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An example of a regular space that is not completely regular

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Abstract. A simpler example of regular space that is not completely regular is attempted.

Keyword. Regular space.

1. Introduction

In 1925 Urysohn [10] posed, but left unanswered, the question of whether or not regular topological spaces exist in which every continuous real-valued function is constant. Tychonoff [9], in an attempt to settle this question, produced an example of a regular space which is not completely regular. Later, making essential use of the example of Tychonoff, Hewitt [5], Novák [7], van Est-Freudenthal [3] and Herrlich [4] constructed regular spaces supporting no non-constant continuous real-valued function. Among the earliest treatises on set topology, Čech [1] gives an account of Novák's example and Vaidyanathaswamy [11] presents Tychonoff's example mentioned above. In recent times, more accessible references are Dugundji [2] and Steen and Seebach [8] which give the same examples under the names "Spiral staircase" or "Tychonoff corkscrew". This example involves the use of the uncountable well-ordered space ω_1. I venture, in this shortnote, on an apparently simpler construction of a regular space that is not completely regular.

2. Construction

For any even integer n let $T_n = \{n\} \times (-1, 1)$ and $X_1 = \cup_{n \text{even}} T_n = \{(n, y): n$ even integer, $-1 < y < 1\}$.

Let $\{\alpha_k, k \geqslant 1\}$ be a strictly increasing sequence of positive real numbers such that $\lim_{k \to \infty} \alpha_k = 1$.

For any odd integer n, set $C_{n,k} = \{(x, y): (x-n)^2 + y^2 = \alpha_k^2\}$, $k = 1, 2, \ldots$ and set $X_2 = \cup_{n \text{odd}} \cup_{k=1}^{\infty} C_{n,k}$. Let a and b be two distinct points not belonging to the union $X_1 \cup X_2$. Form the set $X = X_1 \cup X_2 \cup \{a, b\}$.

Topology of X

We shall define a topology on X by describing the neighbourhoods of each of its points.

For each odd integer n and each $k \geqslant 1$, all points of $C_{n,k}$ except the point $a_{n,k} = (n, \alpha_k)$

are isolated. A neighbourhood of $a_{n,k}$ consists of all but a finite number of points of $C_{n,k}$. Write

$$C_n = \bigcup_{k=1}^{\infty} C_{n,k}, \quad n \text{ odd}.$$

If $p = (n, y) \in X_1$, consider the subset

$$\{(z, y): n-1 < z < n+1\} \cap (C_{n-1} \cup C_{n+1})$$

of X. A neighbourhood of p consists of all but a finite number of points of this subset. A neighbourhood of a consists of all points of $X_1 \cup X_2$ with first coordinate greater than some real number c. A neighbourhood of b consists of points of $X_1 \cup X_2$ with first coordinate less than some real number d. The neighbourhoods describe a T_1 topology on X. It is not difficult to see that under this topology each neighbourhood of a point of X contains a closed neighbourhood of the same point. X is thus regular and Hausdorff.

Failure of complete regularity

We claim that given a real-valued, continuous function f on X, $f(a) = f(b)$. Consequently X fails to be completely regular.

Let us first observe that if h is a continuous real-valued function on $C_{n,k}$, the set

$$\{(x, y) \in C_{n,k}: h(x, y) \neq h(a_{n,k})\}$$
$$= \left\{(x, y) \in C_{n,k}: h(x, y) \notin \bigcap_{m=1}^{\infty} \left(h(a_{n,k}) - \frac{1}{m}, h(a_{n,k}) + \frac{1}{m} \right)\right\}$$

is at most a countable subset of $C_{n,k}$.

Let $f: X \to \mathbb{R}$ be an arbitrary, continuous function. Set $B_{n,k} = \{(x, y) \in C_{n,k}: f(x, y) \neq f(a_{n,k})\}$ and $D_n =$ ordinates of points in $\cup_{k=1}^{\infty} B_{n,k}\}$. In view of the observation above, each $B_{n,k}$ is countable and consequently, each D_n is so. If $D = \cup_{n \text{odd}} D_n$, D is then a countable subset of $(-1, 1)$. Suppose $p \in X_1$ is such that $p \in T_n$ and the ordinate y of p does not belong to D. Consider

$$\{(z, y): n-1 < z < n+1\} \cap (C_{n-1} \cup C_{n+1}).$$

If

$$(z, y) \in C_{n-1,k}, f(z, y) = f(a_{n-1,k})$$

and if

$$(z, y) \in C_{n+1,k}, f(z, y) = f(z, y) = f(a_{n+1,k}).$$

From the structure of neighbourhoods of p it is clear that

$$f(p) = \lim_{k \to \infty} f(a_{n-1,k}) = \lim_{k \to \infty} f(a_{n+1,k}).$$

Let $q \in X_1$ be such that $q \in T_{n+2}$ and the ordinate of q does not belong to D. Considerations as above will lead us to conclude that

$$f(q) = \lim_{k \to \infty} f(a_{n+3,k}) = \lim_{k \to \infty} f(a_{n+1,k}).$$

Hence, $f(p) = f(q)$.

If $G = \{(n, y) : n \text{ even and } y \in (-1, 1) - D\}$, the above argument shows that for any $p \in G$, $f(p) = \lim_{k \to \infty} f(a_{n+1,k}) = \lim_{k \to \infty} f(a_{n-1,k})$ where $p = (n, y)$. Thus f is a constant on G, say, α. Since f assumes the value α in every neighbourhood of each of a and b, $f(a) = \alpha = f(b)$. The claim is thus established.

A few remarks about the space X

(I) The space X has the merit of playing the role of the space Q which enters into the construction, due to Herrlich [4, page 153], of a regular space on which every continuous real-valued function is constant.

(II) The space X admits a proper subspace which is also a regular Hausdorff space that fails to be completely regular. To be precise take the subspace Z of X where

$$Z = \Bigg(\bigcup_{\substack{n \geq 0 \\ n \text{ even}}} T_n \Bigg) \cup \Bigg(\bigcup_{\substack{n \geq 1 \\ n \text{ odd}}} \bigcup_{k=1}^{\infty} C_{n,k} \Bigg) \cup \{a\}.$$

T_0 and $\{a\}$ are disjoint closed subsets of Z. If $g : Z \to \mathbb{R}$ is a continuous function which is 1 on T_0, it can be easily seen that $g(a) = 1$. As a result, Z cannot be completely regular.

(III) At the time of the construction of the space X, the author was not aware of the existence of the paper by Mysior [6] which contains an elementary example of regular space which is not completely regular. However the space X is a different example.

References

[1] Čech E, *Topological Spaces* (1966) (Interscience Pub.)
[2] Dugundji J, *Topology* (1966) (Allyn and Bacon)
[3] Est, Van W T and Freudenthal U H, Trennung durch stetige Funktionen in topologischen Räumen, *Indagationes Math.* **13** (1951) 359–368
[4] Herrlich H, Wann sind alle-stetigen Abbildungen in Y Konstant? *Math. Z.* **90** (1965) 152–154
[5] Hewitt E, On two problems of Urysohn, *Ann. Math.* **47** (1946) 503–509
[6] Mysior A, A regular space which is not completely regular, *Proc. Am. Math. Soc.* **81** (1981) 652–653
[7] Novák J, Regular space, on which every continuous function is constant, *Cas. Pěst. Mat. Fys.* **73** (1948) 58–68
[8] Steen L A and Seebach Jr J A, *Counterexamples in Topology* (1970) (Holt, Rinehart and Winston, Inc.)
[9] Tychonoff A, Über die topologische Erweiterung von Rëumen, *Math. Ann.* **102** (1930) 544–561
[10] Urysohn P, Über die Machtigkeit der zusammenhangenden Mengen, *Math. Ann.* **94** (1925) 262–295
[11] Vaidyanathaswamy R, *Treatise on Set Topology* (1947) Madras