 1$, the point of interest is $t\sqrt{N}/2\pi$, and m_j, the length, satisfies: $t^{1/2+\delta} \ll m_1 \ll t$. We can manage to get the same transformation formulae taking $M(r) = t\sqrt{N}/2\pi$ where r is an approximation to $\sqrt{N}$ which satisfies:

$$|r - 1/\sqrt{N}| \ll t^{1/2}, r = h/m, m \ll t^{1/4}$$

with $(m, N) = 1$.

It can be verified that the order of n_j remains unaltered and so will other estimates which depended on $f'(M(r)) = r$. For example let us look at $|-r - f'(M_1)|$:

$$M_1 = t\sqrt{N}/2\pi - m_1 = t\sqrt{N}/2\pi(1 - 2\pi m_1/t\sqrt{N});$$

So

$$M_1^{-1} = 2\pi/t\sqrt{N}(1 - 2\pi m_1/t\sqrt{N})^{-1} \simeq 2\pi/t\sqrt{N}(1 + 2\pi m_1/t\sqrt{N}) \text{ as } m_1 \ll t^{1-\delta};$$

and $f'(M_1) = -t/2\pi M_1$. Thus

$$\begin{aligned}|-r - f'(M_1)| &= |r - 1/\sqrt{N}(1 + 2\pi m_1/t\sqrt{N})| \\ &= |r - 1/\sqrt{N} - 2\pi m_1/Nt| \asymp m_1 t^{-1} \\ &\times (= m_1 F M_1^{-2}, \text{ as } F = M_1 = t).\end{aligned}$$

We will make use of this remark in our application to 'zeros on the critical line' in the next section.

4. Applications

In this section we give two applications of the transformation formula. The first application deals with the zeros on the critical line of the Dirichlet series $\phi(s)$ associated with holomorphic cusp forms and the second application deals with the order of $\phi(k\backslash 2 + it)$. In all these applications we use only Rankin's meanvalue estimate though in the case of holomorphic forms the estimate $a(n) \ll n^{(k-1)/2+\varepsilon}$ (Ramanujan – Petersson conjecture) is known due to Deligne. Thus these results go through in the case of Maass forms as well where the analogue of Rankin's estimate has been proved but Deligne's estimate has not yet been; the best result known here is $a(n) = O(n^{1/5+\varepsilon})$ due to Serre.

Zeros on the critical line

Consider the Dirichlet series $\phi(s) = \Sigma a(n) n^{-s}$ where $a(n)$'s are Fourier coefficients of a cusp form of weight k, level N and character ε; this series satisfies the following functional equation

$$(2\pi/\sqrt{N})^{-s}\Gamma(s)\phi(s) = C(2\pi/\sqrt{N})^{s-k}\Gamma(k-s)\psi(k-s),$$

where $|C| = 1$ (for a proof take $m = 1$ in theorem 2.1 of § 2). If ε is a real character then $f \to f|_{H(N)}$ is an automorphism of $M(N, k, \varepsilon)$ and since it is an involution we can decompose $M(N, k, \varepsilon)$ further as $M^+(N, k, \varepsilon) + M^-(N, k, \varepsilon)$ where on $M^{\pm}(N, k, \varepsilon) H(N)$ acts by ± 1. Thus if $f \in M^{\pm}(N, k, \varepsilon)$ then $b(n) = \pm a(n)$ in the earlier notation. In this situation if we rewrite the functional equation as

$$\phi(s) = C'\Delta(s)\phi(k-s), \quad \Delta(s) = (2\pi/\sqrt{N})^{2s-k}\Gamma(k-s)/\Gamma(s), C' = \pm C$$

and further assume that $a(n)$'s are real we see that on the critical line $\Delta(s)$ has absolute value 1, $|\Delta((k/2)+it)|=1$. Therefore the function

$$Z_\phi(t)=[C'\Delta((k/2)+it))]^{-1/2}\phi((k/2)+it)$$

is a real function of t. We can now use this function to check whether $\phi(s)$ has any zeros on the critical line for t in an interval $[T-H, T+H]$ by comparing the integrals

$$\left|\int_{-H}^{H} Z_\phi(T+u)\mathrm{d}u\right| \text{ and } \int_{-H}^{H} |Z_\phi(T+u)|\mathrm{d}u$$

for if $\phi(s)$ does not vanish for t in the above interval then these two integrals should coincide.

Theorem 4.1. *Suppose that $a(n)$ is real for all n. Then for all $\varepsilon>0$ there exists a number $T_0=T_0(\varepsilon)$ such that for all $T\geqslant T_0$ the function $\phi(s)$ has a zero $(k/2+i\gamma)$ with $|T-\gamma|\leqslant T^{1/3+\varepsilon}$. A similar statement holds for the function $\phi(s,1/N)$.* □

Proof. We shall first prove the theorem for the function $\phi(s,1/N)$. Observe that for the Dirichlet series $\phi(s,1/N)$ also the corresponding function

$$Z_\phi(t)=[C\Delta(k/2)+it)]^{-1/2}\phi((k/2)+it,1/N)$$

is real by virtue of the functional equation proved in theorem 2.1. Also note that here ε need not be a real character and that the result is true for $\phi(s,h/m)$ where h is such that $h^2\equiv 1(\mathrm{mod}\, m)$. Suppose that $\phi(k/2+it,1/N)$ does not vanish for t in the interval $[\tau-H, T+H]$. Then $Z_\phi(t)$ is of constant sign in the above interval. Let $H=T^{(1/3)+3\varepsilon}$ and consider the integral

$$I=\int_{-H}^{H} Z_\phi(T+u)\exp(-(u/H_0)^2)\mathrm{d}u,$$

where $H_0=T^{1/3+2\varepsilon}$

It is well known that

$$|I|=\int_{-H}^{H}|Z_\phi(T+u)|\exp(-(u/H_0)^2)\mathrm{d}u \gg \int_{-H_0}^{H_0}|Z_\phi(T+u)|\mathrm{d}u\gg H_0$$

See Theorem 3 in [1] for a proof.

We shall estimate I in a different way by making use of the following representation for $\phi(s,1/N)$ on the critical line:

Lemma 4.2 *Let $t\geqslant 2$ and $t^2\ll X\ll t^A$ where A is an arbitrary positive constant. Then we have, putting $a'(n)=a(n)e(1/N)$,*

$$\phi(k/2)+it,1/N)=\sum_{n\leqslant X} a'(n)n^{-(k/2)-it}+$$
$$+(\log 2)^{-1}\sum_{X<n\leqslant 2X} a'(n)\log(2X/n)n^{-(k/2)-it}+O(tX^{-1}).$$

□

Proof. The proof is standard (see for example [6]).

Take $X = T^3$ and let $K \in [T^{2/3-\varepsilon}, 2T^{2/3-\varepsilon}]$. We have

$$\begin{aligned} I = & \sum_{\substack{n \leqslant T^3 \\ |n - TN/2\pi| > K}} a'(n) n^{-(k/2) - iT} C^{-1/2} \\ & \times \int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \exp(-(u/H_0)^2) \mathrm{d}u \\ & + C^{-1/2} \int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} \left(\sum_{|n - TN/2\pi| \leqslant K} a'(n) n^{-(k/2) - i(T+u)} \right) \\ & + \exp(-(u/H_0)^2) \mathrm{d}u + (\log 2)^{-1} \sum_{T^3 < n \leqslant 2T^3} a'(n) \log(2T^3/n) n^{-(k/2) - iT} \times \\ & \times C^{-1/2} \int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \exp(-(u/H_0)^2) \mathrm{d}u + O(1) \\ = & I_1 + I_2 + I_3 + O(1). \end{aligned}$$

We will now show that I_1 and I_3 are small. Let first $n > TN/2\pi + K$, and estimate the integral,

$$\int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \exp(-(u/H_0)^2) \mathrm{d}u,$$

by looking at the corresponding complex integral over the rectangular contour with vertices $\pm H$, $\pm H - iH_0$. By Sterling's formula we have (remember $\Delta(s) = (2\pi/N)^{2s-k} \Gamma(k-s)/\Gamma(s)$)

$$\begin{aligned} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} = & \exp(i(T \log(TN/2\pi) \\ & - T + u \log(TN/2\pi n) + O(1))). \end{aligned}$$

On the vertical sides this is bounded and

$$\exp(-(u/H_0)^2) \ll \exp^{\varepsilon}(-T).$$

On the horizontal side in the lower half-plane $\exp(-(u/H_0)^2)$ is bounded and

$$\Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \ll \exp\{-H_0 \log(2\pi n/NT)\} \ll \exp(-AT^{\varepsilon}).$$

For $n < TN/2\pi - K$ the corresponding integral can be estimated similarly by integrating in the upper half-plane. Thus I_1 and I_3 are $\ll 1$.

Coming to I_2 we have

$$\begin{aligned} I_2 & \ll H \sup_{|T-t| \leqslant H} \left| \sum_{|n - TN/2\pi| \leqslant K} a'(n) n^{-(k/2) - it} \right| \\ & \ll H \sup_{|T-t| \leqslant H} \left| \sum_{|n - tN/2\pi| \leqslant K} a'(n) n^{-(k/2) - it} \right| + O(HT^{-1/30 + 3\varepsilon/2}) \end{aligned}$$

The error was obtained by Rankin's estimate with error term:

$$\sum_{n\leqslant x}|a(n)|^2 = Ax^k + O(x^{k-2/5}).$$

We shall estimate the above sum by applying the transformation formula from § 3 with $r = 1/N$ and $M_j = tN/2\pi + (-1)^j K$. Then $n_j \ll t^{1/3-2\varepsilon}$, and the above sum is $\ll T^{-3\varepsilon/2}$. Thus

$$|I| \ll H_0 T^{-\varepsilon/2}.$$

But this contradicts $|I| \gg H_0$ if T is sufficiently large. Hence the assertion.

Now, coming to the Dirichlet series $\phi(s)$ we have

$$\Delta(k/2+it)^{-1/2} n^{-iu} = \exp(i(T\log(T\sqrt{N}/2\pi) - T + u\log(T\sqrt{N}/2\pi n) + O(1)).$$

Hence the sum which we will have to estimate will be over an interval around $T\sqrt{N}/2\pi$. Here we will have to use the remark made at the end of § 4. Because of the approximation of $\sqrt{N}$ by $r = h/m$ we will have to apply the smoothed version of the transformation formula. So instead of the integral I above we will start with its smoothed version I_J

$$I_J = \int_{-H}^{H} \eta_J(T+u) Z_\phi(T+u)\exp(-(u/H_0)^2)\mathrm{d}u.$$

As in the previous case we have $|I_J| \gg H_0$.

Proceeding as before but breaking the sum at $|n - T\sqrt{N}/2\pi| \leqslant K - v$ where $v = v_1 + v_2 + \cdots + v_J$ is the smoothing parameter, we get

$$I_J = I_1' + I_2' + I_3' + O(1),$$

where now

$$I_2' = \int_{-H}^{H} [C'\Delta((k/2) + i(T+u))^{-1/2}\eta_J(T+u)$$

$$\times \left(\sum_{|n - T\sqrt{N}/2\pi| \leqslant K - v} a(n) n^{-k/2 - i(T+u)}\right)\exp(-(u/H_0)^2)\mathrm{d}u.$$

Thus

$$|I_2'| \ll H \sup_{|T-t|\leqslant H}\left|\sum_{|n - T\sqrt{N}/2\pi|\leqslant K - v} \eta_J(n) a(n) n^{-k/2 - it}\right|.$$

Now estimating as in the previous case but now using the remark at the end of § 3 and smoothed version of the transformation formula we conclude that the above sum is $\ll T^{-3\varepsilon/2}$ and so I_2' is $\ll H_0 T^{-\varepsilon/2}$. The integrals I_1' and I_3' are estimated as before.

Estimation of 'long' sums and order of $\phi(k/2+it)$

Here we are concerned with exponential sums

$$\sum_{M\leqslant n\leqslant M'} a(n) g(n) e(f(n))$$

which are "long" in the sense that the length may be of the order of M itself. It is not practical to transform such sums directly as in § 3 because variations in $f'(x)$ might be too much in the interval $[M, M']$. It is advisable to first partition $[M, M']$ into segments such that $f'(x)$ practically remains a constant in each segment and then transform these short sums. But we need to assume that $f'(x)$ is approximately a power to be able to get some saving in the estimate. The precise result (theorem 4.6 in [6]) is as follows:

Theorem 4.3. *Let $2 \leqslant M < M' \leqslant 2M$ and let f be a holomorphic function in the domain $D = \{z \,|\, |z - x| < cM$ for some $x \in [M, M']\}$ where c is a positive constant. Suppose that $f(x)$ is real in $[M, M']$ and that either*

$$f(z) = Bz^{\alpha}(1 + O(F^{-1/3})), \quad z \in D$$

where $\alpha \neq 0$, 1 is a fixed real number and

$$F = |B| M^{\alpha}$$

or

$$f(z) = B \log z (1 + O(F^{-1/3})), \quad z \in D \text{ with } F = |B|.$$

Let $g \in C^1[M, M']$ and suppose that for $x \in [M, M']$

$$|g(x)| \ll G, \quad |g'(x)| \ll G'.$$

Assume further that $M^{3/4} \ll F \ll M^{3/2}$. Then

$$\left| \sum_{M \leqslant n \leqslant M'} a(n) n^{-(k-1)/2} g(n) e(f(n)) \right| \ll (G + MG') M^{1/2} F^{1/3 + \varepsilon}$$

where $a(n)$'s are Fourier coefficients of a cusp form. □

We will not give a proof here since Jutila's proof for the full modular group case goes through except that a slight modification is required since (unlike in that case in our situation) we do not have transformation formulae for $M_1 < t/2\pi r < M_2$, where $r(= h/m)$ is a rational number, for all r; we need to assume that $(m, N) = 1$ or N (N is the level of the cusp form) to get a transformation formula. The required modification is as follows: Put $M_0 = F^{2/3 + \delta}$ and let $K = (M/M_0)^{1/2}$. We may suppose that $M \geqslant M_0$ for, otherwise the assertion is trivial. Consider the Farey sequence of order K and drop all those fractions h/m with $(m, N) > 1$. Denote this set of fractions by $\mathbb{K}$. If $r = h/m$ and $r' = h'/m'$ are two consecutive fractions in $\mathbb{K}$ let $\rho = (h + h')/(m + m')$ be their 'mediant'. We have

$$\rho - r = (mh' - m'h)/m(m + m')$$

In the usual case we would have $\rho - r = 1/m(m + m')$; but order-wise both are same i.e. $1/mK$. Define the points $M(\rho)$ by $f'(M(\rho)) = \rho$ and break the given sum at points $M(\rho)$ lying in the interval $[M, M']$. The rest of the proof is as in [6].

COROLLARY 4.4

We have

$$|\phi(k/2 + it)| \ll (|t| + 1)^{1/3 + \varepsilon}.$$ □

Proof. We have the following approximate functional equation for $\phi(s)$, for $0 \leqslant \sigma \leqslant k$ and $t \geqslant 10$:

$$\phi(s) = \sum_{n \leqslant x} a(n)n^{-s} + \psi(s) \sum_{n \leqslant y} b(n)n^{s-k} + O(x^{k/2-\sigma}\log t)$$

where

$$x, y \geqslant 1, \quad xy = (t\sqrt{N}/2\pi)^2 \text{ and } \psi(s) = (2\pi/\sqrt{N})^{2s-k}\Gamma(k-s)/\Gamma(s).$$

This reduces the proof of the corollary showing that for all (positive and negative) large values of t and for all M, M' with $1 \leqslant M < M' \leqslant t\sqrt{N}/2\pi$ and $M' \leqslant 2M$ we have

$$\left| \sum_{M \leqslant n \leqslant M'} a(n)n^{-k/2-it} \right| \ll t^{1/3+\varepsilon}.$$

This is precisely the estimate of the theorem 4.3 applied to this sum.

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