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# Proof of some conjectures on the mean-value of Titchmarsh series – III

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**Abstract.** With some applications in view, the following problem is solved in some special case which is not too special. Let $F(s) = \Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$ be a generalized Dirichlet series with $1 = \lambda_1 < \lambda_2 < \ldots, \lambda_n \leqslant Dn$, and $\lambda_{n+1} - \lambda_n \geqslant D^{-1}\lambda_{n+1}^{-\alpha}$ where $\alpha > 0$ and $D(\geqslant 1)$ are constants. Then subject to analytic continuation and some growth conditions, a lower bound is obtained for $(1/H)\int_0^H |F(it)|^2 \mathrm{d}t$. These results will be applied in other papers to appear later.

**Keywords.** Titchmarsh series; mean value; lower bounds.

## 1. Introduction

In the previous papers [1] and [2] with the same title (as the present one) we proved some conjectures made by the second author [4]. In this paper we formulate a new conjecture (which we believe to be true at least in some modified form) and indicate a slight progress towards it.

**Conjecture.** *Let $1 = \mu_1 < \mu_2 < \ldots$ be any sequence of real numbers with $1/C \leqslant \mu_{n+1} - \mu_n \leqslant C$ where $C(\geqslant 1)$ is an integer constant and $n = 1, 2, 3, \ldots$. Let us form the sequence $1 = \lambda_1 < \lambda_2 < \ldots$ of all possible (distinct) finite power products of $1 = \mu_1, \mu_2, \ldots$ with non-negative integral exponents. Let $s = \sigma + it$, $H(\geqslant 10)$ be a real parameter, and $\{a_n\}$ $(n = 1, 2, 3, \ldots)$ with $a_1 = 1$ be any sequence of complex numbers (possibly depending on $H$) such that $F(s) = \Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$ is absolutely convergent at $s = B$ where $B \geqslant 3$ is an integer constant. Suppose that $F(s)$ can be continued analytically in $(\sigma \geqslant 0, 0 \leqslant t \leqslant H)$ and that there exist $T_1, T_2$ with $0 \leqslant T_1 \leqslant H^{3/4}, H - H^{3/4} \leqslant T_2 \leqslant H$ such that for some $K(\geqslant 30)$ there holds*

$$\max_{\sigma \geqslant 0}(|F(\sigma + iT_1)| + |F(\sigma + iT_2)|) \leqslant K.$$

*Finally let $\Sigma_{n=1}^{\infty} |a_n| \lambda_n^{-B} \leqslant H^A$ where $A(\geqslant 1)$ is an integer constant. Then there exists a constant $\delta > 0$ (depending only on $A$, $B$ and $C$) such that for all $H \geqslant H_0(A, B, C)$ there holds*

$$\frac{1}{H}\int_0^H |F(it)|^2 \mathrm{d}t \geqslant \frac{1}{2} \sum_{\lambda_n \leqslant H^\delta} |a_n|^2, \tag{1}$$

*provided that $H^{-1}$ loglog $K$ does not exceed a small positive constant.*

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*Remark* 1. We can strengthen the Conjecture (1) by replacing $\frac{1}{2}$ by a more specific function of $H$ which is asymptotic to 1 as $H \to \infty$.

*Remark* 2. By the method of [1] we can prove that

$$\frac{1}{H}\int_0^H |F(it)|\mathrm{d}t \geqslant \frac{1}{2}. \tag{2}$$

The Remark 1 is also applicable.

*Remark* 3. Under the condition

$$\sum_{\lambda_n \leqslant X} |a_n| \leqslant D_0(\log X)^R \ (R = H^{\varepsilon},\ D_0 \text{ any constant},\ X \geqslant 30), \tag{3}$$

we can prove (2). Remark 1 is also applicable. For both these results the conditions involving $K$ are unnecessary. For the results mentioned in Remarks 2 and 3 we refer the reader to [1] and [5].

*Remark* 4. Actually the proof of (1) in [1] goes through without serious problems until we come to a lower bound for

$$\frac{1}{H}\int_0^H \left| \sum_{n \leqslant H^{\delta}} a_n \lambda_n^{-it} \right|^2 \mathrm{d}t.$$

To apply Montgomery–Vaughan theorem we need good lower bounds for $\lambda_{n+1} - \lambda_n$. These are not available in general. But we can work with $\mu_n = (n_0 + n - 1)/n_0$ where $n_0(\geqslant 2)$ is any integer constant (of course using Montgomery–Vaughan Theorem). Thus in this special case we can prove Conjecture (1). We can also handle $\mu_n = (1+\beta)^{-1}(n+\beta)$ where $\beta(>0)$ is any real algebraic constant.

*Remark* 5. We can formulate Conjecture (1) with no conditions involving $K$, but instead we have to assume condition (3). Remark 1 is also applicable.

Before closing this section we like to make two important remarks. First $\lambda_n \leqslant \mu_n \leqslant Cn$ which is obvious because $\{\lambda_n\}$ contains the subsequence $\{\mu_n\}$. Secondly for $x \geqslant 1$ and $\eta \geqslant 2C + 1$, we have,

$$\begin{aligned}
\sum_{\lambda_n \leqslant x} 1 &\leqslant x^{\eta} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\eta}} \leqslant x^{\eta}\left(1 - \sum_{n=2}^{\infty} \mu_n^{-\eta}\right)^{-1} \\
&\leqslant x^{\eta}\left\{1 - \sum_{n=2}^{\infty}\left(1 + \frac{n-1}{C}\right)^{-\eta}\right\}^{-1} \\
&\leqslant x^{\eta}\left\{1 - \int_0^{\infty}\left(1 + \frac{u}{C}\right)^{-\eta}\mathrm{d}u\right\}^{-1} = x^{\eta}\left(1 - C\int_0^{\infty}\frac{\mathrm{d}u}{(1+u)^{\eta}}\right)^{-1} \\
&= x^{\eta}\left(1 - \frac{C}{\eta - 1}\right)^{-1} \leqslant 2x^{\eta}.
\end{aligned}$$

Hence in (1) the condition $\lambda_n \leqslant H^{\delta}$ is equivalent to a condition of the type $n \leqslant H^{\delta}$ with a different constant $\delta > 0$.

## 2. Main lemma

*Let r be a positive integer, $H \geqslant (r+5)U$ where $U \geqslant 2^{70}(16B)^2$ and M and N are positive integers subject to $N > M \geqslant 1$, and $B(\geqslant 3)$ an integer constant. Let $\{b_m\}$ $(1 \leqslant m \leqslant M)$ and $\{c_n\}$ $(n \geqslant N)$ be two sequences of complex numbers, $1 = \lambda_1 < \lambda_2 < \ldots$ be any increasing sequence of real numbers and let $A(s) = \Sigma_{m \leqslant M} b_m \lambda_m^{-s}$. Let $B(s) = \Sigma_{n \geqslant N} c_n \lambda_n^{-s}$ be absolutely convergent for $s = B$ and continuable analytically in $(\sigma \geqslant 0, 0 \leqslant t \leqslant H)$. Write $g(s) = A(-s)B(s)$,*

$$G(s) = U^{-r}\int_0^U \mathrm{d}u_r \ldots \int_0^U \mathrm{d}u_1 (g(s+i\lambda))$$

*where (here and elsewhere) $\lambda = u_1 + \ldots + u_r$. Assume that there exist real numbers $T_1$ and $T_2$ with $0 \leqslant T_1 \leqslant U$, $H - U \leqslant T_2 \leqslant H$, such that*

$$|g(\sigma + iT_1)| + |g(\sigma + iT_2)| \leqslant \exp\exp\left(\frac{U}{16B}\right).$$

*uniformly in $0 \leqslant \sigma \leqslant B$. Let*

$$S_1 = \sum_{m \leqslant M, n \geqslant N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B 2^r \left(U \log \frac{\lambda_n}{\lambda_m}\right)^{-r}$$

*and*

$$S_2 = \sum_{m \leqslant M, n \geqslant N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B.$$

*Then, we have,*

$$\left|\int_{2U}^{H-(r+3)U} G(it)\mathrm{d}t\right| \leqslant 2B^2 U^{-10} + 54BU^{-1}\int_0^H |g(it)|\mathrm{d}t$$
$$+ (H + 64B^2)S_1 + 16B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

*Remark.* This lemma is borrowed from [1] (see pages 2 to 8).

## 3. Progress towards the conjecture

From now on we assume that $1 = a_1, a_2, a_3, \ldots$ is any sequence of complex numbers. We set $b_m = a_m$ and $c_n = a_n$ and assume that $\Sigma_{n=1}^{\infty} |a_n| \lambda_n^{-B}$ is convergent.

*Lemma* 1. *We have, with $\bar{A}(s) = \Sigma_{m \leqslant M} a_m \lambda_n^{-s}$,*

$$\int_{2U}^{H-(r+3)U} |\bar{A}(it)|^2 \mathrm{d}t \geqslant (H - (r+5)U - 10\lambda_M \Delta(\lambda_M)) \sum_{m \leqslant M} |a_m|^2,$$

*where* $\Delta(\lambda_M) = \max\limits_{\substack{\mu \neq \nu \\ 1 \leqslant \mu, \nu \leqslant M}} |\lambda_\mu - \lambda_\nu|^{-1}$.

*Proof.* Follows from Montgomery-Vaughan Theorem (see [3]).

*Lemma* 2. *We have,*

$$S_2 \leqslant \lambda_M^{2B}\left(\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B}\right)^2$$

*and*

$$S_1 \leqslant 2^r \lambda_M^{2B}\left(\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B}\right)^{-2}(U\lambda_N^{-1}(\lambda_N - \lambda_M))^{-r}.$$

*Proof.* The first inequality is trivial and the second follows from

$$\log\frac{\lambda_N}{\lambda_M} = -\log\left(1-\left(1-\frac{\lambda_M}{\lambda_N}\right)\right) > \frac{\lambda_N - \lambda_M}{\lambda_N}.$$

We now make the following.

*Hypothesis.* $\{\lambda_n\}$ is any increasing sequence of real numbers satisfying $\lambda_1 = 1$, $\lambda_n \leqslant Dn$, $\lambda_{n+1} - \lambda_n \geqslant \lambda_{n+1}^{-\alpha} D^{-1}$, where $D(\geqslant 1)$ is an integer constant and $\alpha$ a positive constant. Also we assume that

$$\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B} \leqslant H^{r\varepsilon/8}$$

where $0 < \varepsilon \leqslant 1/[2(\alpha+1)]$ and $r \geqslant [(200B+200)\varepsilon^{-1}]$ is any integer. Also $F(s) = \Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$ shall be as in the introduction except that the $\{\lambda_n\}$ are not related to the $\{\mu_n\}$. $\{\lambda_n\}$ will now be an independent sequence.

From now on we set $N = M+1$, $M = [H^{(1/(\alpha+1))-\varepsilon}]$, $U = H^{1-(\varepsilon/2)} + 50B \log\log K_1$ where $K_1 = H^r K$. Note that if $H \geqslant (r+5)U$ is not satisfied, our main theorem (to follow) asserts that a positive quantity is non-negative. Also note that

$$\min_{0\leqslant t\leqslant H^{3/4}} \max_{\sigma\geqslant 0} |F(\sigma+it)|$$

$$\geqslant \min_{0\leqslant t\leqslant U} \max_{\sigma\geqslant 0} |F(\sigma+it)|$$

and a similar result holds for the intervals $(H - H^{3/4}, H)$ and $(H-U, H)$.

*Lemma* 3. *We have,*

$$S_2 \leqslant (DH)^{2B} H^{r\varepsilon/4}, S_1 \leqslant 2^r (DH)^{2B} H^{r\varepsilon/4}((2D)^{-\alpha-2}H^{\varepsilon/2})^{-r}$$

*and*

$$\lambda_M \Delta(\lambda_M) \leqslant D^{\alpha+2} H^{1-\varepsilon}.$$

*Proof.* We have $\lambda_M \leqslant DM \leqslant DH$ and this proves the first inequality. Also

$$U\lambda_N^{-1}(\lambda_N - \lambda_M) \geqslant H^{1-(\varepsilon/2)}\lambda_N^{-1-\alpha}D^{-1} \geqslant H^{1-(\varepsilon/2)}D^{-1}(DN)^{-1-\alpha}$$

$$\geqslant H^{1-(\varepsilon/2)}(2DM)^{-1-\alpha}D^{-1} \geqslant H^{1-(\varepsilon/2)}(2D)^{-\alpha-2}H^{-1+\varepsilon},$$

and this proves the second inequality. The third follows from

$$\lambda_M \Delta(\lambda_M) \leqslant \lambda_M D \lambda_M^{\alpha} \leqslant D^{\alpha+2} M^{1+\alpha} \leqslant D^{\alpha+2} H^{1-\varepsilon}.$$

The lemma is completely proved.

Now we apply the main lemma (we closely follow the proof of the first main theorem in [1]). Let

$$A(s) = \sum_{m \leqslant M} \bar{a}_m \lambda_m^{-s}, \quad \bar{A}(s) = \sum_{m \leqslant M} a_m \lambda_m^{-s}$$

and

$$B(s) = \sum_{n \geqslant N} a_n \lambda_n^{-s}.$$

Then, we have, in $\sigma \geqslant B$, $F(s) = \bar{A}(s) + B(s)$ and so

$$\begin{aligned} |F(it)|^2 &= |\bar{A}(it)|^2 + 2\,\mathrm{Re}\,(g(it)) + |B(it)|^2 \\ &\geqslant |\bar{A}(it)|^2 + 2\,\mathrm{Re}\,(g(it)), \end{aligned}$$

where $g(s) = A(-s)B(s)$. Hence

$$\begin{aligned} &\int_0^H |F(it)|^2 \mathrm{d}t \\ &\geqslant U^{-r} \int_0^U \mathrm{d}u_r \ldots \int_0^U \mathrm{d}u_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (|\bar{A}(it)|^2 + 2\,\mathrm{Re}\, g(it))\mathrm{d}t \\ &= J_1 + 2J_2 \text{ say.} \end{aligned}$$

By Lemmas 1 and 3, we have,

$$J_1 \geqslant (H - (r+5)U - 10D^{\alpha+2}H^{1-\varepsilon}) \sum_{n \leqslant M} |a_n|^2.$$

Again, we have, for $0 \leqslant \sigma \leqslant B$,

$$\begin{aligned} |g(s)| &= |A(-s)B(s)| = |A(-s)(F(s) - A(s))| \\ &\leqslant \left(\sum_{n \leqslant M} |a_n| \lambda_n^B\right) K + \left(\sum_{n \leqslant M} |a_n| \lambda_n^B\right)^2 \\ &\leqslant \lambda_M^{2B} H^{r\varepsilon/8} K + \lambda_M^{4B} H^{r\varepsilon/4}. \end{aligned}$$

Hence

$$\begin{aligned} |g(s)|_{t=T_1} + |g(s)|_{t=T_2} &\leqslant 2K\lambda_M^{4B} H^{r\varepsilon/4}(H^{-r\varepsilon/8}\lambda_M^{-2B} + K^{-1}) \\ &\leqslant K(DH)^{4B} H^{r\varepsilon/4} \leqslant H^r K = K_1, \end{aligned}$$

the last two inequalities being true for instance if $H \geqslant 10D$. Observe that

$$\operatorname{expexp}\left(\frac{U}{16B}\right) \geqslant \operatorname{expexp}\left(\frac{50}{16}\operatorname{loglog} K_1\right) \geqslant K_1$$

and hence the condition on $g$ and $U$ required by the main lemma is satisfied. Hence by the main lemma, we have,

$$|J_2| \leqslant \frac{2B^2}{U^{10}} + \frac{54B}{U}\int_0^H |g(it)|\,dt + (H+64B^2)S_1 + 16B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

provided $H \geqslant (r+5)U$ and $U \geqslant 2^{70}(16B)^2$. As remarked already we can ignore the condition $H \geqslant (r+5)U$. Also we will satisfy $H \geqslant (50rBD^{B+\alpha+2})^{8/\varepsilon}$ and we will show later that this implies $U \geqslant 2^{70}(16B)^2$. We can assume that $\int_0^H |F(it)|^2 dt \leqslant H\sum_{n\leqslant M}|a_n|^2$ (otherwise the result asserted by the main theorem to follow, is trivially true). Hence

$$\int_0^H |g(it)|\,dt = \int_0^H |A(-it)B(it)|\,dt$$

$$\leqslant \int_0^H |A(-it)|^2 dt + \int_0^H |B(it)|^2 dt$$

$$\leqslant 3\int_0^H |A(-it)|^2 dt + 2\int_0^H |F(it)|^2 dt$$

(on noting that $B(it) = F(it) - \bar{A}(it)$)

$$\leqslant (5H + 10D^{\alpha+2}H^{1-\varepsilon}) \sum_{n\leqslant M} |a_n|^2$$

by Montgomery–Vaughan Theorem and the third part of Lemma 3. Hence

$$2|J_2| \leqslant \frac{4B^2}{H^5} + \frac{108B}{H^{1-(\varepsilon/2)}}(5H + 10D^{\alpha+2}H^{1-\varepsilon}) \sum_{n\leqslant M}|a_n|^2$$

$$+ (2H + 128B^2)S_1 + 32B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

$$\leqslant \left\{\frac{4B^2}{H^5} + \frac{108B}{H^{1-(\varepsilon/2)}}(5H + 10D^{\alpha+2}H^{1-\varepsilon}) + (2H+128B^2)S_1 + 32B^2 S_2 \exp\left(-\frac{U}{8B}\right)\right\} \sum_{n\leqslant M}|a_n|^2.$$

Thus

$$\int_0^H |F(it)|^2 dt \geqslant (H - S_3) \sum_{n\leqslant M}|a_n|^2,$$

where $S_3 > 0$ and

$$S_3 = (r+5)U + 10D^{\alpha+2}H^{1-\varepsilon} + \frac{4B^2}{H^5} + \frac{108B}{H^{1-\varepsilon/2}}(5H + 10^{\alpha+2}H^{1-\varepsilon})$$

$$+ (2H+128B^2)2^r(DH)^{2B}(2D)^{r(\alpha+2)}H^{-r\varepsilon/4}$$

$$+ 32B^2(DH)^{2B}H^{r\varepsilon/4}r^r(8B)^r H^{-r(2-\varepsilon)/2}.$$

Here we have used $\exp\left(-\frac{U}{8B}\right) \leqslant \frac{r^r(8B)^r}{U^r}$. Note that

$$\log\log K_1 \leqslant \log\log(H^r K) \leqslant \log\log K + \log(r \log H)$$
$$\leqslant \log\log K + \log r + \log H$$

and that $\frac{1}{2}(\log H)^2 \leqslant H$ and so $\log H \leqslant 2H^{1/2}$ and so

$$\frac{\log H}{H^{1-\varepsilon/4}} \leqslant \frac{2}{H^{(1/2)-(\varepsilon/4)}} \leqslant 2H^{-1/4}.$$

Hence

$$(r+5)U \leqslant 100Br(\log\log K + \log r + \log H) + (r+5)H^{1-\varepsilon/2}.$$

Thus

$$S_3 \leqslant 100Br \log\log K + r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}\left\{\frac{100B\log H}{H^{1-(\varepsilon/4)}} + \frac{(r+5)}{r(\log r)D^{\alpha+2}H^{\varepsilon/4}} + \frac{10}{r(\log r)H^{3\varepsilon/4}} + \frac{4B^2}{H^5} + \frac{108B(15D^{\alpha+2})H}{r(\log r)D^{\alpha+2}H^{2-(3\varepsilon/4)}} + \frac{130B^2H2^rD^BH^{2B}(2D)^{r(\alpha+2)}H^{-r\varepsilon/4}}{r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}} + \frac{32B^2(DH)^{2B}r^r(8B)^rH^{-r(1-\varepsilon/4)}}{r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}}\right\}.$$

Denote the expression in the last curly bracket by $S_4$. Then we have

$$S_4 \leqslant \frac{200B}{H^{\varepsilon/4}} + \frac{2r}{H^{\varepsilon/4}} + \frac{10}{H^{\varepsilon/4}} + \frac{4B^2}{H^{\varepsilon/4}} + \frac{1620B}{H^{\varepsilon/4}} + \frac{130B^2D^BH^{2B+1+1}}{H^{\varepsilon/4}}\left(\frac{4D^{\alpha+2}}{H^{\varepsilon/4}}\right)^r + \frac{32B^2D^{2B}}{H^{\varepsilon/4}}H^{2B+1}\left(\frac{8Br}{H^{1/2}}\right)^r.$$

Let $H^{\varepsilon/8} \geqslant 4D^{\alpha+2}$. We have $H^{r\varepsilon/8} \geqslant H^{((200B+200)\varepsilon^{-1}-1)\varepsilon/8} \geqslant H^{2B+2}$. Let $H^{1/4} \geqslant 8Br$. We have $H^{r/4} \geqslant H^{2B+1}$. Now both $H^{\varepsilon/8} \geqslant 4D^{\alpha+2}$ and $H^{1/4} \geqslant 8Br$ are satisfied if

$$H \geqslant (32BrD^{\alpha+2})^{8/\varepsilon}.$$

Hence under this only condition, we have,

$$S_4 \leqslant (200B + 2r + 10 + 4B^2 + 1620B + 130B^2D^B + 32B^2D^{2B})H^{-\varepsilon/4}$$
$$\leqslant rB^2D^{2B}H^{-\varepsilon/4}(200 + 2 + 10 + 4 + 1620 + 130 + 32)$$
$$\leqslant 2000rB^2D^{2B}H^{-\varepsilon/4} \leqslant 1$$

provided $H \geqslant (2000rB^2D^{2B})^{4/\varepsilon}$. Now this last condition and $H \geqslant (32BrD^{\alpha+2})^{8/\varepsilon}$ are both satisfied if $H \geqslant (50rBD^{B+\alpha+2})^{8/\varepsilon}$. Finally $U > H^{1-\varepsilon/2} \geqslant H^{3/4} \geqslant (50 \times 200B)^{(8/\varepsilon)(3/4)} = (10{,}000B\cdot B)^{6/\varepsilon} \geqslant 2^{13(6/\varepsilon)}(16B)^2\ (B^{24}/(16B)^2) \geqslant 2^{70}(16B)^2$ since $B \geqslant 3$. Collecting, we have proved the following

**Main Theorem.** *Let $\{\lambda_n\}(n = 1, 2, 3, \ldots)$ with $\lambda_1 = 1$ be any increasing sequence of real numbers with the properties $\lambda_n \leqslant Dn$ and $\lambda_{n+1} - \lambda_n > D^{-1}\lambda_{n+1}^{-\alpha}$ where $\alpha(>0)$ is a constant and $D(\geqslant 1)$ is an integer constant. Let $\{a_n\}(n = 1, 2, 3, \ldots)$ with $a_1 = 1$ be any sequence* *of complex numbers such that* $F(s) = \Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$ *is absolutely convergent at* $s = B$, *where* $B(\geqslant 3)$ *is an integer constant. Let* $0 < \varepsilon < (2(1+\alpha))^{-1}$ *and let* $r(\geqslant [(200B + 200)\varepsilon^{-1}])$ *be any integer constant. Let*

$$\sum_{n=1}^{\infty} |a_n| \lambda_n^{-B} \leqslant H^{(r\varepsilon/8)}.$$

*Assume that* $F(s)$ *possesses an analytic continuation in* $(\sigma \geqslant 0, 0 \leqslant t \leqslant H)$ *and that there exist* $T_1$, $T_2$ *with* $0 \leqslant T_1 \leqslant H^{3/4}$, $H - H^{3/4} \leqslant T_2 \leqslant H$ *such that for some* $K(\geqslant 30)$ *there holds*

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leqslant K$$

*uniformly in* $0 \leqslant \sigma \leqslant B$. *Let*

$$H \geqslant (50rBD^{B+\alpha+2})^{8/\varepsilon}.$$

*Then, there holds,*

$$\frac{1}{H}\int_0^H |F(it)|^2 \mathrm{d}t \geqslant (1-\phi) \sum_{n \leqslant M} |a_n|^2,$$

*where*

$$M = H^{\theta}, \quad \theta = \frac{1}{1+\alpha} - \varepsilon,$$

*and*

$$\phi = r(\log r) D^{\alpha+2} H^{-\varepsilon/4} + 100H^{-1} Br \operatorname{loglog} K.$$

In view of the two closing remarks at the end of § 1 we can be now deduce some corollaries.

COROLLARY 1.

*Let* $\mu_n = n$. *Then the conjecture is true.*

*Proof.* We can take $C = 1$, $\alpha = \varepsilon$ and $D = 1$.

COROLLARY 2.

*Let* $n_0(\geqslant 2)$ *be an integer constant and* $\mu_n = (n_0 + n - 1)/n_0$. *Then the conjecture is true.*

*Proof.* First, since $\{\mu_n\}$ is a subsequence of $\{\lambda_n\}$ is follows that $\lambda_n \leqslant \mu_n \leqslant n$. To apply the main theorem we have to verify that $\lambda_{n+1} - \lambda_n \geqslant D^{-1} \lambda_{n+1}^{-\alpha}$ holds with some constant $\alpha > 0$ and $D(\geqslant 1)$ an integer constant. To prove this we observe that we can assume that $\lambda_{n+1} - \lambda_n \leqslant 1$. In this case

$$\lambda_{n+1} - \lambda_n = \frac{m_1 \ldots m_k}{n_0^k} - \frac{n_1 \ldots n_l}{n_0^l} \geqslant n_0^{-j}$$

where $j = \max(k, l)$. Now $(1 + (1/n_0))^k \leqslant \lambda_{n+1}$ and $(1 + (1/n_0))^l \leqslant \lambda_n$ and so $j = \max(k, l) \leqslant (\log \lambda_{n+1})(\log((n_0+1)/n_0)^{-1}$. But $\log(n_0+1)/n_0 = -\log(1 - (1/n_0 + 1)) > 1/(n_0 + 1)$.

Thus $j \leqslant (n_0 + 1)(\log \lambda_{n+1})$ and so

$$n_0^{-j} \geqslant \lambda_{n+1}^{-\alpha} \text{ where } \alpha = (n_0 + 1)\log n_0.$$

Plainly we can take $D = 1$.

COROLLARY 3.

*Let $\beta > 0$ be an algebraic constant and $\mu_n = (n + \beta)/(1 + \beta)$. Then the conjecture is true. (The conjecture is also true for the choice $\mu_1 = 1$, $\mu_n = n + \beta - 1$ for $n > 1$).*

*Proof.* As before $\lambda_n \leqslant \mu_n \leqslant (n\beta + 2n)(\beta + 1)^{-1}$. Also considering the norm of $\lambda_{n+1} - \lambda_n$ (in case it is $\neq 0$) we can prove that $\lambda_{n+1} - \lambda_n \geqslant D\lambda_{n+1}^{-\alpha}$. The latter assertion follows similarly.

**Post-script**. The results of this paper were necessitated by a lot of applications to the zeros of generalized Dirichlet series. All these applications will form the subject matter of our forthcoming paper "On the zeros of a class of generalized Dirichlet series-XI".

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