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Subordination properties of certain integrals

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Abstract. Let $B_1(\mu,\beta)$ denote the class of functions $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ that are analytic in the unit disc Δ and satisfy the condition $\mathrm{Re} f'(z)(f(z)/z)^{\mu-1}>\beta$, $z\in\Delta$, for some $\mu>0$ and $\beta<1$. Denote by $S^*(0)$ for $B_1(0,0)$. For μ and c such that $c>-\mu$, let $F=I_{\mu,c}(f)$ be defined by

$$F(z)=\left[\frac{\mu+c}{z^c}\int_0^z f^\mu(t)t^{c-1}\mathrm{d}t\right]^{1/\mu},\quad z\in\Delta.$$

The author considers the following two types of problems:

(i) To find conditions on μ, c and $\rho>0$ so that $f\in B_1(\mu,-\rho)$ implies $I_{\mu,c}(f)\in S^*(0)$;
(ii) To determine the range of μ and $\delta>0$ so that $f\in B_1(\mu,-\delta)$ implies $I_{\mu,o}(f)\in S^*(0)$;

We also prove that if f satisfies $\mathrm{Re}(f'(z)+zf''(z))>0$ in Δ then the nth partial sum f_n of f satisfies $f_n(z)/z\prec-1-(2/z)\log(1-z)$ in Δ. Here, $\prec$ denotes the subordination of analytic functions with univalent analytic functions. As applications we also give few examples.

Keywords. Differential subordination; univalent star-like; convex function.

1. Introduction

Let A denote the class of functions f, $f(z)=z+a_2z^2+\cdots$, that are analytic in the unit disc $\Delta=\{z:|z|<1\}$. For a given real number, $\beta<1$, let $S^*(\beta)$ and $K(\beta)$ represent the subclasses of A consisting of star-like functions of order β and convex functions of order β, respectively. Let $B_1(\mu,\beta)$ be such that

$$\mathrm{Re} f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}>\beta,\quad z\in\Delta$$

for some $\mu>0$ and $\beta<1$; and that let $R(\beta)\equiv B_1(1,\beta)$. For $0\leqslant\beta<1$ functions in these classes are in fact univalent in Δ [1].

For $f\in A$, and μ and c such that $\mu+c>0$, let $F=I_{\mu,c}(f)$ be defined by

$$F(z)=\left[\frac{\mu+c}{z^c}\int_0^z f^\mu(t)t^{c-1}\mathrm{d}t\right]^{1/\mu},\quad z\in\Delta. \tag{1}$$

Many authors have studied this operator in various situations [1, 8, 10]. However from a more general result obtained in [4], the author as a special case concerning

Alexander and Libera transforms established that

$$I_{0,1}(R(-\delta_1)) \subset S^*(0), \quad \text{for } \delta_1 = 0{\cdot}35\ldots; \tag{2}$$

$$I_{0,1}(R(-\delta_2)) \subset S^*(\beta), \quad \text{for } \delta_2 = 0{\cdot}29\ldots; \tag{3}$$

$$I_{1,1}(R(-\delta_3)) \subset S^*(0), \quad \text{for } \delta_3 = 0{\cdot}09\ldots; \tag{4}$$

$$I_{1,1}(R(-\delta_1)) \subset S^*(\beta), \quad \text{for } \delta_4 = 0, \tag{5}$$

where β is the positive root of a cubic polynomial, and it may be observed that the bounds appearing in the above inclusions are not the best possible ones.

These relations actually improve the earlier results of Mocanu [3], and Singh and Singh [11] who respectively proved (4) and (2) with $\delta_1 = -1/4$ and $\delta_3 = 0$. Further, the question concerning the sharpness of the above implications is still open. However a better bound can be found. For example the author [6] recently described a method for improving the bound for (3). Although the members of a class R_1, $R_1 = \{f \in A: |f'(z) - 1| < 1\}$, are univalent and bounded, the author [5] demonstrated a function $f_0 \in R_1$ such that $f_0 \notin S^*$. In spite of this observation, in the present note, our methods in fact further yield an interesting generalization from $I_{1,c}(R(\beta)) \subset S^*(0)$ to $I_{\mu,c}(B_1(\mu,\beta)) \subset S^*(0)$ concerning the integral operator defined by (1).

2. Preliminaries and main results

If f and g are analytic in Δ, we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if g is univalent in Δ, $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

We are going to prove the following results.

Theorem 1. *Let*

$$L_{\mu,c}(z) = -1 + 2\int_0^1 \frac{dt}{1 - zt^{1/(\mu+c)}}, \quad M = [1 - L_{\mu,c}(-1)][1 - L_{\mu,0}(-1)],$$

$$N = [1 - L_{\mu,0}(-1)]^2, \quad S = \frac{2\mu+1}{c^2}[M^2 + 2M(\mu+c)] - 3N,$$

$$T = -2\left(\frac{2\mu+1}{c^2}\right)[(1-M)(M+\mu+c)] - 6N,$$

$$W = \frac{2\mu+1}{c^2}(1-M)^2 - 3N \quad \text{and} \quad \rho = \frac{-T - (T^2 - 4SW)^{1/2}}{2S}.$$

If $f \in A$ satisfies

$$\mathrm{Re}\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > -\rho$$

then for $F(z)/z \neq 0$ in Δ,

$$F = I_{\mu,c}(f) \in S^*(0)$$

for $0 < \mu \leqslant 2/\sqrt{3}$ and $-\mu < c \leqslant \sqrt{(2\mu+1)/3}$.

An adaptation of the same method would give us the following theorem to deal with the case $c = 0$.

Theorem 2. *Let $\mu > 0$ be determined from*

$$N = [1 - L_{\mu,0}(-1)]^2 \geqslant 1.$$

Then for this range of μ, we, for $(I_{\mu,0}(f))(z)/z \neq 0$ in Δ, have

$$\mathrm{Re}\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > -\delta = -\frac{N-1}{N+2\mu} \textit{ implies } I_{\mu,0}(f) \in S^*(0).$$

To prove our theorems, we require the following lemmas:

Lemma A. [2] *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that the function $\psi:\mathbb{C}^2 \times \Delta \to \mathbb{C}$ satisfies the condition $\psi(ix, y; z) \notin \Omega$, for all real $x, y \leqslant -(1 + x^2)/2$ and all $z \in \Delta$. If the function p defined by $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic in Δ and if $\psi(p(z), zp'(z); z) \in \Omega$, then* Re *$p(z) > 0$ in Δ.*

Using the arguments of [6, lemma 1] (see also [4]) it is clear that the following more general lemma holds.

Lemma B. If p is analytic in Δ with $p(0) = \exp(i\gamma)$, (γ real and fixed), $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha \geqslant 0 (\alpha \neq 0)$, *$\beta < 1$ and that*

$$Re\{p(z) + \alpha z p'(z)\} > \beta cos\gamma, \quad z \in \Delta$$

then

$$Re p(z) > \beta cos\gamma + (1 - \beta) cos\gamma [2\delta(Re\alpha) - 1], \quad z \in \Delta$$

where δ is given by

$$\delta = \delta(Re\alpha) = \int_0^1 \frac{\mathrm{d}t}{1 + t^{Re\alpha}}$$

is an increasing function of $Re\alpha$ and $(1 + Re\alpha)/(1 + 2Re\alpha) \leqslant \delta < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Example 1. If we let $R_\gamma(\beta)$ to be the class of analytic functions

$$f(z) = \exp(i\gamma) z + a_2 z^2 + \ldots\ldots, \quad (\gamma \text{ real and fixed})$$

defined on Δ, so that $\mathrm{Re} f'(z) > \beta \cos\gamma$, then

$$f \in R_\gamma(\beta) \text{ implies } f(z)/z \prec \beta\cos\gamma + i \sin\gamma$$
$$+ (1 - \beta)\cos\gamma[-1 - (2/z)\log(1 - z)], \quad z \in \Delta.$$

Lemma C. [6, Corollary 1] *Let p be analytic in Δ, with $p(0) = 1$. Suppose that an analytic function λ on Δ satisfies*

$$|Im\lambda(z)| \leqslant \sqrt{3}(Re\lambda(z) - \sqrt{3}/2), \quad z \in \Delta \tag{6}$$

then

$$Re\{p(z) + \lambda(z) z p'(z)\} > 0 \tag{7}$$

implies $|arg p(z)| < \pi/3$, $z \in \Delta$.

It seems fitting to add the following result at this point, since it is also related to our further investigation.

COROLLARY

Let $c \neq 0$ be a complex number and p and h be analytic in Δ, with $p(z)\cdot h(z) \neq 0$,

$$p(z) + \frac{zp'(z)}{c} = h(z) \tag{8}$$

and

$$\left| arg\left(\frac{p(z)}{cp(z) + zp'(z)} - \frac{\sqrt{3}}{2}\right)\right| < \frac{\pi}{3}, \quad z \in \Delta. \tag{9}$$

Let g be analytic in Δ, $g(0) = 1$ with $Re g(z) > 0$, for $z \in \Delta$. If $G = I(g)$ is defined by

$$G(z) = cz^{-c}p(z)^{-1}\int_0^z g(t)t^{c-1}h(z)\mathrm{d}t, \quad z \in \Delta \tag{10}$$

then G is analytic in Δ, $G(0) = g(0)$ and $|arg\, G(z)| < \pi/3$ for Δ.

Proof. Since $z = 0$ in (8) and (9) gives $p(0) = h(0)$ and $|\arg(1/c - \sqrt{3}/2)| < \pi/3$, the conditions on p, h and g imply that G is analytic in Δ, $G(0) = 1$. Introduce $\lambda(z) = p(z)/(cp(z) + zp'(z))$. Since $\mathrm{Re}\, g(z) > 0$, by differentiating (10) we obtain

$$\mathrm{Re}(G(z) + \lambda(z)zG'(z)) = \mathrm{Re}\, g(z) > 0, \quad z \in \Delta.$$

Hence conditions (6) and (7) of Lemma C is satisfied with $p = G$, and so we conclude that $|\arg G(z)| < \pi/3$. □

Example 2. If we take $p(z) = \exp(\gamma z)$ (and hence $h(z) = (1 + (\gamma/c)z)\exp(\gamma z)$, we obtain: if g is analytic in Δ with $g(0) = 1$ then

$$\mathrm{Re}\, g(z) > 0 \text{ implies} \left|\arg\left(cz^{-c}\exp(\gamma z)\int_0^z g(z)t^{c-1}\exp(\gamma t)(1 + (\gamma/c)t)\mathrm{d}t\right)\right| < \pi/3,$$

provided

$$\left|\arg\left(\frac{1}{c + \gamma z} - \frac{\sqrt{3}}{2}\right)\right| < \pi/3 \text{ for } z \in \Delta.$$

As an immediate application of this result and Lemma *B*, it follows that the function q_α defined by (see also [6])

$$q_\alpha(z) = \frac{1}{\alpha}z^{-1/\alpha}\int_0^z \frac{1+t}{1-t}t^{1/\alpha - 1}\mathrm{d}t = -1 + 2\int_0^1 \frac{\mathrm{d}t}{1 - zt^\alpha}, \quad z \in \Delta \tag{11}$$

satisfies

$$\mathrm{Re}\, q_\alpha(z) > -1 + 2\int_0^1 \frac{\mathrm{d}t}{1 + t^{\mathrm{Re}\,\alpha}}, \text{ provided Re } \alpha > 0; \tag{12}$$

and

$$|\arg q_\alpha(z)| < \pi/3, \quad \text{provided } |\arg(\alpha - \sqrt{3}/2| < \pi/3. \tag{13}$$

Lemma D. If $p(z)$ is analytic in Δ, $p(0)=1$, and $\operatorname{Re} p(z) > 1/2$, $z\in\Delta$, then for any function g, analytic in Δ, the function $p*g$ takes values in the convex hull of the image of Δ under g.

Using Lemma D, which directly follows by using the Herglotz's representation, we draw the following example:

Example 3. For

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \in A, \quad \text{let } f_n(z)=z+\sum_{k=2}^{n} a_k z^k.$$

Then

$$f_n(z)/z = p(z)*(-1+2\varphi(z)),$$

where * is meant for Hadamard product (or convolution),

$$p(z)=1+\frac{1}{2}\sum_{k=2}^{\infty} k^2 a_k z^{k-1} \quad \text{and } \varphi(z)=\left(1+\sum_{k=2}^{n} k^{-2} z^{k-1}\right).$$

If $f\in A$ satisfies

$$\operatorname{Re}(f'(z)+zf''(z))>0, \quad z\in\Delta,$$

then the function p defined above shows that $\operatorname{Re} p(z) > 1/2$ in Δ. From the fact that [7]

$$\operatorname{Re}\left(g(z)=1+\sum_{k=2}^{n} k^{-1} z^{k-1}\right) > 1/2, \quad z\in\Delta$$

we from (11) easily deduce that

$$\varphi(z)=\frac{1}{z}\int_0^z g(t)\mathrm{d}t \prec \tfrac{1}{2}+\tfrac{1}{2}\left[-1+2\int^1 \frac{\mathrm{d}t}{1-zt}\right],$$

or, equivalently

$$2\varphi(z)-1 \prec E(z), \quad z\in\Delta,$$

where $E(z)=-1-(2/z)\log(1-z)$, and $E(\Delta)\subset\{\omega:\operatorname{Re}\omega>2\ln 2-1\}\cap\{w:|\arg\omega|<\pi/3\}\cap\{\omega:|\operatorname{Im}\omega|<\pi\}$.

This from Lemma D proves that if $zf'\in R(0)$ then for every $n\geqslant 1$ we have

$$f_n(z)/z \prec E(z), \quad z\in\Delta.$$

Remark. In 1928, [12] Szegö proved that if $f\in K(0)$ ($S^*(0)$), then all sections f_n are convex (starlike with respect to origin) in $|z|<1/4$. Recently in [9], Ruscheweyh obtained these as very special cases of more general results.

Proof of the Theorem 1. Consider the function P defined by

$$P(z)=F'(z)(F(z)/z)^{\mu-1}, \quad z\in\Delta.$$

Then P is analytic in Δ, $P(0)=1$ and therefore using (1) and a little calculation, we have

$$P(z)+\frac{zP'(z)}{\mu+c}=f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}, \quad z\in\Delta. \tag{14}$$

Since $f\in B_1(\mu,-\rho)$, (14) gives that,

$$P(z)\prec -\rho+(1+\rho)\,L_{\mu,c}(z). \tag{15}$$

Thus (15) becomes, if we use either (12) or Lemma B,

$$\operatorname{Re} F'(z)\left(\frac{F(z)}{z}\right)^{\mu-1} > -\rho+(1+\rho)\,L_{\mu,c}(-1), \quad z\in\Delta. \tag{16}$$

If we substitute $\beta=-\rho+(1+\rho)\,L_{\mu,c}(-1)$ and note that we can now rewrite (16) as $F\in B_1(\mu,\beta)$, we at once see from Lemma B that

$$\left(\frac{F(z)}{z}\right)^{\mu}\prec\beta+(1-\beta)\,L_{\mu,0}(z),$$

holds in Δ. From (11), (12) and (13) it can be observed that the function Q defined by $Q(z)=(F(z)/z)^{\mu}$ satisfies

$$Q(\Delta)\subset\Omega_1\cap\Omega_2, \quad \text{provided } 0<\mu\leqslant 2/\sqrt{3}, \tag{17}$$

where

$$\Omega_1=\{\omega\in\mathbb{C}:\operatorname{Re}\omega>\beta+(1-\beta)\,L_{\mu,0}(-1)\} \quad \text{and}$$

$$\Omega_2=\{\omega\in\mathbb{C}:|\arg(\omega-\beta)|<\pi/3\}.$$

We now apply Lemma A. If we denote

$$p(z)=zF'(z)/F(z)$$

then p is analytic in Δ, $p(0)=1$ and from (1) it follows easily that

$$\frac{Q(z)}{\mu+c}[zp'(z)+\mu p^2(z)+cp(z)]=f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}.$$

Thus if $f\in B_1(\mu,-\rho)$, we get from the above identity that

$$\{\psi(p(z),zp'(z);z):|z|<1\}\subset\Omega=\{\omega\in\mathbb{C}:\operatorname{Re}\omega>-\rho\},$$

where

$$\psi(r,s;z)=Q(z)[s+\mu r^2+cr]/(\mu+c).$$

To prove our theorem, it suffices to show, by Lemma A that

$$\psi(ix,y;z)\notin\Omega \tag{18}$$

for all real x, $y\leqslant-(1+x^2)/2$ and all $z\in\Delta$.

If we set $\operatorname{Re}Q(z)=U$ and $\operatorname{Im}Q(z)=V$, we see that

$$\operatorname{Re}\psi(ix,y;z)=[U(y-\mu x^2)-Vcx]/(\mu+c)$$
$$\leqslant-[U(1+2\mu)x^2+2Vcx+U]/(2(\mu+c)),$$

when x is real and $y \leqslant -(1+x^2)/2$. So (18) holds provided $Q = U + iV$ satisfies

$$\left[\frac{U-\rho(\mu+c)}{\rho(\mu+c)}\right]^2 - \left[\frac{V}{(2\mu+1)^{1/2}(\mu+c)\rho/|c|}\right]^2 \geqslant 1. \tag{19}$$

It is easy to check that

$$\beta + (1-\beta)\, L_{\mu,0}(-1) = 1-(1+\rho)M \quad (= U_0, \text{say}),$$

and

$$\sqrt{3}(1-\beta) = \sqrt{3}(1+\rho)(1-L_{\mu,0}(-1)) \quad (= V_0, \text{say}).$$

So finally, (19) holds if

$$\left[\frac{U_0-\rho(\mu+c)}{\rho(\mu+c)}\right]^2 - \left[\frac{V_0}{(2\mu+1)^{1/2}(\mu+c)\rho/|c|}\right]^2 = 1,$$

which holds because by the hypothesis ρ is the smallest positive root of the equation

$$S\rho^2 + T\rho + W = 0,$$

where S, T and W are as in the statement of Theorem 1. The proof of Theorem 1 is, therefore, complete. □

Proof of Theorem 2. Using the same technique as in the proof of Theorem 1, to prove Theorem 2, by Lemma A, it is sufficient to show that

$$\psi(ix, y; z) \notin \Omega = \{\omega \in \mathbb{C} : \operatorname{Re}\omega > -\delta\},$$

when x real, $y \leqslant -(1+x^2)/2$ and all $z \in U$; where

$$\psi(r, s; z) = Q(z)(s+\mu r^2)/\mu \quad \text{with } Q(z) = (F(z)/z)^{\mu} \text{ and } I_{\mu,0}(f) = F$$

and δ is defined in Theorem 2.

By hypothesis $f \in B_1(\mu, -\delta)$ and so this gives $F \in B_1(\mu, -\delta + (1+\delta)\, L_{\mu,0}(-1))$. In view of this and Lemma B, we, from a simple manipulation, easily obtain that $\operatorname{Re} Q(z) > 2\mu\delta$ in Δ.

Thus, we have for real $x, y \leqslant -(1+x^2)/2$ and all $z \in U$,

$$\operatorname{Re}\psi(ix, y; z) \leqslant -(2\mu\delta)/(2\mu) = -\delta,$$

and so the conclusion of Theorem 2 follows. □

Remark. We observe that Re $f'(z)(f(z)/z)^{\mu-1} > -\delta$, $(\delta > 0)$, $z \in \Delta$, need not imply the univalence of f in Δ. Further it is also clear from the proofs that the bounds of Theorems 1 and 2 are not the best possible ones. Morever, the method of proof allows us to find better estimation if one is able to determine the best values of δ, $(\delta = \delta(\alpha) > 0)$, and η for which

$$\operatorname{Re}(p(z) + \alpha z p'(z)) > -\delta \text{ implies } |\arg p(z)| < \pi\eta/2, \quad z \in \Delta$$

whenever p is analytic in Δ with $p(0) = 1$.

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