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Factors for $|\bar{N}, p_n; \delta|_k$ summability of Fourier series

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Abstract. In this paper two theorems on $|\bar{N}, p_n; \delta|_k$ summability factors, which generalize the results of Bor [4] on $|\bar{N}, p_n|_k$ summability factors, have been proved.

Keywords. Fourier series; summability factors; absolute summability.

1. Introduction

Let Σa_n be a given infinite series with partial sums (s_n) and let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geqslant 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series Σa_n is said to be summable $|\bar{N}, p_n|_k, k \geqslant 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty \tag{3}$$

and it is said to be summable $|\bar{N}, p_n; \delta|_k, k \geqslant 1$ and $\delta \geqslant 0$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}$$

In the special case when $p_n = 1$ for all values of n (resp. $\delta = 0$), $|\bar{N}, p_n; \delta|_k$ summability is the same as $|C, 1; \delta|_k$ (resp. $|\bar{N}, p_n|_k$) summability.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t) \mathrm{d}t = 0 \tag{5}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{6}$$

We write

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}, \quad \varphi_1(t) = \frac{1}{2}\int_0^t \varphi(u)\mathrm{d}u \text{ and } \Delta\lambda_n = \lambda_n - \lambda_{n+1}.$$

2. Quite recently Bor [4] proved the following theorems.

Theorem A. *Let the sequence* (p_n) *be such that*

$$P_n = O(np_n) \tag{7}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{8}$$

If $\varphi_1(t)$ *is of bounded variation in* $(0, \pi)$ *and* (λ_n) *is a sequence such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n|^k < \infty \tag{9}$$

and

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty. \tag{10}$$

then the series $\Sigma A_n(t) P_n \lambda_n (np_n)^{-1}$ *is summable* $|\bar{N}, p_n|_k$ *for* $k \geqslant 1$.

Theorem B. *Let the sequence* (p_n) *be such that conditions* (7) *and* (8) *of Theorem A are satisfied. If* Σa_n *is a series of complex terms such that*

$$B_n \equiv \sum_{v=1}^{n} v a_v = O(n), \tag{11}$$

then the series $\Sigma a_n P_n \lambda_n (np_n)^{-1}$ *is summable* $|\bar{N}, p_n|_k$ *for* $k \geqslant 1$.

3. The aim of this paper is to prove above theorems for $|\bar{N}, p_n; \delta|_k$, with $k \geqslant 1$ and $\delta \geqslant 0$, summability. Now, we shall prove the following theorem.

Theorem 1. *Let the sequence* (p_n) *be such that conditions* (7) *and* (8) *of Theorem A are satisfied and*

$$\sum_{n=v}^{\infty} (P_n/p_n)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left\{(P_v/p_v)^{\delta k} \frac{1}{P_v}\right\}. \tag{12}$$

If $\varphi_1(t)$ *is of bounded variation in* $(0, \pi)$ *and* (λ_n) *is a sequence such that*

$$\sum_{n=1}^{\infty} n^{\delta k - 1} |\lambda_n|^k < \infty \tag{13}$$

and

$$\sum_{n=1}^{\infty} n^{\delta k} |\Delta\lambda_n| < \infty, \tag{14}$$

then the series $\Sigma A_n(t) P_n \lambda_n (np_n)^{-1}$ *is summable* $|\bar{N}, p_n; \delta|_k$ *for* $k \geqslant 1$ *and* $\delta \geqslant 0$.

Theorem 2. *Let the sequence* (p_n) *be such that conditions* (7) *and* (8) *of Theorem A and condition* (12) *of Theorem* 1 *are satisfied. If condition* (11) *of Theorem B is satisfied by the series* Σa_n, *then the series* $\Sigma a_n P_n \lambda_n (np_n)^{-1}$ *is summable* $|\bar{N}, p_n; \delta|_k$ *for* $k \geqslant 1$ *and* $\delta \geqslant 0$, *where* (λ_n) *is as in Theorem* 1.

4. We need the following lemmas for the proof of our theorems.

Lemma 1. *If* $\varphi_1(t)$ *is of bounded variation in* $(0, \pi)$, *then*

$$\sum_{v=1}^{n} vA_v(x) = O(n) \text{ as } n \to \infty. \tag{15}$$

This lemma is a particular case of Lemma due to Prasad and Bhatt ([5], Lemma 9).

Lemma 2. ([3]). *If the sequence* (p_n) *is such that conditions* (7) *and* (8) *of Theorem A are satisfied, then*

$$\Delta\{P_n/(p_n n^2)\} = O(1/n^2) \text{ as } n \to \infty. \tag{16}$$

5. *Proof of Theorem* 2. Let (T_n) denote the $(\bar{N}, p_n)$ mean of the series $\Sigma a_n P_n \lambda_n (np_n)^{-1}$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i P_i \lambda_i (ip_i)^{-1} = \frac{1}{P_n}\sum_{v=0}^{n} (P_n - P_{v-1}) a_v P_v \lambda_v (vp_v)^{-1}.$$

Then, for $n \geqslant 1$, we have that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} P_v a_v \lambda_v (vp_v)^{-1}.$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= B_n \lambda_n n^{-2} - p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} p_v P_v B_v \lambda_v (v^2 p_v)^{-1} \\ &\quad + p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v B_v (v^2 p_v)^{-1} \\ &\quad + p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v B_v \lambda_{v+1} \Delta\{P_v/(v^2 p_v)\} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.} \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality for $k > 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} |T_{n,i}|^k < \infty, \text{ for } i = 1, 2, 3, 4.$$

Firstly, we have that

$$\sum_{n=1}^{m} (P_n/p_n)^{\delta k+k-1} |T_{n,1}|^k = \sum_{n=1}^{m} (P_n/p_n)^{\delta k} (P_n/p_n)^{k-1} |\lambda_n|^k |B_n|^k n^{-2k}$$
$$= O(1) \sum_{n=1}^{m} n^{\delta k} n^{k-1} |\lambda_n|^k n^k n^{-2k}$$
$$= O(1) \sum_{n=1}^{m} n^{\delta k-1} |\lambda_n|^k$$
$$= O(1) \text{ as } m \to \infty,$$

by virtue of (7), (11), (13).

Now, when $k > 1$ applying Hölder's inequality, with indices k and k', where $1/k + 1/k' = 1$, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,2}|^k = \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \left| \sum_{v=1}^{n-1} \frac{p_v P_v B_v \lambda_v}{v^2 p_v} \right|^k$$
$$\leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \left\{ \frac{P_v |B_v| |\lambda_v|}{v^2 p_v} \right\}^k$$
$$\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k (P_v/p_v)^k |B_v|^k v^{-2k}$$
$$\times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^{\delta k} (P_v/p_v)^{k-1} |B_v|^k |\lambda_v|^k v^{-2k}$$
$$= O(1) \sum_{v=1}^{m} v^{\delta k-1} |\lambda_v|^k$$
$$= O(1)$$

as $m \to \infty$, by (7), (11), (12) and (13).

On the other hand, since

$$\sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \leqslant P_{n-1} \sum_{v=1}^{n-1} |\Delta\lambda_v| \Rightarrow \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \leqslant \sum_{v=1}^{n-1} |\Delta\lambda_v| = O(1),$$

by (14), we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,3}|^k \leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \left| \sum_{v=1}^{n-1} \frac{P_v P_v B_v \Delta\lambda_v}{v^2 p_v} \right|^k$$
$$\leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \right.$$

$$\times (P_v/p_v)^k |B_v|^k v^{-2k}\Bigg\} \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v|\right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} P_v |\Delta\lambda_v| (P_v/p_v)^k |B_v|^k v^{-2k}$$

$$\times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{v-1}}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^{\delta k} (P_v/p_v)^k |B_v|^k v^{-2k} |\Delta\lambda_v|$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k} |\Delta\lambda_v|$$

$$= O(1),$$

as $m \to \infty$, by virtue of (7), (11), (12) and (14).

Finally, using the fact that $\Delta\{P_v/(v^2 p_v)\} = O(1/v^2)$, by Lemma 2, we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,4}|^k \leqslant \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k}$$

$$\times \sum_{v=1}^{n-1} P_v |B_v| |\lambda_{v+1}| \Delta\{(P_v/(v^2 p_v)\}|^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \Bigg\{ \sum_{v=1}^{n-1} p_v (P_v/p_v) v^{-2}$$

$$\times |B_v| |\lambda_{v+1}| \Bigg\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \Bigg\{ \sum_{v=1}^{n-1} (P_v/p_v)^k p_v v^{-2k}$$

$$\times |B_v|^k |\lambda_{v+1}|^k \Bigg\} \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^k p_v |\lambda_{v+1}|^k |B_v|^k v^{-2k}$$

$$\times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} (P_v/p_v)^{\delta k} (P_v/p_v)^{k-1} |\lambda_{v+1}|^k v^k v^{-2k}$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k-1} |\lambda_{v+1}|^k$$

$$= O(1)$$

as $m \to \infty$, by (7), (11), (12) and (13). Therefore, we get that

$$\sum_{n=1}^{m} (P_n/p_n)^{\delta k+k-1} |T_{n,i}|^k = O(1) \text{ as } m \to \infty, \text{ for } i = 1, 2, 3, 4.$$

This completes the proof of Theorem 2.

Proof of Theorem 1. Theorem 1 is a direct consequence of Theorem 2 and Lemma 1.

Remark. If we take $\delta = 0$ in our theorems 1 and 2, then we get Theorem A and Theorem B, respectively. Because in this case the conditions (13) and (14) reduce to conditions (9) and (10), respectively. It should be noted that in this case condition (12) is obvious.

If we take $p_n = 1$ for all values of n in Theorem 1, then we get the following corollary.

COROLLARY

If $\varphi_1(t)$ is of bounded variation in $(0, \pi)$ and (λ_n) is a sequence such that conditions (13) *and* (14) *of Theorem* 1 *are satisfied, then the series $\Sigma A_n(t)\lambda_n$, at $t = x$ is summable $|C, 1; \delta|_k, k \geqslant 1$, provided that $1 - \delta k > 0$.*

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