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On the frequency of Titchmarsh's phenomenon for $\zeta(s)$—VIII

R BALASUBRAMANIAN, K RAMACHANDRA* and
A SANKARANARAYANAN*

The Institute of Mathematical Sciences, Madras 600 113, India
*School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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Abstract. For suitable functions $H = H(T)$ the maximum of $|(\zeta(\sigma+it))^z|$ taken over $T \leqslant t \leqslant T+H$ is studied. For fixed $\sigma(\frac{1}{2} \leqslant \sigma \leqslant 1)$ and fixed complex constants z "expected lower bounds" for the maximum are established.

Keywords. Riemann zeta-function; frequency; Titchmarsh's phenomenon.

1. Introduction

It is our object to prove the following three Ω-theorems by applying two fundamental theorems of Ramachandra which he proved in [5] (these theorems will be stated in § 2). Let $G(\sigma, t) = |(\zeta(\sigma+it))^z|$ where σ is a constant in $[\frac{1}{2}, 1]$ and z is a non-zero complex constant. Since z can be written as $z = re^{i\theta}$ where $r > 0$ and θ lies in $[0, 2\pi)$, in order to state Ω theorems for $G(\sigma, t)$ it suffices to assume $r = 1$. *We shall in what follows write* $z = e^{i\theta}$.

Theorem 1. *We have, with* $z = e^{i\theta}$,

$$\max_{T \leqslant t \leqslant T+H} G(1,t) \geqslant e^{\gamma}\lambda(\theta)(\log\log H - \log\log H) + O(1)$$

where γ *is the Euler's constant and*

$$\lambda(\theta) = \prod_p \left\{\left(1 - \frac{1}{p}\right)\left(\frac{p[(p^2 - \sin^2\theta)^{\frac{1}{2}} + \cos\theta]}{p^2 - 1}\right)^{\cos\theta} \times \exp\left(\sin\theta \tan^{-1}\left(\frac{\sin\theta}{(p^2 - \sin^2\theta)^{\frac{1}{2}}}\right)\right)\right\}.$$

The conditions on H and T are $T \geqslant H \geqslant C \log\log\log T$, $T \geqslant T_0$ *where C and* T_0 *are large positive constants.*

Remark 1. Levinson [3] was the first to prove that when $\theta = 0$

$$\max_{1 \leqslant t \leqslant T} G(1,t) \geqslant e^{\gamma}\log\log T + O(1)$$

and when $\theta=\pi$

$$\max_{1\leqslant t\leqslant T} G(1,t)\geqslant \frac{6}{\pi^2}\mathrm{e}^{\gamma}(\log\log T-\log\log\log T)+O(1).$$

Note that $\lambda(\pi)=6/\pi^2$. However, around the same time Ramachandra [6] proved that

$$\max_{T\leqslant t\leqslant T+H} G(1,t)\geqslant \mathrm{e}^{\gamma}\lambda(\theta)\log\log H(1+O(1))$$

when $\theta=0$, and $\theta=\pi$. Later Ramachandra [7] extended the conditions on H to $H\geqslant C\log\log\log\log T$ (without assuming any hypothesis) and to $H\geqslant C\log\log\log\log\log T$ (assuming Riemann hypothesis). These results go through for any θ in $[0,2\pi)$.

Remark 2. This theorem as well as Theorems 2 and 3 has obvious extensions to ordinary L-functions and more generally to L-functions of algebraic number fields and so on. We do not carry out the details here.

Theorem 2. *Let α be a constant in $(\frac{1}{2},1)$. Then*

$$\max_{T\leqslant t\leqslant T+H} G(\alpha,t)>\exp\left(C_1\frac{(\log H)^{1-\alpha}}{\log\log H}\right)$$

where C_1 is a positive constant depending on α and $T^{1/3}\leqslant H\leqslant T$.

The next theorem depends on Riemann hypothesis (R.H.).

Theorem 3. (*on R.H.*). *In Theorem 2 the condition on H can be relaxed to $T\geqslant H\geqslant C\log\log T$. Also under this condition on H, there holds*

$$\max_{T\leqslant t\leqslant T+H} G(\tfrac{1}{2},t)>\exp\left(C_2\left(\frac{\log H}{\log\log H}\right)^{\frac{1}{2}}\right)$$

where $C_2>0$ is a numerical constant.

Remark 1. When $\theta=0$, Theorem 3 can be upheld without assuming R.H.

Remark 2. The results of this paper are inspired by the paper [4] of Montgomery who proves

$$\max_{0\leqslant t\leqslant T} G(\alpha,t)>\exp\left(C_3\frac{(\log T)^{1-\alpha}}{(\log\log T)^{\alpha}}\right)$$

where $C_3\geqslant 0$ depends on α and $\frac{1}{2}<\alpha<1$. In a recent paper [8], Ramachandra and Sankaranarayanan have obtained this result with $C_3=C_4(\alpha-\frac{1}{2})^{\frac{1}{2}}/(1-\alpha)$ where $C_4>0$ is a numerical constant. (The quantity $(\alpha-\frac{1}{2})^{\frac{1}{2}}$ can be replaced by 1 if we assume R.H.). However Montgomery's method does not work for short intervals $[T,T+H]$ and also for L-functions.

2. Ramachandra's theorems

We shall now state a special case of the main theorem of [5].

Theorem 4. *Let $a_1 = 1, a_2, a_3, \ldots$ be a sequence of complex numbers satisfying $|a_n| \leqslant (nH)^A$ where $H \geqslant 10^{10}$ and A is a positive constant. Let $F(s) = \Sigma_{n=1}^{\infty} a_n/n^s$ (where $s = \sigma + it$) admit an analytic continuation in $\sigma > 0$, $T \leqslant t \leqslant T + H$. Then*

$$\max_{\sigma > 0}\left(\frac{1}{H}\int_T^{T+H} |F(\sigma + it)|^2 \,\mathrm{d}t\right) > C(A) \sum_{n \leqslant H/200} |a_n|^2 \left(1 - \frac{\log n}{\log H} + \frac{1}{\log \log H}\right),$$

where $C(A)$ is a positive constant depending only on A.

As a corollary he deduced

Theorem 5. *In addition to the condition of Theorem 4 let us suppose that in $(\sigma > 0, T \leqslant t \leqslant T + H)$ the maximum of $|F(s)|$ be $\leqslant \exp\exp(H/100A)$. Then*

$$\left(\frac{1}{H}\int_T^{T+H} |F(it)|^2 \,\mathrm{d}t\right) \geqslant C(A) \sum_{n \leqslant H/200} |a_n|^2 \left(1 - \frac{\log n}{\log H} + \frac{1}{\log \log H}\right)$$

where $C(A) > 0$ depends only on A, provided the LHS is interpreted in a limiting sense.

Remark 1. In the reference to Ramachandra's paper the theorem proved is slightly different. But it is not hard (from his argument) to prove Theorem 4 and and deduce Theorem 5. In that reference Ramachandra uses the kernel related to $\exp(s^{4a+2})$ where a is a non negative integer. However in deducing Theorem 5 from Theorem 4 we have to use the kernel related to $\exp((\sin s)^2)$.

Remark 2. From Theorem 4 we can also deduce that the maximum of $G(\sigma, t)$ in $(\sigma \geqslant 1, T \leqslant t \leqslant T + H)$ exceeds the right hand side of Theorem 1. Similar remarks holds good for Theorems 2 and 3.

Remark 3. For improvements of Theorems 4 and 5 see the paper [2] by Balasubramanian and Ramachandra.

3. Proof of theorem 1.

We will prove Theorems 1, 2 and 3 in §§3, 4 and 5 respectively. We adopt different notations in each of these sections and we will explain the notations in each of these sections in the respective sections.

Lemma 3.1. *Let k be a positive integer and let*

$$F(s) = (\zeta(1 + s))^{kz} = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.1}$$

then

$$a_1 = 1 \text{ and } |a_n| \leqslant n^2(\zeta(2))^k. \tag{3.2}$$

Proof. Follows by Euler product for $\zeta(1+s)$.

Lemma 3.2. *We have in* $(\sigma \geqslant 0, T \leqslant t \leqslant T+H)$

$$|F(s)| \leqslant \exp(k C_1 \log\log T) \tag{3.3}$$

where c_1 *is a positive constant and* $10 < H \leqslant T$.

Proof. Follows since it is well-known that

$$\log\zeta(\sigma+it) = O(\log\log T)$$

in $(\sigma \geqslant 1, T \leqslant t \leqslant T+H)$.

Lemma 3.3. *The conditions for the application of Theorem 5 are satisfied if* $H \gg \log\log\log T$ *where the implied constant is a large positive constant, and* $k = O(\log H)$.

Proof. Follows from Lemmas 3.1 and 3.2.

Lemma 3.4. *Let* $k_0 = kz$ *and let* $n \geqslant 2$ *and let* $n = \prod_p p^m$ *be the prime power decomposition of* n. *Then*

$$a_1 = 1 \text{ and } a_n = \prod_p \frac{k_0(k_0+1)\dots(k_0+m-1)}{m!\,p^m} \tag{3.4}$$

Proof. Follows by Euler's product for $\zeta(1+s)$.

Lemma 3.5. *Let, for each* $p \leqslant k$,

$$l = k\left(\frac{\cos\theta + (p^2-\sin^2\theta)^{\frac{1}{2}}}{p^2-1}\right) = \frac{k}{q} \text{ say and } m = [l]. \tag{3.5}$$

Then, putting

$$n = \prod_{p\leqslant k} p^m, \text{ we have}$$

$$\frac{1}{2k}\log|a_n|^2 = \frac{1}{2k}\sum_{p\leqslant k}\{-2m\log m + 2m + O(\log m) - 2m\log p + E(k,m)\} \tag{3.6}$$

where

$$E(k,m) = \sum_{v=0}^{m-1}\log(k^2+v^2+2kv\cos\theta). \tag{3.7}$$

Proof. Follows from the formula

$$\log m! = m\log m - m + O(\log m).$$

Lemma 3.6. *We have,*

$$E(k,m) = 2m\log k + k\int_0^{1/q}\log(1+u^2+2u\cos\theta)du + O\left(\frac{1}{p}\right). \tag{3.8}$$

Proof. We have

$$E(k,m) = \sum_{v=0}^{m-1} \left\{ \log(k^2 + v^2 + 2kv\cos\theta) - \int_v^{v+1} \log(k^2 + u^2 + 2ku\cos\theta)\mathrm{d}u \right\} + \int_0^m \log(k^2 + u^2 + 2ku\cos\theta)\mathrm{d}u.$$

Here the sum on the right is easily seen to be $O(1/p)$. The integral on the right is

$$2m\log k + \int_0^m \log\left(1 + \frac{u^2}{k^2} + 2\frac{u}{k}\cos\theta\right)\mathrm{d}u.$$

Here we can replace the upper limit m of the integral by l with an error $O(m/k) = O(1/p)$. The lemma now follows by a change of variable.

Lemma 3.7. *We have,*

$$\frac{1}{k}\sum_{p\leqslant k} \log m = O\left(\frac{1}{\log k}\right) \tag{3.9}$$

and

$$\frac{1}{k}\sum_{p\leqslant k}\frac{1}{p} = O\left(\frac{1}{\log k}\right).$$

Proof. Follows by prime number theorem.

Lemma 3.8. *We have,*

$$\frac{1}{2k}\sum_{p\leqslant k}\{-2m\log m + 2m - 2m\log p + 2m\log k\} = \sum_{p\leqslant k}\left\{-\frac{1}{q}\log\frac{p}{q} + \frac{1}{q}\right\} + O\left(\frac{1}{\log k}\right). \tag{3.10}$$

Proof. On the LHS we can replace m by l with a total error

$$\leqslant \frac{1}{2k}\sum_{p\leqslant k} O(\log m) = O\left(\frac{1}{\log k}\right).$$

The rest is

$$\sum_{p\leqslant k}\left\{-\frac{1}{q}\log\frac{k}{q} + \frac{1}{q} - \frac{1}{q}\log p + \frac{1}{q}\log k\right\}$$

which gives the lemma.

Lemma 3.9. *We have,*

$$\frac{1}{2k}\sum_{p\leqslant k} k\int_0^{(1/q)} \log(1+u^2+2u\cos\theta)\mathrm{d}u$$

$$= \mathrm{Re}\sum_{p\leqslant k}\left(\frac{1+(1/q)\mathrm{e}^{i\theta}}{\mathrm{e}^{i\theta}}\log\left(1+\frac{1}{q}\mathrm{e}^{i\theta}\right)-\frac{1}{q}\right). \tag{3.11}$$

Proof. Trivial.

Lemma 3.10. *We have,*

$$\frac{1}{2k}\log|a_n|^2 = \log\log k + \gamma + \log\lambda(\theta) + O\left(\frac{1}{\log k}\right), \tag{3.12}$$

where $\lambda(\theta)$ *is as in Theorem* 1.

Proof. By Lemmas 3.5, 3.6, 3.7 and 3.8 we see that LHS of (3.12) is, (with an error $O(1/\log k)$),

$$\mathrm{Re}\sum_{p\leqslant k}\left\{-\frac{1}{q}\log\frac{p}{q}+\frac{1}{q}\log\left(1+\frac{1}{q}\mathrm{e}^{i\theta}\right)+\mathrm{e}^{-i\theta}\log\left(1+\frac{1}{q}\mathrm{e}^{i\theta}\right)\right\}.$$

Now the contribution from the first two terms (in the curly bracket) to the sum is

$$\mathrm{Re}\sum_{p\leqslant k}\frac{1}{q}\log\left|\frac{q+\mathrm{e}^{i\theta}}{p}\right| = 0,$$

since

$$|q+\mathrm{e}^{i\theta}|^2 = \left(\frac{p^2-1}{(p^2-\sin^2\theta)^{\frac{1}{2}}+\cos\theta}+\cos\theta\right)^2+\sin^2\theta$$

$$= \left(\frac{p^2-1+\cos^2\theta+\cos\theta(p^2-\sin^2\theta)^{\frac{1}{2}}}{(p^2-\sin^2\theta)^{\frac{1}{2}}+\cos\theta}\right)^2+\sin^2\theta$$

$$= p^2.$$

The third term contributes

$$\sum_{p\leqslant k}\left(\cos\theta\log\frac{p}{q}+\sin\theta\tan^{-1}\left(\frac{\sin\theta}{q+\cos\theta}\right)\right)$$

$$= \sum_{p\leqslant k}\left\{\log\left(1-\frac{1}{p}\right)+\cos\theta\log\frac{p}{q}+\sin\theta\tan^{-1}\left(\frac{\sin\theta}{q+\cos\theta}\right)\right\}$$

$$+\sum_{p\leqslant k}\log\left(1-\frac{1}{p}\right)^{-1}.$$

This together with the well-known formula $\Pi_{p\leqslant k}(1-1/p)^{-1} = \mathrm{e}^{\gamma}\log k + O(1)$ proves the lemma.

Lemma 3.11. *For the n defined in Lemma* 3.5, *we have,*

$$\log n = \sum_{p \leqslant k} m \log p = k \log k + O(k). \tag{3.13}$$

Proof. Replacement of m by l involves an error $O(k)$ by the prime number theorem. Now $l = k/q$ and

$$\begin{aligned} q &= p\left(p - \frac{1}{p}\right)(p[1 - (\sin^2\theta/p^2)]^{\frac{1}{2}} + \cos\theta)^{-1} \\ &= p\left(1 - \frac{1}{p^2}\right)\left([1 - (\sin^2\theta/p^2)]^{\frac{1}{2}} + \frac{\cos\theta}{p}\right)^{-1} \\ &= p + O(1). \end{aligned}$$

This proves the lemma.

Lemma 3.12. *Set* $k = [\log H/(2\log\log H)]$. *Then for all H exceeding a large positive constant, we have,*

$$n \leqslant \frac{H}{200}.$$

Proof. Follows from Lemma 3.11.

Lemma 3.13. *The maximum of* $|(\zeta(1+s))^z|$ *in* $(\sigma = 0,\ T \leqslant t \leqslant T + H)$ *exceeds*

$$\left(\frac{C(A)}{\log\log H}|a_n|^2\right)^{1/2k}.$$

Proof. Follows from Theorem 5.

Lemmas 3.12 and 3.13 complete the proof of Theorem 1, in view of Lemma 3.10.

4. Proof of theorem 2.

Lemma 4.1. *Let* $\frac{1}{2} \leqslant \beta \leqslant 1$ *and* $H = T^{1/3}$. *Then the number of zeros of* $\zeta(s)$ *in* $(\sigma \geqslant \beta,\ T \leqslant t \leqslant T + H)$ *is*

$$\ll H^{4(1-\beta)/(3-2\beta)}(\log T)^{100} \tag{4.1}$$

where the constant implied by the Vinogradov symbol $\ll$ *is absolute.*

Proof. This is a consequence of a deep result of Balasubramanian [1] on the mean-square of $|\zeta(\frac{1}{2} + it)|$. (See Theorem 6 on page 576 of his paper).

Lemma 4.2. *Let* $\frac{1}{2} < \beta < \alpha < 1$. *Then there exists a t-interval* I *contained in* $T \leqslant t \leqslant T + H$, *of length* T^{δ} (*where* $\delta > 0$ *is a constant depending on* β) *such that the region* $(\sigma \geqslant \beta,\ t \in I)$ *is free from zeros of* $\zeta(s)$.

Proof. Follows from Lemma 4.1.

Lemma 4.3. *Let I_0 denote the t-interval obtained from I by removing on both sides intervals of length* (1/100) T^δ. *Then in* $(\sigma \geqslant \alpha, t \in I_0)$ *we have*

$$\log \zeta(s) = O(\log T).$$

Proof. Follows by Borel-Caratheodory theorem.

Lemma 4.4. *We apply Theorem 5 to the interval I_0 in place of $T \leqslant t \leqslant T + H$. Then*

$$\max_{(\sigma=0, t\in I_0)} |(\zeta(\alpha+s))^z| \geqslant \left(\frac{C(A)}{\log\log H}|a_n|^2\right)^{1/2k} \tag{4.2}$$

where $n \leqslant H/200$, $k \geqslant 1$ is any integer which is $O(\log H)$, *and a_n are defined by*

$$F(s) = (\zeta(\alpha+s))^{kz} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{4.3}$$

Proof. It is easily seen (as before) that the conditions for the application of Theorem 5 are satisfied and hence the lemma.

Lemma 4.5. *Let*

$$n = \prod_{(k/4)^{1/\alpha} \leqslant p \leqslant (k/2)^{1/\alpha}} p. \tag{4.4}$$

Then

$$|a_n|^2 \geqslant \exp\left(\frac{C_1 k^{1/\alpha}}{\log k}\right).$$

where C_1 is a positive constant.

Proof. Follows by Euler's product for $\zeta(s)$.

Lemma 4.6. *Let $k = [C_2(\log H)^\alpha]$ where $C_2 > 0$ is a small constant. Then, we have,*

$$n \leqslant \frac{H}{200} \tag{4.5}$$

and so R.H.S. *of* (4.2) *exceeds*

$$\exp\left(\frac{C_3(\log H)^{1-\alpha}}{\log\log H}\right), \tag{4.6}$$

where $C_3 > 0$ is a constant.

Proof. Follows from lemma 4.5.

Theorem 2 now follows from (4.2) and lemma 4.6.

5. Proof of theorem 3

The first part of Theorem 3 follows exactly as in the proof of Theorem 2. It remains to prove only the second part of Theorem 3. We begin with

Lemma 5.1. *Given any $t \geqslant 10$ there exists a real number τ with $|t-\tau| \leqslant 1$ such that*

$$\frac{\zeta'(\sigma+i\tau)}{\zeta(\sigma+i\tau)} = O((\log t)^2) \tag{5.1}$$

uniformly in $-1 \leqslant \sigma \leqslant 2$. Hence

$$\log \zeta(\sigma+i\tau) = O((\log t)^2) \tag{5.2}$$

uniformly in $-1 \leqslant \sigma \leqslant 2$.

Proof. See Theorem 9.6 (A), p. 184 of [9].

Lemma 5.2. *Let*

$$F(s) = ((\zeta(\tfrac{1}{2}+s))^{kz}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} (\sigma \geqslant 2), \tag{5.3}$$

where $k \geqslant 1$ is any integer. Let $x \geqslant 1000$,

$$\prod_{k^3 \leqslant p \leqslant k^4} \left(1+\frac{k^2}{p^s}\right) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}. \tag{5.4}$$

Then

$$\sum_{n \leqslant x} \frac{b_n}{n} \leqslant \sum_{n \leqslant x} |a_n|^2, \tag{5.5}$$

$$\prod_{k^3 \leqslant p \leqslant k^4} \left(1+\frac{k^2}{p}\right) > \exp(k^2 \log 5/4) \tag{5.6}$$

$$\prod_{k^3 \leqslant p \leqslant k^4} \left(1+\frac{k^2}{p^{1+\delta}}\right) < \exp(k^2 e^{-C_1}) \tag{5.7}$$

where $\delta = C_1/\log k$ and C_1 is any positive constant.

$$\sum_{n \leqslant x} \frac{b_n}{n} > \exp(k^2 \log 5/4) - x^{\delta} \sum_{n > x} \frac{b_n}{n^{1+\delta}} \tag{5.8}$$

$$> \frac{1}{2} \exp(k^2 \log 5/4) \tag{5.9}$$

provided

$$k \geqslant C_2 \left(\frac{\log x}{\log \log x}\right)^{\frac{1}{2}} \tag{5.10}$$

where C_2 is a positive constant.

Proof. Equation (5.5) is trivial. Equations (5.6) and (5.7) follow from $\log(1+y) < y$ and $> y - y^2/2$ for $0 < y < 1$. Equation (5.8) is trivial where (5.9) follows if $x^\delta \leqslant \exp(k^2 e^{-C_1})$ and C_1 is large. This leads to the condition (5.10) for the validity of (5.9).

Lemma 5.3. *We have, with* $k = [C_3(\log H/\log\log H)^{\frac{1}{2}}]$,

$$\left(\frac{C(A)}{\log\log H}\sum_{n \leqslant H/200} |a_n|^2\right)^{1/2k} > \exp\left(k\log\frac{100}{99}\right) \tag{5.11}$$

where C_3 *is a certain positive constant.*

Proof. Follows from (5.5), (5.9) and (5.10).

Lemma 5.4. *The condition* $|a_n| \leqslant (nH)^A$ *is satisfied for some* $A > 0$.

Proof. Follows from the Euler product for $\zeta(s)$.

Lemma 5.5. *Without loss of generality we can assume that*

$$\max_{T \leqslant t \leqslant T+H} |F(it)| \leqslant \exp(\log H)^3). \tag{5.12}$$

Proof. Otherwise the required result follows.

Lemma 5.6. *The inequality* (5.12) *implies* (*subject to* $H \geqslant C_4 \log\log T$ *where* $C_4 > 0$ *is a large constant*) *that*

$$\max_{\sigma \geqslant 0,\, T+(H/9) \leqslant t \leqslant T+(8H/9)} |F(\sigma+it)| \leqslant \exp((\log H)^4). \tag{5.13}$$

Proof. Let $s_0 = \sigma_0 + it_0$ where $0 < \sigma_0 \leqslant 1$ and $T + \frac{H}{9} \leqslant t_0 \leqslant T + \frac{8H}{9}$. Consider the analytic function

$$\phi(s) = F(s)\exp\left(\left(\sin\left(\frac{s-s_0}{100}\right)\right)^2\right).$$

For any real $t \geqslant 10$ let t^* denote the real number τ given by Lemma 5.1.

Let R denote the rectangle with the following corners,

$$s_1 = i\left(t_0 - \frac{H}{10}\right)^*,\ s_2 = i\left(t_0 + \frac{H}{10}\right)^*$$

$$s_3 = 2 + i\left(t_0 + \frac{H}{10}\right)^* \text{ and } s_4 = 2 + i\left(t_0 - \frac{H}{10}\right)^*.$$

On the horizontal sides of R we have

$$|F(s)| = \exp(O(k(\log T)^2)).$$

On the vertical sides we have, by lemma 5.5,

$$|F(s)| \leqslant \exp((\log H)^3).$$

Lemma 5.6 now follows since (by maximum modulus principle) $|F(s_0)| = |\phi(s_0)| \leqslant$ maximum of $|\phi(s)|$ on the boundary of R provided $C_4 > 0$ is a large constant.

Lemma 5.7. *The conditions for applying Theorem* 5 *are satisfied for the interval*

$$T + \frac{H}{9} \leqslant t \leqslant T + \frac{8H}{9}.$$

Proof. Follows from Lemmas 5.4 and 5.6.

The second part of Theorem 3 now follows from Lemma 5.3 (by a slight change of notation).

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2)* K Ramachandra, *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$-IX, *Hardy-Ramanujan J*, **13** (1990), 28–33

in [1]* the author has shown that we can take $C_2 = 3/4$.
in [2]* the author has shown that if

$$\min_{|I|=H} \max_{t \in I} G(1,t) = f(H)$$

the minimum being over all intervals I of length H, then, for all $H \geqslant H_0(\theta)$,

$$|f(H)e^{-\gamma}(\lambda(\theta))^{-1} - \log\log H| \leqslant \log\log\log H + O(1)$$

where

$$\lambda(\theta) = \prod_p \lambda_p(\theta),$$

$$\lambda_p(\theta) = \left(1 - \frac{1}{p}\right)\left(\left(1 - \frac{s^2}{p^2}\right)^{1/2} - \frac{c}{p}\right)^{-c} \exp\left(s \sin^{-1}\frac{s}{p}\right)$$

c and s being defined by $c + is = \exp(i\theta)$. It is not hard to see that $\lambda(\theta)$ is the same as before.