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The seminormality property of circular complexes

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Abstract. In this paper we prove that the ring $R[X, Y]/(X \cdot Y, Y \cdot X)$ is seminormal, where R is a Cohen–Macaulay normal domain and X, Y are matrices of indeterminates.

Keywords. Seminormal; Pic(R), $U(R)$.

1. Introduction

We prove the seminormality property of the Buchsbaum–Eisenbud variety of circular complexes having coordinate ring $A = (R[X, Y]/(X \cdot Y, Y \cdot X))$, where R is a Cohen–Macaulay normal domain.

These varieties have attracted the attention of many people like Buchsbaum, Eisenbud, Kempf, Huneke, etc. (Prof. M S Narasimhan asked during Indo-USSR conference whether the ring A is seminormal.) Elisabetta Strickland [2] has given quite a clear picture of the components of A and she has proved that each component of this variety is normal. We express the ideals of their intersections in suitable forms and observe that these intersections are reduced. We use homological criteria of seminormality and prove that their unions (of components in suitable order) are seminormal, which ultimately proves that the variety is seminormal.

In §2, we define seminormality and quote results proved in [2] and [3]. In §3 we prove the main theorem as stated in the first paragraph.

Before proceeding further we set up few notations.

$X := (X_{ij})_{n_0 \times n_1}$ where X_{ij} is an indeterminate.
$Y := (Y_{ij})_{n_1 \times n_0}$ where Y_{ij} is an indeterminate.
$R[X, Y]$ denotes the R-algebra generated by the entries of X and Y.
$X \cdot Y$ and $Y \cdot X$ denote the sets of entries of the matrices $X \cdot Y$ and $Y \cdot X$ respectively.
$I_r(X)(\text{resp.}\, I_r(Y)) = \{(r+1) \times (r+1) \text{ minors of } X (\text{resp. } Y)\}$
$I(k_1, k_2) = \langle X \cdot Y, Y \cdot X, I_{k_1}(X), I_{k_2}(Y) \rangle$
$(x^0, y^0) \in R^N$, where $N = 2(n_0 \times n_1)$, $x^0 \in M_{n_0 \times n_1}(R)$ and $y^0 \in M_{n_1 \times n_0}(R)$.
$W(k_1, k_2) = \{(x^0, y^0) | x^0 \cdot y^0 = 0,\ y^0 \cdot x^0 = 0,\ rk x^0 \leqslant k_1, rk y^0 \leqslant k_2\}$.
For a given ring R, $R^{[n]} := R[Z_1, \ldots, Z_n]$ where Z_i's are indeterminates.
$U(R)$ = the group of units of R.
Pic(R) = the group of isomorphism classes of invertible modules of R under the binary operation '⊗' (tensor product).

2. Preliminaries

We give an algebraic definition of 'seminormality' which is equivalent to the other notions of seminormality (see [3]).

DEFINITION

A commutative ring R is said to be *seminormal* provided, i) it is reduced. ii) for $a \in Q(R)$, the total quotient ring of R, such that $a^3, a^2 \in R$ then $a \in R$.

Remark. From this definition it is obvious that a normal ring is always seminormal.

For the convenience of the reader we state the necessary results of Strickland [2] and Swan [3].

Theorem 1. (i) *The ring $R[X, Y]/I(k_1, k_2)$ is reduced and has required property that its R-valued points correspond to the set $W(k_1, k_2)$.* (ii) *Furthermore, if the ring R is Cohen–Macaulay normal domain with $k_1 + k_2 \leqslant \min(n_0, n_1)$ then the ring $R[X, Y]/I(k_1, k_2)$ is Cohen–Macaulay normal domain and $\dim R[X, Y]/I(k_1, k_2) = [(n_0 + n_1) - (k_1 + k_2)] \cdot (k_1 + k_2)$.*

Proof. For a proof refer to [2].

Theorem 2. *Let R be any commutative ring. Then the following properties are equivalent.*

i) $Pic(R) = Pic\, R[X_1, \ldots, X_n]$ *for some* $n \geqslant 1$.
ii) $Pic(R) = Pic\, R[X_1, \ldots, X_n]$ *for all n.*
iii) R_{red} *is seminormal.*

Proof. For a proof refer to [3].

3. The main theorem

In this section we state and prove the main theorem. We begin with some lemmas which are required for the proof of the main theorem.

Lemma 1. Let R be a commutative ring, let J and K be two ideals of R such that R/J and R/K are seminormal. Then $R/J \cap K$ is seminormal if and only if $R/J + K$ is reduced.

Proof. Consider the following commutative diagram with the canonical maps

$$\begin{array}{ccc} R/J \cap K & \rightarrow & R/J \\ \downarrow & & \downarrow \eta_1 \\ R/K & \overset{\eta_2}{\rightarrow} & R/(J+K) \end{array}$$

This is a cartesian square with surjective maps η_1 and η_2. Therefore one has a Mayer–Vietoris sequences with the following commutative diagram (see [1]).

$$\begin{array}{ccccccc}
U(R/(J\cap K)) & \rightarrow & U(R/J)\oplus U(R/K) & \rightarrow & U(R/(J+K)) & \overset{h_1}{\rightarrow} \\
 & \downarrow f_1 & & \downarrow f_2 & & \downarrow g_1 \\
U(R/(J\cap K))^{[n]} & \rightarrow & U(R/J)^{[n]}\oplus U(R/K)^{[n]} & \rightarrow & U(R/J+K)^{[n]} & \overset{h_2}{\rightarrow}
\end{array}$$

$$\begin{array}{ccccc}
\overset{h_1}{\rightarrow} & \mathrm{Pic}(R/(J\cap K) & \rightarrow & \mathrm{Pic}(R/J)\oplus\mathrm{Pic}(R/K) & \rightarrow \\
 & \downarrow g_2 & & \downarrow g_3 & \\
\overset{h_2}{\rightarrow} & \mathrm{Pic}(R/J\cap K)^{[n]} & \rightarrow & \mathrm{Pic}(R/J)^{[n]}\oplus\mathrm{Pic}(R/K)^{[n]} & \rightarrow
\end{array}$$

where all vertical arrows are injective. Now R/J and R/K reduced implies f_1 is an isomorphism and as the rings are seminormal, we have the isomorphic map g_2 (by theorem 2). Therefore f_2 is an isomorphism if and only if g_1 is. In other words $R/J+K$ is reduced if and only if $R/J\cap K$ is seminormal.

Lemma 2. Let the ring R be a Cohen–Macaulay, normal domain and let I_i denote the ideal $\langle X\cdot Y, Y\cdot X, I_i(X), I_{n_1-i}(Y)\rangle$. Then the ideal $I_0\cap\cdots\cap I_r = \langle X\cdot Y, Y\cdot X, I_r(X)\rangle$ for every $0\leqslant r\leqslant n_1$, where $n_1\leqslant n_0$.

Proof. First we prove the following equality

$$W(r,n_1) = W(0,n_1)\cup W(1,n_1-1)\cup\cdots\cup W(r,n_1-r).$$

We denote the right hand side of the equation by W_r. Take $(x^0,y^0)\in W(r,n_1)$. Then we have $x^0\cdot y^0=0, y^0\cdot x^0=0$ and $rkx^0=i\leqslant r$. But $x^0\cdot y^0=0$ implies $rkx^0+rky^0\leqslant n_1$ which implies $rky^0\leqslant n_1-i$ and therefore $(x^0,y^0)\in W(i,n_1-i)$. Conversely, it is obvious that the set $W_r\subseteq W(r,n_1)$. Hence we have $W(r,n_1)=W_r$. Therefore

$$I(W(r,n_1)) = I(W_r) = I(W(0,n_1))\cap\cdots\cap(I(W(r,n_1-r)).$$

By theorem (1) this is equivalent to

$$\langle X\cdot Y, Y\cdot X, I_r(X)\rangle = I_0\cap\cdots\cap I_r$$

for $1\leqslant r\leqslant n_1$.

Now we prove the main theorem.

Theorem. *Let R be a Cohen–Macaulay normal domain.*

i) *The ring $R[X,Y]/\langle X\cdot Y, Y\cdot X\rangle$ is seminormal.*
ii) *It has n_1+1 components, each of dimension $n_0\cdot n_1$, where we assume that $n_1\leqslant n_0$.*

Proof. Let $A=R[X,Y]$ and $I=\langle X\cdot Y, Y\cdot X\rangle$ and $I_i=I(i,n_1-i)$. By theorem 1, the ring A/I is reduced and A/I_i is normal hence seminormal. By lemma 2, one has $I=I_0\cap\cdots\cap I_{n_1}$. To prove the theorem, it is enough to prove that, the ring $A/I_0\cap\cdots\cap I_{r-1}$ seminormal implies $A/I_0\cap\cdots\cap I_r$ is seminormal for any $r\leqslant n_1$. Let us denote by J the ideal $I_0\cap\cdots\cap I_{r-1}$ and by K the ideal I_r then $J+K=\langle X\cdot Y, Y\cdot X, I_{r-1}(X), I_{n_1-r}(Y)\rangle$.

Therefore by theorem 1, $A/J+K$ is reduced. Hence by lemma 1 $A/J\cap K$ is seminormal. Thus $R[X, Y]/\langle X\cdot Y, Y\cdot X\rangle$ is seminormal.

Now we have $\langle X\cdot Y, Y\cdot X\rangle = I_0\cap\cdots\cap I_{n_1}$. Consider the ideals I_i, I_j where $i<j$. Consider the canonical surjective maps $\eta_1:R[X, Y]\to R[X]$ sending all the entries of Y to zero and $\eta_2:R[X, Y]\to R[Y]$ sending all the entries of X to zero. Since $\eta_1(I_i)=\langle I_i(X)\rangle$ and $\eta_1(I_j)=\langle I_j(X)\rangle$, we have $I_i\not\subseteq I_j$. Similarly, as $\eta_2(I_i)=\langle I_{n_1-i}(Y)\rangle$ and $\eta_2(I_j)=\langle I_{n_1-j}(Y)\rangle$ we have $I_j\not\subseteq I_i$. Therefore $I_0,\cdots,I_{n_1}$ are minimal primes of $\langle X\cdot Y, Y\cdot X\rangle$. Moreover, by theorem 1 $\dim R[X, Y]/I_i = n_0\cdot n_1$.

Hence we conclude that the ring $R[X, Y]/\langle X\cdot Y, Y\cdot X\rangle$ is seminormal with n_1+1 equidimensional, normal, Cohen–Macaulay components.

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References

[1] Bass H, *Albegraic K-Theory*, (New York: Benjamin) 1968

[2] Strickland E, On the conormal bundle of determinantal variety, *J. Algebra* **75** (1982) 523–537

[3] Swan R G, On seminormality. *J. Algebra* **67** (1980) 210–229