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Maximal monotone differential inclusions with memory

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Abstract. In this paper we study maximal monotone differential inclusions with memory. First we establish two existence theorems; one involving convex-valued orientor fields and the other nonconvex valued ones. Then we examine the dependence of the solution set on the data that determine it. Finally we prove a relaxation theorem.

Keywords. Maximal monotone operator; resolvent; resolvent convergence topology; selection theorem; relaxation.

1. Introduction

In this paper we examine maximal monotone differential inclusions with memory, defined in $\mathbb{R}^N$. First we consider the existence problem, and we prove two such theorems. One with a convex-valued orientor field and the other with a nonconvex valued one. Then we examine the dependence of the solution set on the data that determine it; i.e., the maximal monotone operator, the orientor field and the past history information. More precisely, we consider a parametrized family of problems, where all the above data depend on the parameter, and we examine how the solution set responds to variations of the parameter. Finally we prove a "relaxation" result, which says that under reasonable hypotheses on the orientor field, the solution set of the "nonconvex problem" is dense in that of the "convex problem". Our formulation of the problem is general enough to incorporate subdifferential systems. Among them of particular interest, because of their diverse applications, are those for which the maximal monotone operator $A = \partial\delta_K$, with δ_K being the indicator function of a nonempty, closed and convex subset K of $\mathbb{R}^N$(i.e., $\delta_K(x) = 0$ if $x \in K$ and $\delta_K(x) = +\infty$ if $x \notin K$) and $\partial\delta_K(\cdot)$ denotes its subdifferential in the sense of convex analysis. It is well-known (see for example Aubin-Cellina [2]), that $\partial\delta_K(x) = N_K(x)$ for every $x \in K$, with $N_K(x)$ being the normal cone to the set K at x. In this case, the corresponding "differential inclusion" is also called "differential variational inequality" and appears in mathematical economics, in the study of dynamic allocation processes (see Aubin-Cellina [2], Henry [10] and Stacchetti [18]) and in theoretical mechanics in the study of unilateral processes (see Moreau [14]). Our system has a memory feature, since the derivative of the state depends on the past history of it. We should mention, that this memory feature of our system, arises in the so-called "absorption lag" dynamic economic models. It signifies that the growth rate $\dot{x}(t)$ of the capital depends on the past history $x_t(\cdot)$ of the capital. Finally given that every control system, after "deparametrization" (union over all admissible controls of all vector fields), can be described by a differential inclusion, the systems studied in this paper incorporate hereditary control systems, monitored by maximal monotone, multivalued in general operators. Our results also extend the works on differential inclusions done by Aubin-Cellina [2], Bressan [5] and Cellina-Marchi [7].

2. Preliminaries

Let (Ω, Σ) be a measurable space and let X be a separable Banach space. Throughout this paper, we will be using the following notation: $P_{f(c)}(X) = \{A \subseteq X\text{: nonempty, closed (convex)}\}$. A multifunction (set-valued function) $F:\Omega \to P_f(X)$, is said to be measurable, if for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\|: z \in F(\omega)\}$ is measurable. Other equivalent definitions of measurability of a $P_f(X)$-valued multifunction can be found in Wagner [20] (see theorem 4.2). We will say that $G:\Omega \to 2^X\setminus\{\varnothing\}$ is graph-measurable, provided that $\mathrm{Gr}G = \{(\omega, x) \in \Omega \times X: x \in F(\omega)\} \in \Sigma x B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability, while the converse is true if there exists a σ-finite measure $\mu(\cdot)$ on Σ, with respect to which Σ is complete (see Wagner [20], theorem 4.2). Now let (Ω, Σ, μ) be a finite measure space and $F:\Omega \to 2^X\setminus\{\emptyset\}$. By S^1_F we will denote the set of integrable selectors of $F(\cdot)$; i.e., $S^1_F = \{f \in L^1(X): f(\omega) \in F(\omega) \mu - \text{a.e.}\}$. This set may be empty. For a graph measurable multifunction $F(\cdot)$, it is nonempty if and only if $\omega \to \inf\{\|z\|: z \in F(\omega)\} \in L^1_+$. In particular, this is the case if $\omega \to |F(\omega)| = \sup\{\|z\|: z \in F(\omega)\} \in L^1_+$. Such multifunctions are called "integrably bounded".

Next let Y, Z be Hausdorff topological spaces and $G: Y \to 2^Z\setminus\{\emptyset\}$ a multifunction. We will say that $G(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)), if for every $U \subseteq Z$ open $G^+(U) = \{y \in Y: G(y) \subseteq U\}$ (resp. $G^-(U) = \{y \in Y: G(y) \cap U \neq \emptyset\}$) is open in Y. A multifunction $G(\cdot)$ which is both u.s.c. and l.s.c. is said to be continuous. So a continuous multifunction $G(\cdot)$, is one that is continuous from Y into $2^Z\setminus\{\emptyset\}$ equipped with the Vietoris topology (see Klein-Thompson [12]). If Z is a metric space, then on $P_f(Z)$ we can define a (generalized) metric, known in the literature as the Hausdorff metric, by setting $h(A, B) = \max[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)]$, for every $A, B \in P_f(Z)$. If Z is complete, then so is the metric space $(P_f(Z), h)$. A multifunction $F:\Omega \to P_f(Z)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from Y into the metric space $(P_f(Z), h)$. If the multifunction has nonempty compact values, then continuity and h-continuity coincide. This follows from the fact that on the collection of nonempty compact sets of a metric space, the Vietoris and Hausdorff topologies coincide (see Klein-Thompson [12], corollary 4.2.3, p. 41).

Let V be a Banach space and $\{A_n, A\}_{n \geq 1} \subseteq 2^V\setminus\{\emptyset\}$. Denote by s- the strong topology on V and by w- the weak topology. We define:

$$s\text{-}\underline{\lim}\, A_n = \{x \in V: x = s\text{-}\lim x_n, x_n \in A_n, n \geq 1\} = \{x \in V: \lim d(x, A_n) = 0\}$$

$$s\text{-}\overline{\lim}\, A_n = \{x \in V: x = s\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}$$

$$= \{x \in V: \underline{\lim}\, d(x, A_n) = 0\}$$

and

$$w\text{-}\overline{\lim}\, A_n = \{x \in V: x = w\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}.$$

It is clear from the above definitions, that we always have $s\text{-}\underline{\lim}\, A_n \subseteq s\text{-}\overline{\lim}\, A_n \subseteq w\text{-}\overline{\lim}\, A_n$. If $s\text{-}\underline{\lim}\, A_n = w\text{-}\overline{\lim}\, A_n = A$, then we say that the A_n's converge to A in the Kuratowski-Mosco sense, denoted by $A_n \xrightarrow{K-M} A$ (see Mosco [15]). If $\dim V < \infty$, then the weak and strong topologies coincide, and so we recover the well-known Kuratowski mode of set convergence (see Kuratowski [13]). If $s\text{-}\underline{\lim}\, A_n = A = s\text{-}\overline{\lim}\, A_n$, then we write $A_n \xrightarrow{K} A$.

Let us also recall some basic facts about maximal monotone operators. So let H be a Hilbert space. An operator $A:D(A)\subseteq H \to 2^H$ is said to be "monotone", if and only if $(x-x', y-y') \geqslant 0$ for all $[x,y],[x',y'] \in \mathrm{Gr}\, A$ (here $(\cdot,\cdot)$ denotes the inner product in H). It is said to be maximal monotone if and only if $(x-v, y-w) \geqslant 0$ for all $[x,y]\in \mathrm{Gr}\, A$, which implies that $w \in Av$ (i.e., the graph of A is not properly included in any other monotone subset of $H \times H$). From a well-known theorem of Minty, we have that $A(\cdot)$ is maximal monotone if and only if from some $\lambda > 0$, $R(I+\lambda A) = H$. Then for every $\lambda > 0$, $J_\lambda = (I+\lambda A)^{-1}: R(I+\lambda A) = H \to D(A)$, and is called the "resolvent of A". The resolvent $J_\lambda(\cdot)$ is nonexpansive and $J_\lambda x \xrightarrow{s} x$ as $\lambda \to 0^+$ for each $x \in D(A)$. Let $\mathscr{M}$ be the set of all maximal monotone operators in H. The topology of R-convergence on $\mathscr{M}$, is the weakest topology, that makes continuous the maps $\hat{J}_{\lambda,x}: \mathscr{M} \to H$ for every $\lambda > 0$ and $x \in H$, where $\hat{J}_{\lambda,x}(A) = (I+\lambda A)^{-1}x$. We will denote by $\mathscr{M}_R$ (or $\mathscr{M}_R(H)$), the set $\mathscr{M}$ equipped with the topology of R-convergence. If H is separable, then $\mathscr{M}_R$ is a Polish space (i.e., a separable, metrizable, complete space). Furthermore, we know that $A_n \xrightarrow{\mathscr{M}_R} A$ if and only if $\mathrm{Gr}\, A_n \xrightarrow{K-M} \mathrm{Gr}\, A$. For further details, we refer to Attouch [1]. Finally, note that if $A(\cdot)$ is maximal monotone, for every $x \in D(A)$, Ax is closed and convex. Hence for every $x \in D(A)$, Ax contains an element of minimum norm (the projection of the origin on Ax). This unique element is denoted by A^0x. Thus we have $A^0x \in Ax$ and $\|A^0x\| = \inf\{\|y\| : y \in Ax\}$. The single-valued operator $A^0: D(A) \to H$, is called the "minimal section" of A.

3. Existence theorem

Let $b, r > 0$ and set $\hat{T} = [-r, b]$, $T_0 = [-r, 0]$ and $T = [0, b]$. We will be studying the following maximal monotone differential inclusion with memory:

$$\left.\begin{aligned} -\dot{x}(t) &\in Ax(t) + F(t, x_t) \text{ a.e. on } T \\ x(v) &= \varphi(v) \quad v \in T_0. \end{aligned}\right\} \qquad (*)$$

Here $x_t \in C(T_0, \mathbb{R}^N)$ and is defined by $x_t(s) = x(t+s)$. So $x_t(\cdot)$ gives us the history of state $x(\cdot)$ from $t-r$ up to the present time t.

In this section we present two existence results concerning $(*)$. The first assumes that the multivalued perturbation $F(t, x_t)$ is convex valued, while the second that it is nonconvex valued.

For the first existence theorem, we will need the following hypotheses on the data.

$\underline{H(A)}$: $A: D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone operator.

$\underline{H(F)}$: $F: T \times C(T_0, \mathbb{R}^N) \to P_{fc}(\mathbb{R}^N)$ is a multifunction s.t.

(1) $t \to F(t, y)$ is measurable,

(2) $y \to F(t, y)$ is u.s.c.,

(3) $|F(t,y)| = \sup\{\|v\| : v \in F(t,y)\} \leqslant a(t) + b(t)\|y\|_\infty$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

$\underline{H(\varphi)}$: $\varphi \in C(T_0, \mathbb{R}^N)$, $\varphi(0) \in \overline{D(A)}$.

By a solution of $(*)$, we understand a function $x \in C(\hat{T}, \mathbb{R}^N)$ s.t. $x(v) = \varphi(v)$ for $v \in T_0$ and $x|_T$ is a solution of the initial value problem $-\dot{x}(t) \in Ax(t) + F(t, x_t)$ a.e., $x(0) = \varphi(0)$ (i.e., $x: T \to \mathbb{R}^N$ is absolutely continuous and there exists $f \in S^1_{F(\cdot, x.)}$ *s.t.* $-\dot{x}(t) \in Ax(t) + f(t)$ a.e., $x(0) = \varphi(0)$).

Theorem 3.1. *If hypotheses $H(A), H(F)$ and $H(\varphi)$ hold, then $(*)$ admits a solution and the solution set is compact in $C(\hat{T}, \mathbb{R}^N)$.*

Proof. First we will obtain an a priori, uniform bound for the solutions of $(*)$. So let $x(\cdot) \in C(\hat{T}, \mathbb{R}^N)$ be such a solution. Then by definition we can find $f(\cdot) \in L^1(T, \mathbb{R}^N)$, $f(t) \in F(t, x_t)$ a.e. s.t. $-\dot{x}(t) \in Ax(t) + f(t)$ a.e. From Benilan's inequality (see for example Vrabie [19], corollary 1.7.1, p. 35), we have that

$$\|x(t)\| \leqslant \|S(t)\varphi(0)\| + \int_0^t \|f(s) \mathrm{d}s,$$

where $\{S(t)\}_{t \in T}$ is the nonlinear semigroup of contractions generated by the maximal operator $A(\cdot)$. Recalling that $t \to \|S(t)\varphi(0)\|$ is continuous on T, we can find $M > 0$ s.t. $\|S(t)\varphi(0)\| \leqslant M$ for all $t \in T$. Hence using growth hypothesis $H(F)(3)$, we get

$$\|x(t)\| \leqslant M + \int_0^t (a(s) + b(s)\|x_s\|_\infty) \mathrm{d}s.$$

Let $h(t) = \|x_t\|_\infty$. Then clearly $h(\cdot) \in C(T, \mathbb{R}^N)$ and we have

$$h(t) \leqslant \hat{M} + \int_0^t (a(s) + b(s)h(s)) \mathrm{d}s, \quad t \in T,$$

with $\hat{M} = \max[M, \|\varphi\|_\infty]$. Invoking Gronwall's inequality we get $M_1 > 0$ s.t. for all $t \in T$ we have

$$h(t) = \|x_t\|_\infty \leqslant M_1.$$

Then consider the following, modified orientor field

$$\hat{F}(t, y) = \begin{cases} F(t, y) \text{ if } \|y\|_\infty \leqslant M_1 \\ F\left(t, \dfrac{M \hat{y}}{\|y\|_\infty}\right) \text{ if } \|y\|_\infty > M_1. \end{cases}$$

Note that $\hat{F}(t, y) = F(t, p_{M_1}(y))$, where $p_{M_1}(\cdot)$ is the M_1-radial retraction on the Banach space $C(T_0, \mathbb{R}^N)$. Hence, $t \to \hat{F}(t, y)$ is measurable, while since $p_{M_1}(\cdot)$ is Lipschitz continuous, $y \to \hat{F}(t, y)$ is u.s.c. (see Klein-Thompson [12], theorem 7.3.11, p. 87). Furthermore we have:

$$|\hat{F}(t, y)| = \sup\{\|v\| : v \in \hat{F}(t, y)\} \leqslant a(t) + b(t)M_1 = \psi(t) \text{ a.e.}$$

with $\psi(\cdot) \in L^1_+$. Then we consider problem $(*)$ with $\hat{F}(t, y)$ instead of $F(t, y)$. Set $V = \{g \in L^1(T, \mathbb{R}^N): \|g(t)\| \leqslant \psi(t) \text{ a.e.}\}$. From the Dunford-Pettis theorem, we know that V is sequentially weakly compact. In what follows V will be equipped with the relative weak-$L^1(T, \mathbb{R}^N)$ topology. Let $R: V \to P_{fc}(V)$ be the multifunction defined by

$R(g)=S^1_{\hat{F}(\cdot,p(g).)}$, where $p: L^1(T,\mathbb{R}^N)\to C(T,\mathbb{R}^N)$ is the map that to each $g\in L^1(T,\mathbb{R}^N)$ assigns the unique solution of $-\dot{x}(t)\in Ax(t)+g(t)$a.e., $x(0)=\varphi(0)$ (its existence is guaranteed by the Benilan-Brezis theorem; see Vrabie [19], theorem 1.9.1, p. 41). We claim that $R(\cdot)$ is an u.s.c. multifunction. Given that V equipped with the relative weak-$L^1(T,\mathbb{R}^N)$ topology is compact, metrizable (see Dunford-Schwartz [8], theorem 3, p. 434), it is enough to show that $\mathrm{Gr}R$ is sequentially closed in $V\times V$ (see Klein-Thompson [12], theorem 7.1.16, p. 78). To this end, let $[g_n,f_n]\in\mathrm{Gr}R n\geqslant 1$ and assume that $[g_n,f_n]\to[g,f]$ in $V\times V$. From corollary 2.3.1, p. 67 of Vrabie [19], we know that $p(\cdot)$ is sequentially continuous from $L^1(T,\mathbb{R}^N)$ equipped with the weak topology into $C(T,\mathbb{R}^N)$ with the strong topology. So if we define $\hat{p}(g)(\cdot)\in C(\hat{T},\mathbb{R}^N)$ by setting $\hat{p}(g)(t)=p(g)(t)$ for $t\in T$ and $\hat{p}(g)(v)=\varphi(v)$ for $v\in T_0$, then we have $\hat{p}(g_n)_t\xrightarrow{s}\hat{p}(g)_t$ in $C(T_0,\mathbb{R}^N)$ for all $t\in T$. Hence applying theorem 4.2. of [16] we have

$$f\in w\text{-}\overline{\lim}\, S^1_{\hat{F}(\cdot,\hat{p}(g).)}\subseteq S^1_{\hat{F}(\cdot,\hat{p}(g)\cdot)}$$

$$\Rightarrow [g,f]\in\mathrm{Gr}\,R$$

$\Rightarrow R(\cdot)$ is indeed u.s.c. as claimed.

Since $R(\cdot)$ is closed, convex valued we can apply the Kakutani-KyFan fixed point theorem to get $g\in R(g)$. Let $\hat{p}(g)(\cdot)=x(\cdot)\in C(\hat{T},\mathbb{R}^N)$. Then this function solves $(*)$ with $\hat{F}(t,y)$ instead of $F(t,y)$. But as in the beginning of the proof, via Gronwall's inequality (see Vrabie [19], p. 3) we can get that $\|x_t\|_\infty\leqslant M_1$. Hence $\hat{F}(t,x_t)=F(t,x_t)$. Thus $x(\cdot)$ solves $(*)$.

Since the solution set of $(*)$ lies in $\hat{p}(V)$ and the latter is compact in $C(\hat{T},\mathbb{R}^N)$, to establish the compactness of the solution set, it suffices to show that it is closed. So let $\{x_n\}_{n\geqslant 1}\subseteq C(\hat{T},\mathbb{R}^N)$ be solutions of $(*)$ and assume that $x_n\to x$ in $C(\hat{T},\mathbb{R}^N)$. Then $x_n=\hat{p}(f_n)$ for some $f_n\in S^1_{F(\cdot,(x_n).)}$. Because of hypothesis $H(F)(3)$ and the Dunford-Pettis theorem, we have that $\{f_n\}_{n\geqslant 1}$ is relatively sequentially w-compact in $L^1(T,\mathbb{R}^N)$. So by passing to a subsequence if necessary, we may assume that $f_n\xrightarrow{w}f$ in $L^1(T,\mathbb{R}^N)$ and so $\hat{p}(f_n)\to\hat{p}(f)=x$ in $C(\hat{T},\mathbb{R}^N)$. Also from theorem 4.2 of [16], we have that $f\in S^1_{\hat{F}(\cdot,x.)}$. Hence $x(\cdot)=\hat{p}(f)(\cdot)$ is also a solution of $(*)$, establishing the compactness in $C(\hat{T},\mathbb{R}^N)$ of $(*)$. **Q.E.D.**

We can also have an existence result for the case where the orientor field $F(\cdot,\cdot)$ is not necessarily convex-valued. For this we will need the following hypotheses on the $F(t,y)$.

<u>$H(F)_1$</u>: $F: T\times C(T_0,\mathbb{R}^N)\to P_f(\mathbb{R}^N)$ is a multifunction s.t.
- (1) $(t,y)\to F(t,y)$ is graph measurable,
- (2) $y\to F(t,y)$ is l.s.c.,
- (3) $|F(t,y)|\leqslant a(t)+b(t)\|y\|_\infty$ a.e., with $a(\cdot)$, $b(\cdot)\in L^1_+$.

Theorem 3.2. *If hypotheses $H(A)$, $H(F)_1$, and $H(\varphi)$ hold, then $(*)$ admits a solution.*

Proof. Let $V\subseteq L^1(T,\mathbb{R}^N)$ and $\hat{F}(t,y)$ be defined as in the proof of theorem 3.1. Let $K=\hat{p}(V)$ (i.e., $K=p(V)$ on T and $K=\{\varphi\}$ on T_0). From the continuity property of $\hat{p}(\cdot)$ (see the proof of theorem 3.1), we have that K is compact in $C(\hat{T},\mathbb{R}^N)$. Hence by Mazur's theorem so is $\hat{K}=\overline{\mathrm{conv}}K$. Let $R:\hat{K}\to P_f(L^1(T,\mathbb{R}^N))$ be defined by $R(y)=S^1_{\hat{F}(\cdot,y.)}$. Using theorem 4.1 of [16], we get that $R(\cdot)$ is l.s.c. Applying Fryszkowski's selection theorem [9], we get $r:\hat{K}\to L^1(T,\mathbb{R}^N)$ continuous *s.t.* $r(y)\in R(y)$ for all $y\in\hat{K}$. Then let $q:\hat{K}\to\hat{K}$ be defined by $q=\hat{p}\circ r$. Clearly $q(\cdot)$ is continuous. So applying Schauder's fixed point theorem, we get $x\in\hat{K}$ s.t. $x=q(x)$. Again through the definition of $\hat{F}(t,y)$ and Gronwall's inequality, we can check that $\|x_t\|_\infty\leqslant M_1\Rightarrow\hat{F}(t,x_t)=F(t,x_t)\Rightarrow$ $x(\cdot)\in C(\hat{T},\mathbb{R}^N)$ solves $(*)$. **Q.E.D.**

Remark. Theorem 3.2 above, improves the result of Cellina-Marchi [7], who considered memoryless systems and assumed that the orientor field was h-continuous in both variables, also extends the existence result of Bressan [5], who also considered memoryless systems and assumed that $A\equiv 0$.

4. A continuous dependence result

In this section, we investigate the dependence of the solution set of $(*)$ on the data that determine it; namely the maximal monotone operator, the orientor field and the function φ.

So let E be a metric space. We consider the following family of problems parametrized by elements in E:

$$\left.\begin{array}{l} -\dot{x}(t)\in A(r)x(t)+F(t,x_t,r) \text{ a.e. on } T \\ x(v)=\varphi(r)(v)\ \ v\in T_0,\ r\in E. \end{array}\right\}\ (*)_r$$

We denote the solution set of $(*)_r$ by $P(r)\subseteq C(\hat{T},\mathbb{R}^N)$. We want to examine the dependence of $P(\cdot)$ on $r\in E$.

To this end we will need the following two auxiliary results. Recall that if $A:D(A)\subseteq\mathbb{R}^N\to 2^{\mathbb{R}^N}$ is a maximal monotone operator, then we can define the realization of A on $L^2(T,\mathbb{R}^N)$, $\hat{A}:D(\hat{A})\subseteq L^2(T,\mathbb{R}^N)\to 2^{L^2(T,\mathbb{R}^N)}$ by

$$\hat{A}x=\{y\in L^2(T,\mathbb{R}^N):y(t)\in Ax(t) \text{ a.e. on } T\}$$

for each $x\in D(A)=\{v\in L^2(T,\mathbb{R}^N):v(t)\in D(A)$ a.e. and there exists $\omega\in L^2(T,\mathbb{R}^N)$ s.t. $\omega(t)\in Av(t)$ a.e.$\}$. It is well-known and easy to prove that the realization $\hat{A}(\cdot)$ is maximal monotone too.

Lemma 4.1. *If $A:E\to\mathcal{M}_R(\mathbb{R}^N)$ is continuous, then so is $\hat{A}:E\to\mathcal{M}_R(L^2(T,\mathbb{R}^N))$.*

Proof. Let $r_n\to r$ in E and let $s(t)=\Sigma_{k=1}^m\chi_{C_k}\omega_k$, with $C_k\in\Sigma$, $\omega_k\in\mathbb{R}^N$ (a simple function). Then since by hypothesis $A(r_n)\to A(r)$ in $\mathcal{M}_r(\mathbb{R}^N)$, we have

$$J_\lambda^{A(r_n)}\omega_k\to J_\lambda^{A(r)}\omega_k \text{ as } n\to\infty$$

for all $k\in\{1,\dots,m\}$, $\lambda>0$. Thus

$$\sum_{k=1}^m\chi_{C_k}(t)J_\lambda^{A(r_n)}\omega_k\to\sum_{k=1}^m\chi_{C_k}(t)J_\lambda^{A(r)}\omega_k \text{ as } n\to\infty$$

for all $t\in T$. From this we deduce that

$$J_\lambda^{\hat{A}(r_n)}(s)\xrightarrow{s}J_\lambda^{\hat{A}(r)}(s) \text{ as } n\to\infty \text{ in } L^2(T,\mathbb{R}^N)$$

for all $\lambda > 0$. Since $s(\cdot)$ was an arbitrary simple function, simple functions are dense in $L^2(T, \mathbb{R}^N)$ and the resolvent operator is nonexpansive, we get

$$J_\lambda^{\hat{A}(r_n)}(x) \xrightarrow{s} J_\lambda^{\hat{A}(r)}(x) \text{ as } n \to \infty \text{ in } L^2(T, \mathbb{R}^N)$$

for all $x \in L^2(T, \mathbb{R}^N)$ and all $\lambda > 0 \Rightarrow \hat{A}(r_n) \to \hat{A}(r)$ in $\mathcal{M}_R(L^2(T, \mathbb{R}^N))$ (see § 2) $\Rightarrow \hat{A}: E \to \mathcal{M}_R(L^2(T, \mathbb{R}^N))$ is indeed continuous. **Q.E.D.**

The second auxiliary result that we will need is the following:

Lemma 4.2. *If X is a Banach space, $\{A_n\}_{n \geq 1} \subseteq P_f(X)$, $A_n \subseteq K$ for all $n \geq 1$ with $K \subseteq X$ compact, and $A_n \xrightarrow{K} A$ as $n \to \infty$, then $A_n \xrightarrow{h} A$ as $n \to \infty$.*

Proof. Let $a_n \in A_n$, $n \geq 1$ s.t. $d(a_n, A) = \sup_{b \in A_n} d(b, A)$. It exists since $A_n, n \geq 1$ is compact (being a closed subset of the compact set K). Note that $\{a_n\}_{n \geq 1} \subseteq K$. So by passing to a subsequence if necessary, we may assume that $a_n \to a$. Because $A_n \xrightarrow{K} A$, we have $a \in A$. Then $d(a_n, A) \to d(a, A) = 0 \Rightarrow \sup_{b \in A_n} d(b, A) \to 0$ as $n \to \infty$.

Similarly, let $a_n \in A$ $n \geq 1$ s.t. $\sup_{b \in A} d(b, A_n) = d(a_n, A_n)$. Since A is compact and $\{a_n\}_{n \geq 1} \subseteq A$, we may assume that $a_n \to a \in A$. Then we have

$$|d(a_n, A_n) - d(a, A_n)| \leq \|a_n - a\| \to 0.$$

But since $A_n \xrightarrow{K} A$, we have $d(a, A_n) \to 0$. So $d(a_n, A_n) \to 0$ as $n \to \infty \Rightarrow \sup_{b \in A} d(b, A_n) \to 0 \Rightarrow A_n \xrightarrow{h} A$. **Q.E.D.**

To prove our continuous dependence result, we will need the following hypotheses on the data.

$\underline{H(A)_1}$: $A: E \to \mathcal{M}_R(\mathbb{R}^N)$ is continuous, for every $r \in E$ $D(A(r))$ is closed and $A^0(r)$ is bounded on compact subsets of $D(A(r))$, uniformly in $r \in B \subseteq E$ nonempty, compact.

Remark. This hypothesis is clearly satisfied if $A(r) = \partial \delta_{K(r)}$ with $K: E \to P_{fc}(\mathbb{R}^N)$ continuous, or if for every $r \in E$, $D(A(r)) = \mathbb{R}^N$ and $r \to A^0(r)$ is bounded on compact sets.

$\underline{H(F)_2}$: $F: T \times C(T_0, \mathbb{R}^N) \times E \to P_{fc}(\mathbb{R}^N)$ is a multifunctions s.t.

(1) $t \to F(t, y, r)$ is measurable,
(2) $(y, r) \to F(t, y, r)$ is continuous,
(3) $h(F(t, y, r), F(t, y', r)) \leq \eta(t) \|y - y'\|_\infty$ a.e. with $\eta(\cdot) \in L^1_+$,
(4) $|F(t, y, r)| \leq a_B(t) + b_B(t) \|y\|_\infty$ with $a_B(\cdot), b_B(\cdot) \in L^2_+$ and for all $r \in B \subseteq E$ nonempty, compact.

$\underline{H(\varphi)_1}$: $\varphi: E \to C(T_0, \mathbb{R}^N)$ is continuous and for all $r \in E$ $\varphi(r)(0) \in \overline{D(A(r))}$.

Theorem 4.3. *If hypotheses $H(A)_1, H(F)_2$ and $H(\varphi)_1$ hold, then $r \to P(r)$ from E into the nonempty compact subsets of $C(\hat{T}, \mathbb{R}^N)$ is continuous and h-continuous.*

Remark. The hypotheses of this theorem and theorem 3.1 guarantee that for every $r \in E$, $P(r)$ is nonempty and compact in $C(\hat{T}, \mathbb{R}^N)$.

Proof. Let $r_n \to r$ in E and let $x \in s\text{-}\overline{\lim}\, P(r_n)$. Then by denoting subsequences with the same index as the original sequences for economy in the notation, we know that there exist $x_n \in P(r_n)$ $n \geqslant 1$ s.t. $x_n \to x$ in $C(\hat{T}, \mathbb{R}^N)$. Then by definition

$$-\dot{x}_n(t) \in A(r_n)x_n(t) + f_n(t) \text{ a.e. on } T$$
$$x_n(v) = \varphi(r_n)(v) \quad v \in T_0,$$

where $f_n \in L^1(T, \mathbb{R}^N)$ and $f_n(t) \in F(t, (x_n)_t, r_n)$ a.e.. Then from the Benilan-Brezis theorem (see Vrabie [19], theorem 1.9.1, p. 41), we have

$$\dot{x}_n(t) = [f_n(t) - A(r_n)x_n(t)]^0 \text{ a.e.}$$
$$\Rightarrow \dot{x}_n(t) = f_n(t) - A(r_n)^0 x_n(t) \text{ a.e.}$$

But from hypothesis $H(A)_1$, we know that there exists $M_2 > 0$ s.t. for all $n \geqslant 1$ and all $t \in T$, we have $\|A(r_n)^0 x_n(t)\| \leqslant M_2$. Hence we have:

$$\|\dot{x}_n(t)\| \leqslant \|f_n(t)\| + M_2$$
$$\leqslant a_B(t) + b_B(t)M_1 + M_2 = h_B(t) \text{ a.e.}$$

with $h_B(\cdot) \in L^2_+$ and $B = \{r_n, r\}_{n \geqslant 1} \subseteq E$ (note that the bound $M_1 > 0$ derived in the proof of theorem 3.1 holds for all $x_n(\cdot)$ $n \geqslant 1$, because of hypothesis $H(F)_2$). From this last inequality, we deduce that $\{\dot{x}_n\}_{n \geqslant 1} \subseteq L^2(T, \mathbb{R}^N)$ is relatively sequentially w-compact. Also as in the proof of theorem 3.1, we can get that $\{x_n\}_{n \geqslant 1} \subseteq \hat{p}(V)$, where $V = \{g \in L^2(T, \mathbb{R}^N)\colon \|g(t)\| \leqslant \psi_B(t) = a_B(t) + b_B(t)M_1 \text{ a.e.}\}$. Since $\hat{p}(V)$ is compact in $C(T, \mathbb{R}^N)$ (see the proof of theorem 3.1), we get that $\{x_n\}_{n \geqslant 1} \subseteq C(T, \mathbb{R}^N)$ is relatively compact. Finally note that for $n \geqslant 1$, $\|f_n(t)\| \leqslant \psi_B(t)$ a.e. Hence $\{f_n\}_{n \geqslant 1}$ is relatively sequentially w-compact in $L^2(T, \mathbb{R}^N)$. Thus by passing to an appropriate subsequence if necessary, we may assume that $x_n \xrightarrow{s} x$ in $C(\hat{T}, \mathbb{R}^N)$, $\dot{x}_n \xrightarrow{w} v$ *in* $L^2(T, \mathbb{R}^N)$ and $f_n \xrightarrow{w} f$ in $L^2(T, \mathbb{R}^N)$. Clearly $v = \dot{x}$ on T. Then note that for all $n \geqslant 1$

$$[x_n, -\dot{x}_n - f_n] \in \mathrm{Gr}\, \hat{A}(r_n)$$

and

$$[x_n, -\dot{x}_n - f_n] \xrightarrow{s \times w} [x, -\dot{x} - f] \text{ in } L^2(T, \mathbb{R}^N) \times L^2(T, \mathbb{R}^N).$$

Also from lemma 4.1, we have that $\hat{A}(r_n) \to \hat{A}(r)$ in $\mathcal{M}_R(L^2(T, \mathbb{R}^N)) \Rightarrow \mathrm{Gr} \cdot \hat{A}(r_n) \xrightarrow{K-M} \mathrm{Gr}\, \hat{A}(r)$ (see Attouch [1], theorem 3.6.2, p. 365). Hence in the limit as $n \to \infty$, we get

$$[x, -\dot{x} - f] \in \mathrm{Gr}\, \hat{A}(r)$$
$$\Rightarrow -\dot{x}(t) \in Ax(t) + f(t) \text{ a.e. on } T$$

and $x(v) = \varphi(r)(v)$ $v \in T_0$ (because of hypothesis $H(\varphi_1)$).

Furthermore because of hypothesis $H(F)_2(2)$ and theorem 4.4 of [16], we have that $f(t) \in F(t, x_t, r)$ a.e. Therefore $x \in P(r)$ and so we have proved that

$$s\text{-}\overline{\lim}\, P(r_n) \subseteq P(r). \tag{1}$$

Next let $x \in P(r)$. Then by definition, we have

$$-\dot{x}(t) \in A(r)x(t) + f(t) \text{ a.e. on } T$$
$$x(v) = \varphi(r)(v) \; v \in T_0$$

with $f \in L^1(T, \mathbb{R}^N)$, $f(t) \in F(t, x_t, r)$ a.e. Let

$$m_n(t) = \operatorname{proj}[f(t); F(t, x_t, r_n)]$$

and

$$u(t, y, r_n) = \operatorname{proj}[m_n(t); F(t, y, r_n)], y \in C(T_0, \mathbb{R}^N),$$

where for every $C \in P_{fc}(\mathbb{R}^N)$, $\operatorname{proj}(\cdot; C)$ denotes the metric projection function. From theorem 4.2 of [11], we know that $m_n(\cdot)$ is measurable, while from theorem 3.33, p. 322 of Attouch [1], we have that $(t, y, r) \to u(t, y, r)$ is measurable on t, continuous in (y, r) (i.e. a Caratheodory function), hence jointly measurable. Consider the following differential inclusions $n \geq 1$:

$$\left.\begin{aligned} &-\dot{x}_n(t) \in A(r_n)x_n(t) + u(t, (x_n)_t, r_n) \text{ a.e. on } T \\ &x_n(v) = \varphi(r_n)(v) \quad v \in T_0. \end{aligned}\right\}.$$

For each $n \geq 1$, we can find a solution $x_n(\cdot) \in C(\hat{T}, \mathbb{R}^N)$ of the above problem. From the bounds obtained in the first half of the proof, we know that by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{s} \hat{x}$ in $C(\hat{T}, \mathbb{R}^N)$ and $\dot{x}_n \xrightarrow{w} \dot{\hat{x}}$ in $L^2(T, \mathbb{R}^N)$. Let $\gamma_n(\cdot)$, $\gamma(\cdot) \in L^2(T, \mathbb{R}^N)$ s.t. $\gamma_n(t) \in A(r_n)x_n(t)$ a.e., $\gamma(t) \in A(r)x(t)$ a.e. and $\gamma_n(t) = -\dot{x}_n(t) - u(t, (x_n)_t, r_n)$, $\gamma(t) = -\dot{x}(t) - f(t)$ a.e. Note that $\gamma_n \xrightarrow{w} \hat{\gamma}$ in $L^2(T, \mathbb{R}^N)$ with $\hat{\gamma}(t) = -\dot{\hat{x}}(t) - u(t, \hat{x}_t, r)$ a.e. Then we have:

$$\begin{aligned} (-\dot{x}_n(t) + \dot{x}(t), x(t) - x_n(t)) &= (\gamma_n(t) - \gamma(t), x(t) - x_n(t)) \\ &\quad + (u(t, (x_n)_t, r_n) - f(t), x(t) - x_n(t)) \text{ a.e.} \end{aligned}$$

Recalling that $\operatorname{Gr} \hat{A}(r_n) \xrightarrow{K-M} \operatorname{Gr} \hat{A}(r)$ in $L^2(T, \mathbb{R}^N) \times L^2(T, \mathbb{R}^N)$ (since $\hat{A}(r_n) \to \hat{A}(r)$ in $\mathcal{M}_R(L^2(T, \mathbb{R}^N))$), we can find $\beta_n \in L^2(T, \mathbb{R}^N)$, $\beta_n(t) \in A(r_n)x(t)$ a.e. s.t. $\beta_n \xrightarrow{w} \gamma$ in $L^2(T, \mathbb{R}^N)$. Then we have

$$\begin{aligned} (\gamma_n(t) - \gamma(t), x(t) - x_n(t)) &= (\gamma_n(t) - \beta_n(t), x(t) - x_n(t)) + (\beta_n(t) - \gamma(t), x(t) - x_n(t)) \\ &\leq (\beta_n(t) - \gamma(t), x(t) - x_n(t)) \text{ a.e.} \end{aligned}$$

the last inequality, being a consequence of the monotonicity of the operator $A(r_n)(\cdot)$. Thus we get:

$$\begin{aligned} \|x_n(t) - x(t)\|^2 &\leq 2 \int_0^t (\beta_n(s) - \gamma(s), x(s) - x_n(s)) \mathrm{d}s \\ &\quad + 2 \int_0^t (u(s, (x_n)_s, r_n) - f(s), x(s) - x_n(s)) \mathrm{d}s \\ &\leq 2 \int_0^t (\beta_n(s) - \gamma(s), x(s) - x_n(s)) \mathrm{d}s \\ &\quad + 2 \int_0^t \|u(s, (x_n)_s, r_n) - f(s)\| \cdot \|x(s) - x_n(s)\| \mathrm{d}s. \end{aligned}$$

Note that

$$\begin{aligned} \|u(s, (x_n)_s, r_n) - f(s)\| &\leq \|u(s, (x_n)_s, r_n) - u(s, x_s, r_n)\| + \|u(s, x_s, r_n) - f(s)\| \\ &\leq h(F(s, (x_n)_s, r_n), F(s, x_s, r_n)) + h(F(s, x_s, r_n), F(s, x_s, r)). \end{aligned}$$

Hence we have:

$$\|x_n(t)-x(t)\|^2 \leqslant 2\int_0^t (\beta_n(s)-\gamma(s), x(s)-x_n(s))\,ds$$
$$+2\int_0^t h(F(s,(x_n)_s,r_n), F(s,x_s,r_n))\cdot\|x(s)-x_n(s)\|\,ds$$
$$+2\int_0^t h(F(s,x_s,r_n), F(s,x_s,r))\cdot\|x(s)-x_n(s)\|\,ds.$$

Recalling that $\beta_n \xrightarrow{w} \gamma$ in $L^2(T,\mathbb{R}^N)$ and $x_n \xrightarrow{s} \hat{x}$ in $C(\hat{T},\mathbb{R}^N)$, and using hypotheses $H(F)_2(2)$ and (3), in the limit as $n\to\infty$, we get

$$\|\hat{x}(t)-x(t)\|^2 \leqslant 2\int_0^t \eta(s)\|\hat{x}_s-x_s\|_\infty\cdot\|x(s)-\hat{x}(s)\|\,ds$$
$$\leqslant 2\int_0^t \eta(s)\|\hat{x}_s-x_s\|_\infty^2\,ds.$$

Set $\theta(s)=\|\hat{x}_s-x_s\|_\infty^2$. Then we have

$$\theta(t)\leqslant 2\int_0^t \eta(s)\theta(s)\,ds.$$

Invoking Gronwall's inequality, we deduce that $\theta(t)=0$ for all $t\in T \Rightarrow \hat{x}=x\in C(\hat{T},\mathbb{R}^N)$. But note that $x_n\in P(r_n) n\geqslant 1$ and $x_n \xrightarrow{s} x$ in $C(T,\mathbb{R}^N)$. Therefore

$$P(r)\subseteq s\text{-}\underline{\lim}\,P(r_n). \tag{2}$$

From (1) and (2) above we get that

$$P(r_n)\xrightarrow{K} P(r) \text{ as } n\to\infty.$$

But $P(r_n)\subseteq \hat{p}(V)$ $n\geqslant 1$ and the latter is compact in $C(\hat{T},\mathbb{R}^N)$. So lemma 4.2 tells us that $P(r_n)\xrightarrow{h} P(r)$. Therefore $P(\cdot)$ is continuous for both the Vietoris and Hausdorff metric topologies as claimed by the theorem. **Q.E.D.**

5. Relaxation

In § 3 we established the existence of solutions for both the "convex" and "nonconvex" problems. In this section, we show that under some additional regularity hypotheses on the orientor field $F(t,y)$, we can in fact show that the solutions of the nonconvex problem are dense in the $C(\hat{T},\mathbb{R}^N)$-topology to those of the "convex" problem. Such a result is usually called in the literature "relaxation theorem". In optimal control, the "relaxed" (i.e. convexified) problem, plays an important role, because on the one hand captures the asymptotic behavior of the minimizing sequences of the original problem and on the other hand, thanks to its convex structure, always has a solution under very general hypotheses on the data (see Avgerinos-Papageorgiou [3], [4], Warga [21] and references therein).

So we consider problem $(*)$ and its "convexified" version:

$$\left.\begin{array}{l} -\dot{x}\in Ax(t)+\overline{\text{conv}}\,F(t,x_t)\text{a.e. on } T \\ x(v)=\varphi(v), v\in T_0. \end{array}\right\}. \quad (*)_c$$

Denote the solution set of $(*)$ by $P(\varphi)$ and that of $(*)_c$ by $P_c(\varphi)$.

We will need the following hypothesis on the orientor field $F(t,y)$.

$\underline{H(F)_3:}$ $F\colon TxC(T_0,\mathbb{R}^N)\to P_f(\mathbb{R}^N)$ is a multifunction s.t.

(1) $t\to F(t,y)$ is measurable,

(2) $h(F(t,y),F(t,z))\leqslant \eta(t)\|y-z\|_\infty$ a.e. with $\eta(\cdot)\in L^1_+$,

(3) $|F(t,y)|\leqslant a(t)+b(t)\|y\|_\infty$ a.e. with $a(\cdot),b(\cdot)\in L^1_+$.

Theorem 5.1. *If hypotheses $H(A)$, $H(F)_3$ and $H(\varphi)$ hold, then $\overline{P(\varphi)}=P_c(\varphi)$ in $C(\hat{T},\mathbb{R}^N)$.*

Proof. From theorem 3.2 we know that $P(\varphi)\neq\emptyset$ and so $P_c(\varphi)\neq\emptyset$. Furthermore, from theorem 3.1 we have that $P_c(\varphi)$ is compact in $C(\hat{T},\mathbb{R}^N)$.

Let $x(\cdot)\in P_c(\varphi)$. Then by definition, we have

$$-\dot{x}(t)\in Ax(t)+f(t)\text{a.e. on } T$$

$$x(v)=\varphi(v), v\in T_0$$

with $f\in L^1(T,\mathbb{R}^N)$, $f(t)\in F(t,x_t)$a.e.

Recall that the map $p\colon L^1(T,\mathbb{R}^N)\to C(\hat{T},\mathbb{R}^N)$, which to each $g\in L^1(T,\mathbb{R}^N)$ assigns the unique solution of the initial value problem $-\dot{x}(t)\in Ax(t)+g(t)$a.e., $x(0)=\varphi(0)$, is sequentially continuous from $L^1(T,\mathbb{R}^N)$ with the weak topology into $C(T,\mathbb{R}^N)$ with the strong topology. As before $\hat{p}\colon L^1(T,\mathbb{R}^N)\to C(\hat{T},\mathbb{R}^N)$ is defined by $\hat{p}(f)(t)=p(f)(t)$ $t\in T$ and $\hat{p}(f)(v)=\varphi(v)$ $v\in T_0$. Let $V\subseteq L^1(T,\mathbb{R}^N)$ be as in the proof of theorem 3.1. Then V equipped with the relative weak-$L^1(T,\mathbb{R}^N)$ topology, is compact, metrizable. So $\hat{p}\colon V\to C(\hat{T},\mathbb{R}^N)$ is "weak-to-strong" continuous. Thus given $\varepsilon>0$, we can find a symmetric, weak neighborhood of the origin in $C(\hat{T},\mathbb{R}^N)$s.t. if $f-f_1\in U\cap V$, then $\|x-\hat{p}(f_1)\|=\|x-x_1\|_\infty<\varepsilon$ (here we have set $x_1=\hat{p}(f_1)$). From theorem 4.2 of [17], we know that we can choose $f_1\in S^1_{F(\cdot,x.)}$. Next, via a straightforward application of Aumann's selection theorem (see Wagner [20], theorem 5.10), we can find $f_2\in S^1_{F(\cdot,x_1.)}$ s.t.

$$d(f_1(t),F(t,(x_1)_t))=\|f_1(t)-f_2(t)\|\text{ a.e.}$$

$$\Rightarrow\|f_1(t)-f_2(t)\|\leqslant h(F(t,x_t),F(t,(x_1)_t))\leqslant\eta(t)\|x_t-(x_1)_t\|_\infty<\eta(t)\varepsilon\text{a.e.}$$

Suppose $f_1,\ldots,f_n\in L^1(T,\mathbb{R}^N)$ have been chosen

$$f_{k+1}(t)\in F(t,(x_k)_t)\text{a.e.}\,k=0,1,\ldots,n-1\,(x_0=x) \quad (3)$$

$$x_k=\hat{p}(f_k)\text{ and }\|f_k(t)-f_{k+1}(t)\|\leqslant\frac{\varepsilon\eta(t)}{(k-1)!}\left[\int_0^t\eta(s)\mathrm{d}s\right]^{k-1}\text{ a.e.} \quad (4)$$

Again through Aumann's selection theorem, we can choose $f_{n+1}\in S^1_{F(\cdot,(x_n).)}$s.t.

$$\|f_{n+1}(t)-f_n(t)\|=d(f_n(t),F(t,x_n)_t))\text{a.e.}$$

$$\leqslant h(F(t,(x_{n-1})_t),F(t,(x_n)_t))\leqslant\eta(t)\|(x_n)_t-(x_{n-1})_t\|_\infty\text{ a.e.}$$

But from corollary, 1.7.1, p. 35 of Vrabie [19] (Benilan's inequality), we have that

$$\|(x_n)_t-(x_{n-1})_t\|_\infty \leqslant \int_0^t \|f_n(s)-f_{n-1}(s)\|\mathrm{d}s \leqslant \int_0^t \frac{\varepsilon\eta(s)}{(n-2)!}\left[\int_0^s \eta(\tau)\mathrm{d}\tau\right]^{n-2}\mathrm{d}s$$

$$\leqslant \int_0^t \frac{\varepsilon}{(n-1)!}d\left[\int_0^s \eta(\tau)d\tau\right]^{n-1}$$

$$=\frac{\varepsilon}{(n-1)!}\left[\int_0^t \eta(s)\mathrm{d}s\right]^{n-1}$$

$$\Rightarrow \|f_{n+1}(t)-f_n(t)\| \leqslant \frac{\varepsilon\eta(t)}{(n-1)!}\left[\int_0^t \eta(s)\mathrm{d}s\right]^{n-1} \text{ a.e.}$$

Thus by induction, we get a sequence $\{f_k\}_{k\geqslant 1}\subseteq L^1(T,\mathbb{R}^N)$ satisfying (3) and (4) above. Clearly then $\{f_n\}_{n\geqslant 1}\subseteq L^1(T,\mathbb{R}^n)$ is Cauchy. So $f_n \xrightarrow{s} f$ in $L^1(T,\mathbb{R}^N)$. Then $x_n=\hat{p}(f_n)\xrightarrow{s} p(\hat{f})=\hat{x}$ in $C(\hat{T},\mathbb{R}^N)$, and from theorem 4.5 of [16] we have that $\hat{f}(t)\in F(t,\hat{x}_t)$ a.e. Thus $\hat{x}=\hat{p}(\hat{f})\in P(\varphi)$. Also exploiting the monotonicity of $A(\cdot)$, we have

$$(\dot{x}_{k+1}(t)-\dot{x}_k(t),x_{k+1}(t)-x_k(t))\leqslant(f_{k+1}(t)-f_k(t),x_{k+1}(t)-x_k(t))\text{a.e.}$$

$$\Rightarrow \frac{1}{2}\|x_{k+1}(t)-x_k(t)\|^2 \leqslant \int_0^t \|f_{k+1}(s)-f_k(s)\|\cdot\|x_{k+1}(s)-x_k(s)\|\mathrm{d}s$$

$$\leqslant \int_0^t \frac{\varepsilon\eta(s)}{(k-1)!}\left(\int_0^s \eta(\tau)\mathrm{d}\tau\right)^{k-1}\|x_{k+1}(s)-x_k(s)\|\,\mathrm{d}s.$$

Invoking lemma A.5, p. 157 of Brezis [6], we get

$$\|x_{k+1}(t)-x_k(t)\| \leqslant \frac{\varepsilon}{k!}\left(\int_0^t \eta(s)\mathrm{d}s\right)^k k\geqslant 1.$$

Hence the triangle inequality gives us

$$\|x_{k+1}(t)-x(t)\| \leqslant \varepsilon\exp\|\eta\|_1$$

$$\Rightarrow \|\hat{x}(t)-x(t)\| \leqslant \varepsilon\exp\|\eta\|_1$$

$$\Rightarrow \|\hat{x}-x\|_\infty \leqslant \varepsilon\exp\|\eta\|_1.$$

Since $\hat{x}\in P(\varphi)$ and $\varepsilon>0$ is arbitrary, we will conclude that $\overline{P(\varphi)}=P_c(\varphi)$, the closure taken in $C(\hat{T},\mathbb{R}^N)$. **Q.E.D.**

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