

Self-dual connections, hyperbolic metrics and harmonic mappings on Riemann surfaces

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Abstract. The Sampson-Wolf model of Teichmüller space (using harmonic mappings) is shown to be exactly the same as the more recent Hitchin model (utilizing self-dual connections). Indeed, it is noted how the self-duality equations become the harmonicity equations. An interpretation of the modular group action in this model is mentioned.*

Keywords. Teichmüller space; self-dual connection; hyperbolic metric; harmonic maps.

1. Introduction

Let M be a compact Riemann surface of genus $g (g \geq 2)$ with hyperbolic (Poincaré) metric on M denoted by σ . We will consider (M, σ) as the base point of the Teichmüller space $T(M) = T_g$. As usual, $T(M)$ parametrizes all the possible hyperbolic metrics (constant negative curvature -4) on the smooth surface M up to pullbacks by diffeomorphisms homotopic to the identity. The canonical bundle on (M, σ) will be called K .

In his Stanford thesis, Wolf [5], following on results of Sampson [4], discovered a natural homeomorphism of the Teichmüller space onto the full vector space, $H^0(M, K^2)$, of holomorphic quadratic differentials on the base Riemann surface (M, σ) . The method is to look at the unique harmonic diffeomorphism $w = w(\sigma, \rho)$, homotopic to the identity, from (M, σ) to (M, ρ) where ρ is another hyperbolic metric on the smooth surface M . The $(2, 0)$ part of the pullback of ρ by w is a *holomorphic* quadratic differential on the base surface by virtue of w being a *harmonic* diffeomorphism. Wolf associates this quadratic differential to the point of $T(M)$ represented by (M, ρ) . Thus one has the homeomorphism

$$\mathscr{W}: T(M) \rightarrow H^0(M, K^2). \quad (1)$$

Of course, \mathscr{W} is not a complex analytic parametrization of T_g since it is well-known

* Acknowledgement: I noted the above identity between Wolf's model and Hitchin's model in 1990. Some experts in the area have pointed out to me that this relationship was privately noted by other mathematicians before me; in particular M Wolf knew it, and communicated it to N J Hitchin, and it was also known to J Eells—and surely others. However, this cute fact is not in published form anywhere to my knowledge, so I think it may be worthwhile to point it out to the larger community of geometers and analysts in this field.

that holomorphically T_g is a bounded domain. See Nag [3] for the basic facts needed from Teichmüller theory.

Also recently, using self-dual connections on the compact Riemann surface M , Hitchin [1, 2] has given a method to describe the various possible hyperbolic metrics on the smooth surface M . The choice of the self-dual connection, and hence of the corresponding hyperbolic metric, depends only on the choice of an arbitrary holomorphic quadratic differential on M . Thus Hitchin's result again provides a natural parametrization of the Teichmüller space $T(M)$ by the full vector space of holomorphic quadratic differentials $H^0(M, K^2)$. Namely one gets Hitchin's homeomorphism

$$\mathcal{H}: H^0(M, K^2) \rightarrow T(M). \quad (2)$$

Our main remark is that Hitchin's model of Teichmüller space as a homeomorphic copy of the vector space $H^0(M, K^2)$ is precisely that discovered by Sampson and Wolf. Namely *the mappings \mathcal{W} and \mathcal{H} are simply inverses to each other*. In fact, we show by a simple change of variables how the self-duality equation becomes the main equation studied by Wolf.

2. Hitchin's method in summary

In the present context, the equation that Hitchin uses to describe the relevant "self-dual connections", depends on the choice of an arbitrary $q (\in H^0(M, K^2))$, and produces a certain hermitian metric $h = h(q)$ on the Riemann surface M as the solution. The self-duality equation asks for this self-dual hermitian metric, h , on M having curvature 2-form $F(h)$ and Kahler form $\omega(h)$ satisfying:

$$F(h) = -2(1 - \|q\|^2)\omega(h)^2. \quad (3)$$

Here $\|q\|^2 = |q|^2/h^2$, represents the squared-norm of q as a function on M with respect to the sought-for metric h . In local holomorphic coordinates z on the base Riemann surface M this reads as follows:

Let $h = h(z)dz \otimes d\bar{z}$ and $q = q(z)dz^2$. Then $\omega(h) = h(z)dz \wedge d\bar{z}$, and $F(h) = -\Delta \log h \cdot dz \wedge d\bar{z}$. So (3) becomes:

$$\Delta \log h = 2(1 - |q|^2/h^2)h \quad (4)$$

with

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2, (z = x + iy).$$

Remark. Notice that if q is the zero quadratic differential then the conformal (hermitian) metric h has constant negative curvature -4 , (as is obvious from (4)). In this case, h is itself the Poincaré metric of M .

Remark. We refer to Hitchin [1, 2] for the reason why (3) arises as the self-dual connections equation for a certain vector bundle over M .

3. The main idea

The interesting discovery by Hitchin is that for arbitrary choice of the holomorphic quadratic differential q on M one can produce a new hyperbolic metric on M , called $h^* = h^*(q)$, via the formula

$$h^* = q + (h + q\bar{q}/h) + \bar{q}. \tag{5}$$

This h^* is, indeed, of constant negative curvature -4 , again. Hitchin’s homeomorphism maps q to the Teichmüller point represented by $(M, h^*(q))$.

Thus h^* is a hyperbolic Riemannian metric on M which is smooth with respect to the original complex structure of M , but is not hermitian with respect to that complex structure for any non-zero choice of the quadratic differential q . Indeed, note that h^* has been written in (5) above in its standard type decomposition into $(2, 0) + (1, 1) + (0, 2)$ parts on M . Because of the presence of non-vanishing $(2, 0)$ and $(0, 2)$ parts, (the q and its conjugate respectively), this h^* cannot be a hermitian (conformal) metric for M .

Note. For some choices of q , however, h^* will produce the same conformal structure on M as the original one, up to diffeomorphism. These quadratic differentials q will be precisely the ones that are Teichmüller modular group translates of the zero differential in this Sampson–Wolf–Hitchin model of Teichmüller space $T(M)$. It would be quite interesting to investigate the nature of these q ’s that are modular equivalent to zero.

Proof that $\mathcal{W} = \mathcal{H}^{-1}$. Consider the identity diffeomorphism $1:(M, \sigma) \rightarrow (M, h^*(q))$. Equation (5) shows that $\rho = h^*(q)$ is a hyperbolic metric such that the $(2, 0)$ part of the pullback $1^*(\rho)$ is the holomorphic quadratic differential q on the base surface. Thus 1 must be the unique harmonic diffeomorphism in its homotopy class, and the definition of Wolf’s map \mathcal{W} shows that $\mathcal{W}(\rho) = q$, as needed. ■

Remark. Hitchin has used the idea above in §11 of his paper [1].

4. The equations of self-duality and harmonicity

In Wolf’s work the equation for the harmonic diffeomorphism $w = w(\sigma, \rho)$ is studied in terms of the “holomorphic energy density” function $H = H(w)$ on the base (M, σ) :

$$\mathcal{H} = \{ \rho(w(z))/\sigma(z) \} \cdot |w_z|^2 \tag{6}$$

H satisfies the p.d.e.:

$$(1/\sigma)\Delta \log H = 2(H - |q|^2/(\sigma^2 H)) - 2 \tag{7}$$

(see [5] Chapter I §2). It is not surprising that Hitchin’s model of T_θ coincides with Wolf’s since the self-duality eq. (3) or (4) is easily related to eq. (7) of harmonicity. In fact we have:

PROPOSITION

The energy density function H for the harmonic diffeomorphism w is strictly positive on M , and is given by the ratio $H = h/\sigma$, where h is the “self-dual” hermitian metric determined by (3).

Proof. Set $h = \sigma H$ and substitute in Hitchin’s equation (3) or (4). A short calculation (which uses the fact that σ is a hyperbolic metric) shows that H satisfies (7), as required. The fact that H is positive is well-known (see [5]). ■

From the above Proposition we can now write the expression for the new hyperbolic metric $\rho = h^*(q)$ from (5):

$$\rho = h^* = q + \sigma(H + |q|^2/(\sigma^2 H)) + \bar{q} \quad (8)$$

utilizing the solution H of Wolf’s equation (7). This expression for ρ can actually be seen in [5] Chapter II—thus proving again the identity of the two methods of describing all the hyperbolic structures on M .

In concluding this note we remark that it is interesting to see the ubiquitous role played by the Laplace operator in the various elliptic p.d.e.’s that describe the hyperbolic uniformization of Riemann surfaces. Thus, we see its presence in (4), again in (7), as well as in the classical Liouville equation for hyperbolic metrics.

See [6] for further related work by Wolf.

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