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Remark on Gronwall's inequality

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Abstract. Gronwall's inequality has many extensions and analogues among them the discrete one. In this paper we present theorems which look like Gronwall's lemma in the classical propositional calculus.

Keywords. Gronwall's inequality; classical propositional calculus.

1. Introduction

One of the most famous inequalities in the theory of differential equations is Gronwall's inequality. Extended and generalized in many directions (see e.g. [1–3, 5]) this inequality has also discrete version embodied in the following theorem.

Theorem. (*discrete analogue of Gronwall's lemma*).

Let

$$x = \{x_n\}_{n=1}^{\infty}, \quad a = \{a_n\}_{n=1}^{\infty}$$

be any real sequences with a – non-negative, c – any real constant. If

h. $$x_{n+1} \leqslant c + \sum_{j=1}^{n} a_j x_j$$

holds for every $n = 1, 2, \ldots$, *then*

c. $$x_{n+1} \leqslant (c + a_1 x_1) \prod_{j=2}^{n} (1 + a_j)$$

for $n = 1, 2, \ldots$

For understanding the meaning of "look like" Gronwall's lemma, let us see that in the hypothesis *h.* terms of the unknown sequence x appear on both sides of the inequality. In the thesis (consequence) *c.* the terms of x are estimated, bounded by the terms of a, c, and (generally not necessary) the first element x_1. The theorems presented below look similar. The main result of this note is to show that the theorems have their analogues in many branches of mathematics. We construct our theorems in the classical propositional calculus $\mathfrak{Z}_2$. However, some of them can be considered in other languages or in metalanguage. Furthermore it seems that the results can be used as a manner of proving theorems as well, direct and indirect.

2. Preliminaries

We denote by $p, q, s, t, a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots, x_1, x_2, \ldots$, the infinite set of statement variables and by $\sim$, $\wedge$, $\vee$, $\supset$, $\equiv$ connectives in $\mathfrak{Z}_2$. Furthermore we shall use the symbols $\wedge_{j=1}^{n} a_j$ and $\vee_{j=1}^{n} a_j$ to denote generalized conjunction and disjunction i.e. $(a_n \wedge (a_{n-1} \wedge (.. \wedge (a_2 \wedge a_1)..)))$, $(a_n \vee (a_{n-1} \vee (.. \vee (a_2 \vee a_1)..)))$ respectively. We suppose $\wedge_{j=k}^{k} a_j$, $\vee_{j=k}^{k} a_j$ means a_k. Since we use the axioms in the proofs we recall them

a.1 $(p \supset q) \supset [(q \supset s) \supset (p \supset s)]$
a.2 $[p \supset (p \supset q)] \supset (p \supset q)$
a.3 $p \supset (q \supset p)$
a.4 $p \wedge q \supset p$
a.5 $p \wedge q \supset q$
a.6 $(p \supset q) \supset [(p \supset s) \supset (p \supset q \wedge s)]$
a.7 $p \supset p \vee q$
a.8 $q \supset p \vee q$
a.9 $(p \supset s) \supset [(q \supset s) \supset (p \vee q \supset s)]$
a.10 $(p \equiv q) \supset (p \supset q)$
a.11 $(p \equiv q) \supset (q \supset p)$
a.12 $(p \supset q) \supset [(q \supset p) \supset (p \equiv q)]$
a.13 $(\sim q \supset \sim p) \supset (p \supset q)$

The rules of inference are substitution, modus ponens, and the derivable rule hypothetical syllogism use of which is denoted by /, °C, r_{syl} respectively (see e.g. [4 pp. 170]). For making the proofs shorter we apply different statement forms which are known or can be easily inferred from the axioms. In the sequel, whenever we use some known formula of the classical propositional calculus, we note this *f.x*. The formulae we use in the presented proofs are

f.1 $(p \supset q) \supset (s \vee p \supset s \vee q)$
f.2 $(p \supset q) \supset (s \wedge p \supset s \wedge q)$
f.3 $p \supset p$
f.4 $p \wedge q \vee p \wedge s \supset p \wedge (q \vee s)$
f.5 $(p \wedge q) \wedge s \supset p \wedge (q \wedge s)$
f.6 $(p \vee q) \supset [p \vee (s \vee q)]$
f.7 $(p \supset q) \supset [(s \supset t) \supset (s \vee p \supset t \vee q)]$
f.8 $(p \vee q) \vee s \supset (p \vee s) \vee q$
f.9 $p \supset p \wedge p$
f.10 $(p \supset q) \supset [(s \supset t) \supset (p \wedge s \supset q \wedge t)]$

N denotes the set of positive integers.

3. Main results

We start with inequality which is easy to prove and is embodied in the following

$$\underset{n \in N}{\forall} a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j \supset x_{n+1} \forall \Vdash \underset{n \in N}{\forall} a_n \vee b_1 \wedge x_1 \supset x_{n+1}.$$

We prove this by applying n-times $a.8$ and r_{syl} and get

1. $$b_1 \wedge x_1 \supset \bigvee_{j=1}^{n} b_j \wedge x_j.$$

Now by f.1 using the rule of substitution

$$\text{f.1}\ p/b_1 \wedge x_1, \quad q\Big/\bigvee_{j=1}^{n} b_j \wedge x_j, \quad s/a_n - 2.$$

we obtain

2. $$\left(b_1 \wedge x_1 \supset \bigvee_{j=1}^{n} b_j \wedge x_j\right) \supset \left(a_n \vee b_1 \wedge x_1 \supset a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j\right).$$

Hence using modus ponens 2., 1. we have

$$2.^{\circ}\text{C}\ 1. - 3.$$

3. $$a_n \vee b_1 \wedge x_1 \supset a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j.$$

By premises (pr.), 3., and the rule of syllogism we obtain

$$r_{\text{syl}}\, 3. - \text{pr.} - c.$$

c. $$a_n \vee b_1 \wedge x_1 \supset x_{n+1}.$$

Remark. If we introduce the connective $\subset$ defined by the truth table

p	q	$p \subset q$
T	T	T
T	F	T
F	T	F
F	F	T

then $p \subset q$ is logically equivalent to $q \supset p$, or by writing $q \supset p$ we denote this statement by $p \subset q$. Then the theorem we have proved above can be expressed in the form more familiar for specialists of differential equations and in fact similiar to the inequality considered in the introduction i.e.

If

h.n. $$x_{n+1} \subset a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j$$

holds (has logical value truth) for every $n \in N$ then

c.n. $$x_{n+1} \subset a_n \vee b_1 \wedge x_1$$

is also true for every $n \in N$.

In such a meaning our theorem ought to be understood. Notice that the statement considered is true if we replace both in the premise (hypothesis) and conclusion $n \in N$ by $n \in Nm := \{1, 2, \ldots, m\}$. It is evident that by f.1 and

$$c_1 \supset \bigvee_{j=1}^{n} c_j$$

we can obtain many similar statements; e.g.

$$\underset{n\in N}{\forall}\left(a_n \vee \bigvee_{j=1}^{n}(b_j \supset x_j) \supset x_{n+1}\right) \Vdash \underset{n\in N}{\forall}(a_n \vee (b_1 \supset x_1) \supset x_{n+1})$$

or using f.2 instead of f.1

$$\underset{n\in N}{\forall}\left(a_n \wedge \bigvee_{j=1}^{n} b_j \vee x_j \supset x_{n+1}\right) \Vdash \underset{n\in N}{\forall}(a_n \wedge (b_1 \vee x_1) \supset x_{n+1})$$

$$\underset{n\in N}{\forall}\left(a_n \wedge \bigvee_{j=1}^{n} b_j \wedge x_j \supset x_{n+1}\right) \Vdash \underset{n\in N}{\forall}(a_n \wedge (b_1 \wedge x_1) \supset x_{n+1}).$$

Regarding axioms a.7 and a.8 we state that the conclusion of the proved statement will be better if many disjuncts appear in the conclusion's antecedent. This leads to the problem of finding the conclusion. The question that whether such a conclusion exists is left open. In the differential equations theory this problem reduces to solving instead inequality respective equation.

Theorem 1. *If*

h.n $\quad a_n \vee \bigvee_{j=1}^{n} b_j \wedge x_j \supset x_{n+1}$

has logical value truth for every $n \in N$ *(for some interpretation) then*

c.1 $\quad a_1 \vee b_1 \wedge x_1 \supset x_2$

c.n $\quad a_n \vee \left[\bigvee_{j=1}^{n-1} b_{j+1} \wedge a_j \vee \left(\bigwedge_{j=1}^{n} b_j\right) \wedge x_1\right] \supset x_{n+1}$

for $n = 2, 3, \ldots$ *is also true.*

Proof.

r_{syl} f.3 $p/a_1 \vee b_1 \wedge x_1$ – h.1 – c.1

Therefore c.1 holds. We shall prove that c.2 is true

f.2 $p/a_1 \vee b_1 \wedge x_1, \; q/x_2, \; s/b_2$ °C h.1 – 1.

1. $\quad b_2 \wedge (a_1 \vee b_1 \wedge x_1) \supset b_2 \wedge x_2$

r_{syl} f.4 $p/b_2, \; q/a_1, \; s/b_1 \wedge x_1$ – 1. – 2.

2. $\quad b_2 \wedge a_1 \vee b_2 \wedge (b_1 \wedge x_1) \supset b_2 \wedge x_2$

f.1 $p/(b_2 \wedge b_1) \wedge x_1, \; q/b_2 \wedge (b_1 \wedge x_1), \; s/b_2 \wedge a_1$ °C f.5 $p/b_2, \; q/b_1, \; s/x_1$ – 3.

3. $\quad b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1 \supset b_2 \wedge a_1 \vee b_2 \wedge (b_1 \wedge x_1)$

r_{syl} 3. – 2. – 4.

4. $\quad b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1 \supset b_2 \wedge x_2$

r_{syl} 4. – a.7 $p/b_2 \wedge x_2, \; q/b_1 \wedge x_1$ – 5.

5. $b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1 \supset b_2 \wedge x_2 \vee b_1 \wedge x_1$

$\text{f.1}\, p/b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1,\ q/b_2 \wedge x_2 \vee b_1 \wedge x_1,\ s/a_2\, {}^\circ\text{C}\ 5. - 6.$

6. $a_2 \vee [b_2 \wedge a_1 \vee (b_2 \wedge b_1) \wedge x_1] \supset a_2 \vee [b_2 \wedge x_2 \vee b_1 \wedge x_1]$

$r_{\text{syl}}\ 6. - \text{h.}2 - \text{c.}2$

This means that the theorem holds for $n = 2$.

Suppose $c.m$ is satisfied for $m = 1, 2, \ldots, k$. We prove that $c.k + 1$ holds.

$r_{\text{syl}}\ \text{a.}7\, p/a_1,\ q/b_1 \wedge x_1 - \text{c.}1 - 7.$

7. $a_1 \supset x_2$

$r_{\text{syl}}\ \text{a.}7\, p/a_m,\ q\Big/\left[\bigvee_{j=1}^{m-1} b_{j+1} \wedge a_j\right] \vee \left[\bigwedge_{j=1}^{m} b_j\right] \wedge x_1 - c.m - 8.$

8. $a_m \supset x_{m+1} \quad m = 2, 3, \ldots, k$

$\text{f.2}\, p/a_m,\ q/x_{m+1},\ s/b_{m+1}\, {}^\circ\text{C}\ 8. - 9. \quad m = 2, 3, \ldots, k$

$\text{f.2}\, p/a_1,\ q/x_2,\ s/b_2\, {}^\circ\text{C}\ 7. - 9.$

9. $b_{m+1} \wedge a_m \supset b_{m+1} \wedge x_{m+1} \quad m = 1, 2, \ldots, k$

$r_{\text{syl}}\ \text{f.}6\, p/a_k,\ q\Big/\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1,\ s\Big/\bigvee_{j=1}^{k-1} b_{j+1} \vee a_j - c.k - 10.$

10. $a_k \vee \left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1 \supset x_{k+1}$

$\text{f.2}\, p/a_k \vee \left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1,\ q/x_{k+1},\ s/b_{k+1}\, {}^\circ\text{C}\ 10. - 11.$

11. $b_{k+1} \wedge \left[a_k \vee \left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1\right] \supset b_{k+1} \wedge x_{k+1}$

$\text{f.1}\, p\Big/\left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1,\ q/b_{k+1} \wedge \left[\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1\right],\ s/b_{k+1} \wedge a_k$

${}^\circ\text{C}\ \text{f.}5\, p/b_{k+1},\ q\Big/\bigwedge_{j=1}^{k} b_j,\ s/x_1 - 12.$

12. $b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1 \supset b_{k+1} \wedge a_k \vee b_{k+1} \wedge \left[\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1\right]$

$r_{\text{syl}}\ 1.2 - \text{f.}4\, p/b_{k+1},\ q/a_k,\ s\Big/\left(\bigwedge_{j=1}^{k} b_j\right) \wedge x_1 - 11. - 13.$

13. $b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1 \supset b_{k+1} \wedge x_{k+1}$

$r_{\text{syl}}\ 9.(m = 1) - \text{a.}7\, p/b_2 \wedge x_2,\ q/b_1 \wedge x_1 - 14.$

14. $b_2 \wedge a_1 \supset b_2 \wedge x_2 \vee b_1 \wedge x_1$

$\text{f.7}\, p/b_2 \wedge a_1,\ q/b_2 \wedge x_2 \vee b_1 \wedge x_1,\ s/b_3 \wedge a_2,\ t/b_3 \wedge x_3\, {}^\circ\text{C}\ 14. -$

$- {}^\circ\text{C}\ 9.(m = 2) - 15.$

15. $$b_3 \wedge a_2 \vee b_2 \wedge a_1 \supset b_3 \wedge x_3 \vee \bigvee_{j=1}^{2} b_j \wedge x_j$$

f.7 $p\Big/\bigvee_{j=1}^{2} b_{j+1} \wedge a_j,\ q\Big/\bigvee_{j=1}^{3} b_j \wedge x_j,\ s/b_4 \wedge a_3,\ t/b_4 \wedge x_4$ °C 15. – – °C 9.$(m=3)$ – 16.

16. $$\bigvee_{j=1}^{3} b_{j+1} \wedge a_j \supset \bigvee_{j=1}^{4} b_j \wedge x_j.$$

Repeating the above reasoning we get

17. $$\bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j \supset \bigvee_{j=1}^{k} b_j \wedge x_j.$$

Hence

f.7 $p\Big/\bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j,\ q\Big/\bigvee_{j=1}^{k} b_j \wedge x_j,\ s/b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1,$

$t/b_{k+1} \wedge x_{k+1}$ °C 17. – °C 13. – 18.

18. $$\left[b_{k+1} \wedge a_k \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1\right] \vee \bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j \supset \bigvee_{j=1}^{k+1} b_j \wedge x_j$$

r_{syl} f.8 $p/b_{k+1} \wedge a_k,\ q\Big/\bigvee_{j=1}^{k-1} b_{j+1} \wedge a_j,\ s\Big/\left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1$ – 18. – 19.

19. $$\left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1 \supset \bigvee_{j=1}^{k+1} b_j \wedge x_j$$

f.1 $p\Big/\left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1,\ q\Big/\bigvee_{j=1}^{k+1} b_j \wedge x_j,\ s/a_{k+1}$ °C 19. – 20.

20. $$a_{k+1} \vee \left[\left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1\right] \supset a_{k+1} \vee \bigvee_{j=1}^{k+1} b_j \wedge x_j$$

r_{syl} 20. – $h.k+1$ – $c.k+1$.

We obtain $c.k+1$ holds so the proof is complete by induction.

Remark. Note that by a.5, a.8, f.1, and f.2

$$a_{k+1} \vee b_1 \wedge x_1 \supset a_{k+1} \vee \left[\left(\bigvee_{j=1}^{k} b_{j+1} \wedge a_j\right) \vee \left(\bigwedge_{j=1}^{k+1} b_j\right) \wedge x_1\right].$$

Therefore the statement we have proved at the beginning of this chapter follows from Theorem 1.

In the next theorem we consider a statement wherein the premise instead generalized wedge stands generalized inverted wedge. We start with similar remarks as before.

Theorem 1. *It is evident by a.7 that*

$$\bigvee_{n\in N} a_n \vee \bigwedge_{j=1}^{n} b_j \vee x_j \supset x_{n+1} \Vdash \bigvee_{n\in N} a_n \supset x_{n+1}.$$

and e.g.

$$\underset{n\in N}{\forall} a_n \vee \bigwedge_{j=1}^{n} (b_j \supset x_j) \supset x_{n+1} \ \Vdash \ \underset{n\in N}{\forall} a_n \supset x_{n+1}.$$

We suppose that the set of premises is semantically consistent. By a.4 we see that if $p \supset x_{n+1}$ then by the rule of syllogism $p \wedge q \supset x_{n+1}$. Therefore the conclusion would be better if less conjunction appears in the antecedent of consequence.

Theorem 2. *If*

h.n $\quad a_n \vee \bigwedge_{j=1}^{n} (b_j \vee x_j) \supset x_{n+1}$

has logical value truth for every $n \in N$, *then*

c.n $\quad a_n \vee (b_1 \vee x_1) \supset x_{n+1} \quad n = 1, 2, ..$

is also true.

Proof.

r_{syl} f.3 $p/a_1 \vee (b_1 \vee x_1)$ – h.1 – c.1.

The theorem holds for $n = 1$. Suppose *c.n* is true for $n = 1, 2, \ldots, k$ then the proof of this yields $c.k+1$.

r_{syl} $a.8\, p/a_1,\ q/b_1 \vee x_1$ – h.1 – $a.8\, p/b_2,\ q/x_2$ – 1.

1. $\quad b_1 \vee x_1 \supset b_2 \vee x_2$

f.10 $p/b_1 \vee x_1,\ q/b_2 \vee x_2,\ s/b_1 \vee x_1,\ t/b_1 \vee x_1$ °C 1. – °C f.3 $p/b_1 \vee x_1$ – 2.

2. $\quad (b_1 \vee x_1) \wedge (b_1 \vee x_1) \supset (b_2 \vee x_2) \wedge (b_1 \vee x_1)$

r_{syl} f.9 $p/b_1 \vee x_1$ – 2. – 3.

3. $\quad b_1 \vee x_1 \supset (b_2 \vee x_2) \wedge (b_1 \vee x_1)$

f.1 $p/b_1 \vee x_1,\ q/(b_2 \vee x_2) \wedge (b_1 \vee x_1),\ s/a_2$ °C 3. – 4.

4. $\quad a_2 \vee (b_1 \vee x_1) \supset a_2 \vee (b_2 \vee x_2) \wedge (b_1 \vee x_1)$

r_{syl} 4. – h.2 – c.2

r_{syl} $a.8\, p/a_2,\ q/b_1 \vee x_1$ – 4. – h.2 – $a.8\, p/b_3,\ q/x_3$ – 5.

5. $\quad b_1 \vee x_1 \supset b_3 \vee x_3$

f.10 $p/b_1 \vee x_1,\ q/b_3 \vee x_3,\ s/b_1 \vee x_1,\ t/(b_2 \vee x_2) \wedge (b_1 \vee x_1)$

°C 5. – °C 3. – 6.

6. $\quad (b_1 \vee x_1) \wedge (b_1 \vee x_1) \supset (b_3 \vee x_3) \wedge [(b_2 \vee x_2) \wedge (b_1 \vee x_1)]$

r_{syl} f.9 $p/b_1 \vee x_1$ – 6. – 7.

7. $$b_1 \vee x_1 \supset \bigwedge_{j=1}^{3} b_j \vee x_j.$$

Following this way we obtain

$$b_1 \vee x_1 \supset a_k \vee (b_1 \vee x_1) \supset x_{k+1} \supset b_{k+1} \vee x_{k+1}$$

and similarly as we obtain 3. and 7. applying f.9, f.10 we get

8. $$b_1 \vee x_1 \supset \bigwedge_{j=1}^{k+1} b_j \vee x_j.$$

Now

$$\text{f.1}\ p/b_1 \vee x_1,\ q\Big/\bigwedge_{j=1}^{k+1} b_j \vee x_j,\ s/a_{k+1}{}^\circ\text{C } 8. - 9.$$

9. $$a_{k+1} \vee (b_1 \vee x_1) \supset a_{k+1} \vee \bigwedge_{j=1}^{k+1} b_j \vee x_j.$$

Hence

$$r_{\text{syl}}\ 9. - \text{h.k} + 1 - \text{c.k} + 1$$

proves that the theorem holds for $k+1$. By the induction argument theorem is proved.

Remark. By a.7 p/a_n, $q/b_1 \vee x_1 - a_n \supset a_n \vee (b_1 \vee x_1)$. Hence by Theorem 2 we get $a_n \supset x_{n+1}$ what we have noticed before this theorem. Applying f.10 we have

$$\text{f.10}\ p/b_2,\ q/b_2 \vee x_2,\ s/b_1,\ t/b_1 \vee x_1{}^\circ\text{C a.7}\ p/b_2,\ q/x_2$$
$$-{}^\circ\text{C a.7}\ p/b_1,\ q/x_1 - 1.$$

1. $$b_2 \wedge b_1 \supset (b_2 \vee x_2) \wedge (b_1 \vee x_1).$$

Continuing this reasoning we obtain

2. $$\bigwedge_{j=1}^{n} b_j \supset \bigwedge_{j=1}^{n} b_j \vee x_j.$$

From there

$$\text{f.1}\ p\Big/\bigwedge_{j=1}^{n} b_j,\ q\Big/\bigwedge_{j=1}^{n} b_j \vee x_j,\ s/a_n{}^\circ\text{C } 2. - 3.$$

3. $$a_n \vee \bigwedge_{j=1}^{n} b_j \supset a_n \vee \bigwedge_{j=1}^{n} b_j \vee x_1$$

$$r_{\text{syl}}\ 3. - \text{h.n} - 4.$$

4. $$a_n \vee \bigwedge_{j=1}^{n} b_j \supset x_{n+1}.$$

The conclusion we have just proved is different from the one given in Theorem 2.

In a similar way we can prove

Theorem 3. *If*

h.n $$a_n \wedge \bigwedge_{j=1}^{n} b_j \vee x_j \supset x_{n+1}$$

has logical value truth for every $n \in N$ then

c.1 $a_1 \wedge (b_1 \vee x_1) \supset x_2$

c.n $a_n \wedge \left[(b_1 \vee x_1) \wedge \bigwedge_{j=1}^{n-1} (b_{j+1} \vee a_j) \right] \supset x_{n+1}$

for $n = 2, 3, \ldots$

is also true.

Some of the theorems we can construct in such a manner have their premises which look like hypothesis in the Gronwall inequality while the conclusion has more different forms. We present as an example one such statement

Theorem 4.

$$\bigvee_{n \in N} \bigwedge_{j=1}^{n} b_j \vee \sim x_j \supset x_{n+1} \Vdash \bigvee_{n \in N} x_{n+1} \vee \sim \bigwedge_{j=1}^{n} b_j.$$

We omit the proof of this theorem.

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