Proc. Indian Acad. Sci. (Math. Sci.), Vol. 101, No. 3, December 1991, pp. 199–213.

Curves on threefolds with trivial canonical bundle

KAPIL H PARANJAPE

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Present address: Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago IL 60637, USA

MS received 3 March 1991

Abstract. C H Clemens has shown that homologically trivial codimension two cycles on a general hypersurface of degree five and dimension three form a subgroup of infinite rank inside the intermediate jacobian. We generalize this to other complete intersection threefolds with trivial canonical bundle.

Keywords. Algebraic cycles; algebraic threefolds; intermediate jacobian; rational curves.

1. Introduction

This paper is devoted to the study of rational curves on complex threefolds with trivial canonical bundle. Clemens ([5] and [4]) has asked if a simply connected threefold which has trivial canonical bundle always contains smooth rational curves. As pointed out by V Srinivas, the étale quotient of a product of three elliptic curves constructed by Igusa [7] is an example of a threefold with trivial canonical bundle and vanishing first Betti number which contains no rational curves; thus the hypothesis of simple connectivity is necessary.

In an earlier paper [2] Clemens has shown the existence of *rigid* rational curves on the generic quintic hypersurface. Further, it is shown (*loc. cit.*) that these curves generate a subgroup of infinite rank inside the Griffiths group of the generic quintic.

These results naturally raise the question as to whether the phenomenon of rigidity of all rational curves and infinite generation of the Griffiths group occurs for all *generic* simply connected K-trivial threefolds. However, it was pointed out by C Schoen that if the Picard number is greater than one, rational curves are not in general rigid. Hence the class of varieties for which one can expect the results on quintics to generalize is that of simply-connected K-trivial threefolds with Picard number 1.

In this paper we study some special examples of such varieties—the complete intersections in $\mathbf{P}^5$. We prove results analogous to those of Clemens for these complete intersection threefolds.

The organization of the paper is as follows:

In §2 we give a summary of the results of Clemens [2] which allow one to prove infinite rank. The methods are completely general and ought to find applications in other dimensions as well.

In § 3 we give a general construction to which the results of § 2 can be applied. Here again the basic construction is for curves on a general hyperplane section of a del Pezzo fourfold and ought to be generalizable.

In § 4 we show that the methods of the previous two sections apply to the complete intersections in $\mathbf{P}^5$. We also summarize the arguments in the form of a theorem. It should be possible to refine these methods to prove the results for complete intersection subvarieties of Grassmanians and other homogeneous spaces. However, for other simply connected K-trivial threefolds, there does not appear to be a method available.

In an Appendix we prove a Bertini type result which is needed in § 2.

2. Summary of Clemens results

Let S be a smooth curve, $\pi:\mathscr{X}\to S$ be a projective family of threefolds with χ smooth, and π smooth except at $o\in\mathscr{X}$ which is an ordinary double point in the fibre X_0 over $o\in S$. Let $\tilde{X}_0\to X_0$ be the blow up of the singular point and E be the exceptional divisor for p; then we have $E\cong\mathbf{P}^1\times\mathbf{P}^1$. Let $t\in S$ be given by an inclusion of the function field of S in the complex numbers, henceforth we refer to such a t as a geometric generic point of S. Let X_t be the geometric generic fibre of π.

Lemma 1. *With notation as above, the following are equivalent*:

(i) *The action of monodromy on* $H^3(X_t,\mathbf{Z})$ *is non-trivial.*
(ii) *The vanishing cycle* $\ker(H_3(X_t,\mathbf{Z})\to H_3(X_0,\mathbf{Z}))$ *is non-zero.*
(iii) *The Hodge structure* $H^3(X_0)$ *is* not *pure.*
(iv) *The morphism* $\operatorname{Pic}(\tilde{X}_0)\otimes\mathbf{Q}\to\operatorname{Pic}(E)\otimes\mathbf{Q}$ *is not surjective.*

Proof. Let $\delta\in H^3(X_t,\mathbf{Z})$ denote the "co-vanishing" cohomology class. The action of monodromy on $H^3(X_t,\mathbf{Z})$ is given by $x\mapsto x+(x,\delta)\delta$. Thus δ is trivial if and only if the monodromy action is trivial. This gives the equivalence of (i) and (ii).

In the following exact sequence of mixed Hodge structures

$$0\to H^3(X_0)\to H^3(X_t)_{\lim}\to\mathbf{Z}(-2),$$

the latter map is given by $x\mapsto(x,\delta)\delta$. Note that $H^3(X_t)_{\lim}$ is self-dual up to a twist and so $H^3(X_0)$ contains a $\mathbf{Z}(-1)$ if and only if δ is non-zero. In other words, the purity of $H^3(X_0)$ is equivalent to the triviality of δ. Thus we have the equivalence of (ii) and (iii).

Finally, we have the exact sequence of mixed Hodge structures

$$H^2(\tilde{X}_0)\to H^2(E)\to H^3(X_0)\to H^3(\tilde{X}_0),$$

which shows the $H^3(X_0)$ is not pure if and only if (iv) holds. □

Assume that one of the above equivalent conditions holds. Let $d:T\to S$ be a double cover ramified at $o\in T$ lying over $o\in S$. The normalization of $\mathscr{X}\times_S T$ has an ordinary double point; let $\mathscr{Y}$ be the result of blowing up this ordinary double point. The special fibre of $\mathscr{Y}\to T$ is the union of $\tilde{X}_0$ and a smooth quadric threefold Q which meet transversally along E.

In this situation Clemens [3] has constructed a Néron family $\mathcal{T} \to T$ of intermediate Jacobians over T such that the special fibre J_0 has two components $J_0^{\pm}$. Here

$$J_0^+ = H^3(X_0, \mathbf{C})/(F^2H^3(X_0, \mathbf{C}) + H^3(X_0, \mathbf{Z})),$$

is an extension of a compact torus by $\mathbf{G}_m$. Further, in the diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}(-1) & \to & H^3(X_0) & \to & H^3(\tilde{X}_0) & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & H^3(Q, E) & \to & H^3(\tilde{X}_0 \cup Q) & \to & H^3(\tilde{X}_0) & \to 0 \end{array}$$

the vertical maps are isomorphisms. Thus, this $\mathbf{G}_m$ can be identified with $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbf{Z}(-2), H^3(Q, E))$.

Choose $p \in E$ and let C be the quadric cone in Q with vertex p. The intersection $C \cap E$ is a pair of lines meeting in p. We have an isomorphism

$$H^1(C - p, C \cap E - p)\cdot(-1) \cong H^3(Q, E).$$

Any pair of distinct lines L_1, L_2 which are distinct from the lines in $C \cap E$ give a non-trivial extension

$$0 \to H^1(C - p, C \cap E - p) \to H^1(C - L_1 - L_2, C \cap E - p) \to \mathbf{Z}(-1)[L_1 - L_2] \to 0$$

and hence a non-trivial point of $\mathbf{G}_m$.

Let $\iota: T \to T$ be the involution corresponding to the double cover $d: T \to S$. This gives us $\iota: \mathcal{Y} \to \mathcal{Y}$ as well. The action on the special fibre of $\mathcal{Y} \to T$ is described as follows: ι acts trivially on $\tilde{X}_0$ and on Q it acts as the unique involution which fixes E. Further, we can lift ι to an action $\tilde{\iota}$ on $\mathcal{T}$. The action of $\tilde{\iota}$ on the special fibre is trivial on the identity component J_0^+ and is non-trivial on the remaining component J_0^-.

Lemma 2. In the situation of Lemma 1 assume that the action of monodromy is non-trivial. Suppose in addition that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \stackrel{i}{\hookrightarrow} & \mathcal{X} \\ p\downarrow & & \downarrow p \\ T & \stackrel{d}{\to} & S \end{array}$$

where d is the double cover as above and $\mathcal{C} \to T$ is a smooth family of connected curves which embeds into $\mathcal{X}$ in such a way that $o \in X_0$ lies on the special fibre C_0.

For each $t \in T$ different from o, the difference $\sigma(t) = C_t - C_{\iota(t)}$ gives a point in the intermediate Jacobian of $X_{d(t)}$. This extends to a section $\sigma: T \to \mathcal{T}$ such that $\sigma(o)$ is a non-trivial two-torsion class in the identity component J_0^+ of the special fibre.

Proof. The surface $\mathcal{C} \subset \chi$ is smooth and meets X_0 in C_0 with multiplicity two. Hence, we get two maps $m: \mathcal{C} \to \mathcal{X} \times T$ and $\iota(m): \mathcal{C} \to \mathcal{X} \times T$ where the images meet along C_0 with multiplicity one. Let $\tilde{\mathcal{C}}$ be the blow up of $\mathcal{C}$ at $o \in \mathcal{C} \subset \mathcal{X}$ and let D denote the exceptional divisor of $\tilde{\mathcal{C}} \to \mathcal{C}$; then m and $\iota(m)$ give us two maps

$n:\mathscr{C} \to y$ and $\imath(n):\tilde{\mathscr{C}} \to y$, such that $n(D)$ and $\imath(n)(D)$ are distinct lines L_1, L_2 in Q that meet E in the same point p. Their difference then gives us a non-trivial class α in the $\mathbf{G}_m$ part of J_0^+.

For each $t \in T$ different from o, we see by continuity that the curves C_t and $C_{\imath(t)}$ are homologically equivalent in the fibre $X_{d(t)}$. Thus we have a class $\sigma(t) = C_t - C_{\imath(t)}$ which is homologically trivial in the Chow group of $X_{d(t)}$. The limiting class is

$$\sigma(o) = n(D) - \imath(n)(D) \in \mathrm{CH}^2(\tilde{X}_0 \cup Q),$$

for a suitable definition of the latter Chow group. Furthermore, $\sigma(t)$ gives a point in the intermediate Jacobian of the fibre $X_{d(t)}$ which extends to a section $\sigma: T \to \mathscr{T}$. Clearly $\sigma(o) = \alpha$ which is non-trivial. Now $\sigma(o) \in J_0^+$ is fixed by $\imath$; on the other hand from the expression above $\imath(\sigma(o)) = -\sigma(o)$, hence it is a non-trivial two torsion class in the identity component J_0^+. □

Let X be a smooth projective threefold with $H^4(X, \mathbf{Z}) = \mathbf{Z}$, and $\{C_d \subset X\}$ be an infinite collection of curves. For any codimension 2 linear section $C_0 \subset X$ we get classes

$$e_d = C_d - \frac{\deg C_d}{\deg C_0} C_0 \in J(X)$$

where $J(X)$ is the intermediate Jacobian of X. As in Clemens [2] we now give a sufficient condition for these classes to generate a subgroup of infinite ranks in $J(X)$.

Assume that (X, C_d) is the pair corresponding to the geometric generic point of S_d in a situation

$$\begin{array}{ccc} \mathscr{C}_d & \hookrightarrow & \mathscr{X}_d \\ p\downarrow & & \downarrow p \\ T^d & \xrightarrow{d} & S_d \end{array}$$

as in Lemma 2; here we have used the subscript to indicate dependence on d. Further assume that for each $l \neq d$ we have a commutative diagram which specializes to (X, C_l) at the geometric generic point of S,

$$\begin{array}{ccc} \mathscr{C}_{l,d} & \hookrightarrow & \mathscr{X}_d \\ \pi\downarrow & & \downarrow\pi \\ S_d & \cong & S_d \end{array}$$

where $\mathscr{C}_l \to S$ is a smooth family of connected curves with embeds into $\mathscr{X}$ in such a way that it misses the ordinary double point of X_0.

All the above data is defined for all d over a countably generated field over $\mathbf{Q}$. Hence it makes sense to assume that there is a geometric generic point $s_d \in S_d - o$ for each d, where the above data specializes to $(X, \{C_l\})$.

Lemma 3. *If* $(X, \{C_l\})$ *are as above then classes* e_l *generate a subgroup of infinite rank in the intermediate Jacobian* $J(X)$ *of* X.

Proof. As in Lemma 2 the action of $\imath$ fixes the class of C_0 since we have assumed that $H^4(X, \mathbf{Z}) = \mathbf{Z}$. Thus we have an additional class

$$\imath(e_d) = e_d' = \imath(C_d) - \frac{\deg C_{d,\imath(t)}}{\deg C_0} C_0$$

in the intermediate Jacobian of X.

With notation as in the proof of Lemma 2, we have $e_d - e_d' = \sigma(t)$. The action of $\tilde{\imath}$ on the classes e_l for $l \neq d$ is trivial since the class of C_0 is fixed and C_l is fixed. Suppose that we have a relation $\Sigma_{\text{finite}} n_d e_d = 0$ then by applying $\tilde{\imath}$ to this relation we get $n_d e_d' + \Sigma_{l \neq d} n_l e_l = 0$. Thus we see that $n_d \sigma(t) = 0$. Then by degeneration we have $n_d \sigma(0) = 0$ and by Lemma 2 we see that n_d must be even.

For any relation $\Sigma_{\text{finite}} n_d e_d = 0$ in the intermediate Jacobian of X, this shows that n_d is even for all d. Thus e_d are independent mod 2. Now let G be the group generated by e_d. We can apply the following easy lemma to G to show that its rank is infinite. □

Lemma 4. *If G is an abelian group such that its torsion subgroup G_{tors} is a subgroup of $(\mathbf{Q}/\mathbf{Z})^r$, then we have*

$$\operatorname{rank}_{\mathbf{Q}}(G \otimes \mathbf{Q}) + r \geqslant \operatorname{rank}_{\mathbf{Z}/2\mathbf{Z}}(G \otimes \mathbf{Z}/2\mathbf{Z}).$$

Using Lemma 1 we can characterize the families $\mathscr{X}_d \to S_d$ by means of conditions on the special fibres $X_0 = X_{0,d}$. A precise meaning will be given to the deformation schemes in the examples considered in § 4.

1. X_0 has at most ordinary double points as singularities.
2. For $l \neq d$, the curves C_l are smooth and lie in the smooth locus of X_0 and the morphism $\mathbf{Def}(X_0, C_l) \to \mathbf{Def}(X_0)$ from the space of deformations of the pair (X_0, C_l) to the space of deformations of X_0 is étale at the point corresponding to (X_0, C_l).
3. C_d is a smooth curve in X_0 passing through *exactly one* ordinary double point $p \in X_0$ and the morphism $\mathbf{Def}(X_0, C_d) \to \mathbf{Def}(X_0)$ is doubly ramified along a divisor containing the point corresponding to (X_0, C_d).
4. Let $\tilde{X}_0$ be the blow-up of X_0 at all its ordinary double points, and let $\{E_q\}_{q \in (X_0)_{\text{sing}}}$ denote the exceptional divisors. If $p \in C_d$ is the special point then the image of

$$\operatorname{Pic}(\tilde{X}_0) \otimes \mathbf{Q} \to \otimes_{q \in (X_0)_{\text{sing}}} \operatorname{Pic}(E_q) \otimes \mathbf{Q}$$

does not contain $\operatorname{Pic}(E_p)$.

The next section will give a general procedure for constructing examples of such degenerations.

3. The principal construction

Let Y be a smooth del Pezzo fourfold, i.e. $Y \subset \mathbf{P}^n$ and $K_Y^{-1} \cong \mathcal{O}_Y \otimes \mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{O}_Y(1)$. Let $S \subset Y$ be a smooth surface such that S is the scheme theoretic intersection of Y with a linear subspace, i.e. if $V = \Gamma(Y, I_{S/Y} \otimes \mathcal{O}_Y(1))$, then we have a surjection

$$V \otimes \mathcal{O}_Y \twoheadrightarrow I_{S/Y} \otimes \mathcal{O}_Y(1) = I_{S/Y}(1).$$

Let $E \subset S$ be an exceptional divisor of the first kind, i.e. $E \cong \mathbf{P}^1$ and $N_{E/S} \cong \mathcal{O}_{\mathbf{P}^1}(-1)$.

Lemma 5. Let Y, S and E be as above. We can find a hyperplane section X of Y containing S and smooth along E.

In this situation, if **Hilb** $((X,E);Y)$ *denotes the space of deformations of the pair (X,E) in Y and* **Hilb** $(X;Y)$ *the deformation of X in Y then the natural morphism*

$$\mathbf{Hilb}\,((X,E);Y) \to \mathbf{Hilb}\,(X;Y)$$

is étale at the point corresponding to (X,E).

Proof. $N^*_{S/Y}(1)$ is generated by its global sections, in fact we have a surjection $V \otimes \mathcal{O}_S \twoheadrightarrow N^*_{S/Y}(1)$. Thus on restricting this to E we have

$$V \otimes \mathcal{O}_E \twoheadrightarrow N^*_{S/Y}(1)|_E.$$

Hence we can find a section $v \in V$ such that this gives a nowhere vanishing section of $N^*_{S/Y}(1)|_E$. Let X_v be the corresponding hyperplane section of Y. Then X_v contains S and is smooth along E.

Now we have an exact sequence of vector bundles on E.

$$0 \to N_{E/S} \to N_{E/X_v} \to N_{S/X_v}|_E \to 0.$$

Furthermore $K_{X_v} = K_Y \otimes \mathcal{O}_Y(X_v) \otimes \mathcal{O}_{X_v} \cong \theta_{x_v}$thus det $N_{E/X_v} = K_E = \mathcal{O}_{\mathbf{P}^1}(-2)$, and so

$$N_{S/X_v}|_E = \det N_{E/X_v} \otimes N_{E/S}^{-1} = \mathcal{O}_{\mathbf{P}^1}(-1).$$

As a result the above sequence splits and $N_{E/X_v} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$.

The infinitesimal deformations of the pair (X_v, E) in Y are given by

$$U = \ker(H^0(E, N_{E/Y}) \oplus H^0(X, N_{X_v/Y}) \to H^0(E, N_{X_v/Y}|E)).$$

From the exact sequence

$$0 \to N_{E/X_v} \to N_{E/Y} \to N_{X_v/Y}|_E \to 0$$

we see that $H^0(E, N_{E/Y}) \cong H^0(E, N_{X_v/Y}|E)$ which yields the isomorphism $U \cong H^0 \cdot (X, N_{X_v/Y})$ under the natural morphism. □

Let G be the vector bundle on S defined by the sequence

$$0 \to G^* \to V \otimes \mathcal{O}_S \to N^*_{S/Y}(1) \to 0.$$

Let f: $\mathbf{P}_S(G) \to \mathbf{P}(V^*)$ denote the natural map. For any point $v \in \mathbf{P}(V^*)$ such that f is étale over v, the set $f^{-1}(v)$ consists of finitely many points. The projections of these points give the singularities of X_v along S. It is easily seen (see Appendix A) that these are ordinary double points. We now need to choose v so that exactly one of these singularities lies on E and also ensure the rigidity of E in X_v for this choice of v. The first step is

Lemma 6. Let N be a vector bundle on a smooth projective curve E, $V \subset \Gamma(E, N)$ be a *space of sections such that $g:\mathbf{P}_E(N)\to\mathbf{P}(V)$ is an embedding. Let G be the vector bundle defined by the exact sequence*

$$0\to G^*\to V\otimes\mathcal{O}_E\to N\to 0.$$

Then the map $f:\mathbf{P}_E(G)\to\mathbf{P}(V^*)$ is birational to its image and the ramification locus of this morphism has codimension two in $\mathbf{P}_E(G)$.

Proof. Now $f^*(\mathcal{O}_{\mathbf{P}(V^*)}(1))=\mathcal{O}_G(1)$ is the tautological line bundle on $\mathbf{P}_E(G)$ which is a quotient of π_1^*G where $\pi_1:\mathbf{P}_E(G)\to E$ is the natural projection. Observe the following diagram of Euler sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \mathcal{O}_G & = & \mathcal{O}_G & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0\to & & \pi_1^*G^*\otimes\mathcal{O}_G(1) & \to & V\otimes\mathcal{O}_G(1) & \to & \pi_1^*N\otimes\mathcal{O}_G(1) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\to & & T_{\mathbf{P}_E(G)/E} & \to & f^*T_{\mathbf{P}(V^*)} & \to & \mathcal{K} & \to 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

and the following diagram of exact sequences

$$\begin{array}{cccccccc} 0\to & T_{\mathbf{P}_E(G)/E} & \to & T_{\mathbf{P}_E(G)} & \to & \pi_1^*T_E & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0\to & T_{\mathbf{P}_E(G)/E} & \to & f^*T_{\mathbf{P}(V^*)} & \to & \mathcal{K} & \to 0. \end{array}$$

These show us that df is computed by a map on $\mathbf{P}_E(G)$

$$\phi:\pi_1^*T_E\to\pi_1^*N\otimes\mathcal{O}_G(1).$$

Similarly, if $\pi_2:\mathbf{P}_E(N)\to E$ denotes the natural projection, one shows that dg is computed by a map on $\mathbf{P}_E(N)$

$$\gamma:\pi_2^*T_E\to\pi^*G\otimes\mathcal{O}_N(1).$$

In fact, if $\pi:\mathbf{P}_E(G)\times_E\mathbf{P}_E(N)\to E$ is the fibre product we have a natural morphism

$$\psi:\pi^*(T_E)\to\mathcal{O}_G(1)\otimes\mathcal{O}_N(1)$$

such that $\phi=(p_1)_*(\psi)$ and $\gamma=(p_2)_*(\psi)$.

We are given that g is an embedding, and thus dg and also γ are inclusions of vector bundles. This gives us a subvariety

$$D=\mathbf{P}_{\mathbf{P}_E(N)}(\operatorname{coker}\gamma)\subset\mathbf{P}_E(G)\times_E\mathbf{P}_E(N)$$

which is precisely the vanishing locus of ψ. It follows that $D\subset\mathbf{P}(V^*)\times\mathbf{P}_E(N)$ is precisely the collection of pairs (v,n), such that the hyperplane section of $\mathbf{P}_E(N)$ defined by v is singular at n. Let $D'\subset\mathbf{P}(V^*)$ be the image of D; this is the dual variety of

$\mathbf{P}_E(N) \to E$ and the divisor $P_E(\mathscr{T})$, for the quotient $N \twoheadrightarrow (N/v\cdot\mathcal{O}_E)/\text{torsion} \cong \mathscr{T}$ with (v,n), $v \in D'$ a smooth point and $n \in \mathbf{P}(V)$ such that the hyperplane in $\mathbf{P}(V^*)$ defined by n is tangent to D' at v. The fibre of the map $D \to D'$ is a projective space at the general point of D'.

If $v \in \mathbf{P}(V^*)$ is in the image $f(\mathbf{P}_E(G))$, then the corresponding hyperplane section of $\mathbf{P}_E(N)$ contains a fibre of π. Thus this hyperplane section is singular. Furthermore, for any $v \in \mathbf{P}(V^*)$, the hyperplane section is the union of finitely many fibres of $\mathbf{P}_E(N) \to E$ and the divisor $P_E(\mathscr{T})$, for the quotient $N \twoheadrightarrow (N/v\cdot\mathcal{O}_E)/\text{torsion} \cong \mathscr{T}$ with rank 1 kernel. Thus, for any $v \in \mathbf{P}(V^*)$ the singularities of the corresponding hyperplane section are contained in finitely many fibres of $\mathbf{P}_E(N) \to E$. In particular, D' is the image of $\mathbf{P}_E(G)$ and the map $\mathbf{P}_E(G) \to D'$ is generically finite. Combined with the fact that $D \to D'$ has general fibre a projective space we see that D, $\mathbf{P}_E(G)$ and D' are birational.

The cokernel of ϕ is supported on the subset of $\mathbf{P}_E(G_E)$ where the birational map $D \to \mathbf{P}_E(G_E)$ has at least 1-dimensional fibres. Since D is irreducible this is of codimension $\geqslant 2$ in $\mathbf{P}_E(G_E)$. □

Let X be a hyperplane section of Y which contains S and has exactly one ordinary double point lying on E and no other singularities on E; such an X will be provided using the above lemma. We must find a condition for

$$\mathbf{Hilb}((X,E);\, Y) \to \mathbf{Hilb}(X;\, Y)$$

to be ramified to order two along a divisor containing (X,E).

Let $\varepsilon: \tilde{Y} \to Y$ be the blow up of Y along S; the exceptional divisor is $P = \mathbf{P}_S(N^*_{S/Y}(1))$; we have a ruled surface $Q = \mathbf{P}_E(N^*_{S/Y}(1)|_E)$ contained in P. Let $\tilde{X}$ be the strict transform of X in $\tilde{Y}$. Then $\tilde{X}$ meets P in a smooth surface $\tilde{S}$ which is the blow up of S at the finitely many ordinary double points of X; one of these is a point $e \in E$. Let F_e be the exceptional divisor of $\tilde{S} \to S$ over e and $\tilde{E}$ be the strict transform of E in $\tilde{S}$. Then $\tilde{X}$ meets Q in $\tilde{E} \cup F_e$; note that $\tilde{E}$ is a section of $Q \to E$ and F_e is the fibre of $Q \to E$ over e. Further, $\tilde{X}$ is smooth along $\tilde{E}$, thus we see that there is a natural map $N_{\tilde{E}/\tilde{X}} \to N_{Q/\tilde{Y}}|_E$, the cokernel of which can be canonically identified with the fibre of $N_{\tilde{X}/\tilde{Y}}$ at $\tilde{e} = \tilde{E} \cap F_e$. We have a diagram of exact sequences,

$$\begin{array}{ccccccccc}
0 & \to & N_{\tilde{E}/\tilde{S}} & \to & N_{\tilde{E}/\tilde{X}} & \to & N_{\tilde{S}/\tilde{X}}|_{\tilde{E}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N_{Q/P} & \to & N_{Q/\tilde{Y}}|_{\tilde{E}} & \to & N_{P/\tilde{Y}}|_{\tilde{E}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N_{E/S} & \to & N_{E/Y} & \to & N_{S/Y}|_E & \to & 0
\end{array}$$

where $N_{P/\tilde{Y}}|_{\tilde{E}}$ can be computed to be $\mathcal{O}_E$ and the inclusion $N_{P/\tilde{Y}}|_{\tilde{E}} \subsetneq N_{S/Y}|_E$ is the one induced by the given morphism $E \cong \tilde{E} \subset Q$.

The last row of the diagram gives us $\Gamma(E, N_{E/Y}) \cong \Gamma(E, N_{S/Y}|_E)$ and thus we have a lift $N_{P/\tilde{Y}}|_{\tilde{E}} \to N_{E/Y}$, in fact it is easily shown that the section actually lifts to $N_{Q/\tilde{Y}}|_{\tilde{E}}$ to split the middle sequence. In order to show that $N_{\tilde{E}/\tilde{X}}$ has no sections we must show that the image of this splitting maps non-trivially under the morphism $N_{Q/\tilde{Y},\tilde{e}} \to N_{\tilde{X}/\tilde{Y},\tilde{e}}$. We shall show this by varying the choice of $\tilde{X}$.

Lemma 7. *Let* $f: \mathbf{P}_S(G) \to \mathbf{P}(V^*)$ *be the natural map. Assume that* $N = \mathbf{P}_E(G|_E) \subset \mathbf{P}_S(G)$ *is not entirely contained in the ramification locus of* f. *Further assume that* $f|_N$ *is birational to its image and is unramified outside codimension 2.*

Then, there is a hyperplane section X *of* Y *containing* S *which has at most ordinary double points as singularities. Further, exactly one of these singularities lies on* E *and* **Hilb** $((X, E); Y) \to$ **Hilb**$(X; Y)$ *is ramified to order two along a divisor containing the point* (X, E).

Proof. Let $\Gamma_\pi \subset N \times E$ denote the graph of $\pi: N \to E$. On $N \times E$ we have a map

$$\Psi: \mathcal{O}_{N \times E}(\Gamma_\pi) \to p_1^* \mathcal{O}_G(1) \otimes p_2^*(N_{S/Y}^*(1)|_E)$$

which restricts to ψ on $\Gamma_\pi \cong N$. Using the isomorphisms

$$\mathcal{O}_{N \times E}(\Gamma_\pi) \cong p_1^* \pi^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(1) \text{ and } N_{S/Y}^*(1)|_E \cong N_{S/Y} \otimes \mathcal{O}_{\mathbf{P}^1}(1)$$

this is equivalent to a map

$$p_1^*(\mathcal{L}) = p_1^*(\mathcal{O}_N(-1) \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(1)) \to p_2^*(N_{S/Y}|_E).$$

Let Z be the vanishing locus for Ψ. Then Z meets Γ_π in the vanishing locus for ψ which is given to be of codimension 2. Further, Γ_π meets every effective divisor in $N \times E$ and thus Z is also of codimension 2 in $N \times E$; in particular, the map $\mathcal{L} \hookrightarrow p_2^*(N_{S/Y}|_E)$ is saturated. Hence, if $U = N \times E - Z$, we have a morphism $p: U \to Q = \mathbf{P}_E(N_{S/Y}^*(1)|_E)$ such that $p^* \mathcal{O}_Q(1) = \mathcal{L}^{-1} \otimes p_2^*(\mathcal{O}_Y(1)|_E)$.

The sequence $0 \to N_{Q/P} \to N_{Q/\tilde{Y}} \to N_{P/\tilde{Y}}|_Q \to 0$ on Q pulls back under p to

$$0 \to p_2^* N_{E/S} \to p^* N_{Q/\tilde{Y}} \to p_1^*(\mathcal{L}) \to 0$$

since $N_{P/\tilde{Y}}|_Q \cong \mathcal{O}_Q(1) \otimes \pi^*(\mathcal{O}_Y(-1)|_E)$. Taking direct images under p_1 we see that this sequence splits to give a map $p_1^*(\mathcal{L}) \to p^* N_{Q/\tilde{Y}}$. As seen in the arguments preceding the lemma we have a natural surjection on Γ_π from the restriction of $N_{Q/\tilde{Y}}$ to the restriction of the pull back $p^* \mathcal{O}_{\tilde{Y}}(\tilde{X})$. Note that $\mathcal{O}_{\tilde{Y}}(\tilde{X}) \cong \mathcal{O}_{\tilde{Y}}(-P) \otimes \varepsilon^* \mathcal{O}_Y(1)$ which restricts on Q to $\mathcal{O}_Q(1)$. In order to show that we have rigidity for $\tilde{E}$ in $\tilde{X}$ we need to show that the composite morphism on $\Gamma_\pi \cong N$

$$\mathcal{L} \to p^* N_{Q/\tilde{Y}}|_{\Gamma_\pi} \to p^* \mathcal{O}_Q(1)|_{\Gamma_\pi}$$

is non-zero. The kernel of the second morphism can be computed to be

$$(\pi^* N_{E/S} \otimes \mathcal{L}) \otimes (\mathcal{L}^{-1} \otimes \pi^*(\mathcal{O}_Y(1)|_E))^{-1} \cong \mathcal{L}^{\otimes 2} \otimes \pi^*(\mathcal{O}_Y(1)|_E \otimes N_{E/S}).$$

So, if $\mathcal{L}$ landed completely inside this on U, we would have a non-trivial section of $\mathcal{L} \otimes \pi^*(\mathcal{O}_Y(-1)|_E \otimes N_{E/S})$ which is isomorphic to $\mathcal{O}_N(-1) \otimes \pi^*(\mathcal{O}_Y(-1)|_E)$. Since the complement of U is of codimension two any such section would extend to all of N and that is clearly impossible.

To summarize, at some suitably chosen point of n we have

1. If $v = f(n) \in \mathbf{P}(V^*)$ then X_v has no singularities other than finitely many ordinary double points on S.

2. There is exactly one ordinary double point of X_v which lies on E.
3. The curve $\tilde{E} \subset \tilde{X}$ is rigid; in fact, from the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1}(-2) \to N_{\tilde{E}/\tilde{X}} \to \mathcal{O}_{\mathbf{P}^1} \to 0$$

and the fact that $\Gamma(E, N_{\tilde{E}/\tilde{X}}) = 0$, we see that $N_{\tilde{E}/\tilde{X}} \cong \mathcal{O}_{\mathbf{P}^1}(-1)^{\otimes 2}$.

The curve $\tilde{E} \cup F_e$ is an exceptional tree of curves of the first kind on $\tilde{S}$ and thus by an argument similar to the one in Lemma 5 it deforms into nearby $\tilde{X}$'s. Thus each of the curves $\tilde{E}$ and $\tilde{E} \cup F_e$ deforms to nearby $\tilde{X}$'s. This gives the result. □

Now, assume that $V = \Gamma(Y, I_{S/Y}(1)) \cong \Gamma(\tilde{Y}, \mathcal{O}_Y(1) \otimes \mathcal{O}_{\tilde{Y}}(-P))$ gives a very ample linear system on $\tilde{Y}$. Then V is also very ample on $Q = \mathbf{P}_E(N^*_{S/Y}(1)|_E)$ so that we can apply Lemma 6. Furthermore, for a general $v \in V$, if $\tilde{X}_v$ denotes the corresponding hyperplane section of $\tilde{Y}$, then we have $\mathrm{Pic}(X_v) \cong \mathrm{Pic}(\tilde{Y}) = \mathrm{Pic}(Y) \oplus \mathbf{Z}[P]$. Now, the blow up of the singularities of X_v gives the same result as blowing up all the fibres of $P \to S$ which lie in $\tilde{X}_v$ from this we see that the hypothesis (4) at the end of § 2 is satisfied.

Finally, assume that S has infinitely many exceptional curves of the first kind $\{E_d\}$. We can then choose an infinite subcollection $\{C_d\}$ so that the images of $\mathbf{P}_{C_d}(G|_{C_d})$ in $\mathbf{P}(V^*)$ are distinct. Then by the above lemmas and subsequent discussion, it is possible to choose, for each d a point v_d so that the hyperplane section X_{v_d} satisfies the conditions stated at the end of § 2.

To summarize the hypothesis on S and Y:

1. $Y \subset \mathbf{P}^n$ is such that $K_Y^{-1} \cong \mathcal{O}_Y \otimes \mathcal{O}_{\mathbf{P}^n}(1)$
2. $S \subset Y$ is such that if $V = \Gamma(Y, I_{S/Y}(1))$ then, V generates $I_{S/Y}(1)$ at stalks
3. If $\tilde{Y}$ is the blow up of Y along S, then the map $\tilde{Y} \to \mathbf{P}(V)$ is an embedding
4. S contains infinitely many exceptional curves of the first kind

We shall produce such examples in the next section.

4. Examples

Let $b: S \to \mathbf{P}^2$ be the surface obtained by blowing up 9 points in general position. Let $H = b^* \mathcal{O}_{\mathbf{P}^2}(1)$ and let E_i denote the exceptional curves in S over the points in $\mathbf{P}^2$ which have been blown up. Let C be the unique elliptic curve in the linear system $|3H - \Sigma_{i=1}^9 E_i|$.

Lemma 8. *With notation as above, S can be embedded in $\mathbf{P}^5$ by the linear system $|4H - \Sigma_{i=1}^9 E_i|$. Further, we have a surjection*

$$\mathcal{O}_{\mathbf{P}^5}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbf{P}^5}(-3) \to I_{S/\mathbf{P}^5}.$$

Proof. We have a short exact sequence on S

$$0 \to \mathcal{O}_S(H) \to \mathcal{O}_S\left(4H - \sum_{i=1}^{9} E_i\right) \to \mathcal{O}_C\left(4H - \sum_{i=1}^{9} E_i\right) \to 0.$$

Since $H^1(S, \mathcal{O}_S(H)) = H^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) = 0$, the associated long exact sequence on cohomology shows that the linear system $D = |4H - \Sigma_{i=1}^9 E_i|$ on S is of dimension five and has no base points on C. In addition, the linear system D contains all curves of the form $C + H$ where H is the pull-back of a line in $\mathbf{P}^2$, so it has no base points outside C. Hence we have a base-point free linear system on S and a map to $\mathbf{P}^5$. We can use the results of Nagata [8] to show that the linear system D in fact embeds S in $\mathbf{P}^5$ as a surface of degree 7.

Let A be a general curve in the linear system D. Now, $H^1(S, \mathcal{O}_S) = 0$, so that D restricts to a complete linear system of degree 7 on A. The linear system on A given by $|\Sigma_{i=1}^9 (E_i \cap A) - (H \cap A)|$ is of degree 5 and thus has a section consisting of five points $\{q_j\}_{j=1}^5$ on A; since A is general in the linear system we may assume that none of these three q_j's are collinear.

Let S' be the surface obtained by blowing up $\mathbf{P}^2$ at these five points and F_j the corresponding exceptional divisors. We have an embedding of S' in $\mathbf{P}^4$ by the linear system $|3H' - \Sigma_{j=1}^5 F_j|$, where H' is the pullback to S' of a general line in $\mathbf{P}^2$. It is well known that this surface is the complete intersection of two quadrics in $\mathbf{P}^4$ (see [1]).

Let A' be the strict transform to S' of the curve A in S; then there is a natural isomorphism between A and A'. Furthermore, by the choice of q_i's, we see that the embedding of A' in $\mathbf{P}^4$ is by the same linear system as the one that embeds A as a hyperplane section of S in $\mathbf{P}^5$; thus we may identify this $\mathbf{P}^4$ with the hyperplane in $\mathbf{P}^5$ which cuts out A in S. The line bundle $\mathcal{O}_{S'}(-A') \otimes \mathcal{O}_{\mathbf{P}^4}(n)$ is generated by global sections for $n \geqslant 3$ (see Nagata *loc. cit.*). For $n = 2$ this is $\mathcal{O}_{S'}(2H' - \Sigma_{j=1}^5 F_j)$ which has exactly one section Q, a line in $\mathbf{P}^4$. The union $A' \cup Q$ is then defined by quadrics in $\mathbf{P}^4$ and A' is defined by cubics. Thus we have a surjection

$$\mathcal{O}_{\mathbf{P}^4}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbf{P}^4}(-3) \to I_{A'/\mathbf{P}^4} = I_{A/\mathbf{P}^4}.$$

Since A is a general hyperplane section of S, we have the result. □

With notation as above, let P be the plane spanned by the elliptic curve C. For Q as in the proof above we have $Q = P \cap \mathbf{P}^4$. From this one can see that the net N of quadrics containing S also contains P, and in fact $S \cup P$ is the complete intersection of these three quadrics. Hence we have a sequence

$$0 \to \mathcal{O}_S^{\oplus 3} \otimes \mathcal{O}_{\mathbf{P}^5}(-2) \to N^*_{S/\mathbf{P}^5} \to T \to 0$$

where $T = N^*_{C/P} = \mathcal{O}_C \otimes \mathcal{O}_{\mathbf{P}^5}(-3)$. The dual sequence is

$$0 \to N_{S/\mathbf{P}^5} \to \mathcal{O}_S^{\oplus 3} \otimes \mathcal{O}_{\mathbf{P}^5}(2) \to \mathcal{O}_C \otimes \mathcal{O}_S(C) \otimes \mathcal{O}_P(C) \to 0.$$

The last surjection induces a map $C \to N$. A point outside the image of this gives a quadric containing S which is smooth along S. Similarly, we take the sequence

$$0 \to \mathcal{O}_P(-2)^{\oplus 3} \to N^*_{P/\mathbf{P}^5} \to T' \to 0$$

where $T' = N^*_{C/S}$. The dual sequence is

$$0 \to N_{P/\mathbf{P}^5} \to \mathcal{O}_P(2)^{\oplus 3} \to \mathcal{O}_C \otimes \mathcal{O}_S(C) \otimes \mathcal{O}_P(C) \to 0$$

which again induces the same map $C \to N$ and so a point outside the image gives us a quadric which is smooth along $S\cup P$. Since this is the base locus of N, there is a smooth quadric Y containing S.

Choose such a smooth quadric. Then S is defined by cubic equations in Y, i.e. if $V_1=\Gamma(Y,I_{S/Y}\otimes\mathcal{O}_{\mathbf{P}^5}(3))$ then we have $V_1\otimes\mathcal{O}_Y\twoheadrightarrow I_{S/Y}\otimes\mathcal{O}_{\mathbf{P}^5}(3)$ is surjective. Further, by adjunction we have $K_Y\cong\mathcal{O}_Y\otimes\mathcal{O}_{\mathbf{P}^5}(-4)$. Now we apply the following lemma

Lemma 9. *Let $S\subset Y$ be a pair of projective varieties, and L a very ample line bundle on Y. If $V_1=\Gamma(Y,I_{S/Y}\otimes L^{\otimes n})$ generates at stalks then $V=\Gamma(Y,I_{S/Y}\otimes L^{\otimes(n+1)})$ is very ample on $\tilde{Y}$, the blow up of Y along S.*

Proof. The surjection $V_1\otimes\mathcal{O}_Y\to I_{S/Y}\otimes L^{\otimes n}$ induced by the evaluation map gives an inclusion $\tilde{Y}\subset Y\times\mathbf{P}(V_1)$. The line bundle $M=p_1^*L\otimes p_2^*\mathcal{O}_{\mathbf{P}(V_1)}(1)$ is very ample on $Y\times\mathbf{P}(V_1)$. Restricting this to $\tilde{Y}$ and taking direct image to Y we see that $\Gamma(\tilde{Y},M|_{\tilde{Y}})=\Gamma(Y,I_{S/Y}\otimes L^{\otimes(n+1)})$. Hence the result. □

Similarly, we can choose Y to be a smooth cubic containing S and then S is defined by quadric equations in Y, i.e. if $V_1=\Gamma(Y,I_{S/Y}\otimes\mathcal{O}_{\mathbf{P}^5}(2))$ then we have a surjection $V_1\otimes\mathcal{O}_Y\to I_{S/Y}\otimes\mathcal{O}_{|\mathbf{P}^5}(2)$. Also note that by adjunction we have $K_Y\cong\mathcal{O}_Y\otimes\mathcal{O}_{\mathbf{P}^5}(-3)$ so that we can again apply the above lemma.

Finally, to produce the exceptional curves of the first kind on S we use

Lemma 10. *Let S be the surface obtained by blowing up* $\mathbf{P}^2$ *at at-least* 9 *general points. Then S contains infinitely many exceptional curves of the first kind.* □

The proof can be found in [8].

We are now in a position to state and prove

Theorem 11. *Let Y be smooth quadric or a smooth cubic in* $\mathbf{P}^4$. *The anticanonical bundle K_Y^{-1} of Y is very ample. Let X be the geometric generic divisor in the corresponding linear system. Then, the Griffiths group of X contains a subgroup of infinite rank.*

Proof. A simple dimension count shows that every smooth cubic contains a surface S as in Lemma 8. Since all smooth quadrics are isomorphic, the same is true for quadrics as well. Further, as a consequence of Lemma 8, if $V=\Gamma(Y,I_{S/Y}\otimes K_Y^{-1})$, then we have a surjection

$$V\otimes\mathcal{O}_Y\twoheadrightarrow I_{S/Y}\otimes K_Y^{-1}.$$

Furthermore, by Lemma 9, if $\tilde{Y}$ is the blow up of Y along S, then we have a natural embedding $\tilde{Y}\hookrightarrow\mathbf{P}(V)$. We note that this implies that the map $P=\mathbf{P}(N_{S/Y}^*)\to\mathbf{P}(V)$ is also an embedding. Now Y is simply connected and has $\mathrm{Pic}(Y)=\mathbf{Z}$; as a consequence $\mathrm{Pic}(\tilde{Y})=\mathbf{Z}\oplus\mathbf{Z}[P]$. Let us adopt the notation $\mathcal{O}_Y(1)=K_Y^{-1}$.

First of all we use Lemma 5 to find X_0 which contains S and is smooth along all the exceptional curves in S. Since there are countable many such curves X_0 is defined over some countably generated field.

Let G be the vector bundle defined by the exact sequence

$$0\to G^*\to V\otimes\mathcal{O}_S\to N_{S/Y}^*\otimes\mathcal{O}_Y(1)\to 0.$$

We have a map $\mathbf{P}_S(G) \to \mathbf{P}(V^*)$. Since $\mathrm{Pic}(Y) = \mathbf{Z}$, the same is true for any smooth divisor D in the linear system $|K_Y^{-1}|$. Then by the adjunction formula, a smooth divisor in D is of general type; in particular, no smooth D can contain S. From this it follows that the map $\mathbf{P}_S(G) \to \mathbf{P}(V^*)$ is generically finite, let B_1 denote its ramification locus. Let B_2 denote the locus in $\mathbf{P}(V^*)$ consisting of v such that the corresponding hyperplane section X_v has singularities outside S, and $B = B_1 \cup B_2$.

By Lemma 10 and S has infinitely many exceptional curves of the first kind. We need to choose among these, curves $E \subset S$ such that the image of $\mathbf{P}_E(G|_E)$ is not contained in B. There are clearly infinitely many of these. Further, we choose an infinite subcollection $\{E_d\}$ such that, in addition, the images of $\mathbf{P}_{E_d}(G|_{E_d}) \xrightarrow{f_d} \mathbf{P}(V^*)$ are distinct. Since the map $\mathbf{P}_S(N^*_{S/Y}(1)) \to \mathbf{P}(V)$ is an embedding we can apply Lemma 6 to show that f_d's are birational to their images.

Now for each d, we choose a point v_d in the image of f_d, which is not in the image of f_l for any $l \neq d$. Further we may assume that v_d is not in B. Let X_d be the corresponding hyperplane section of Y; then $(X_d)_{\mathrm{sing}}$ is a finite collection of ordinary double points lying in S. We apply Lemmas 5 and 7 to conclude that X_d satisfies conditions (1)–(3) listed at the end of § 2.

Let X'_d denote the strict transform of X_d in $\tilde{Y}$; this is the small resolution of X_d. Since it is a hyperplane section of $\tilde{Y}$ it has $\mathrm{Pic}(X'_d) = \mathbf{Z} \oplus \mathbf{Z}[S'_d]$; where $S'_d = P \cap X'_d$ is the strict transform of S in X'_d. The result of blowing up the finitely many exceptional curves of the map $S'_d \to S$ in X'_d is $\tilde{X}_d$, which is the blow up of the finitely many ordinary double points of X_d. From this we see that condition (4) of § 2 is also satisfied.

Note that $\mathbf{Hilb}(X; Y)$ is just the projective space $|K_Y^{-1}|$. Let A_0 be a curve in $\mathbf{Hilb}(X; Y)$ joining X to X_0. We use the second part of Lemma 5 to construct infinitely many rigid rational curves in X, by deforming along A_0 all the exceptional curves of the first kind in S. For each d we choose a curve B_d in $\mathbf{P}(V^*)$ joining X_d to X_0 which is not entirely contained in any of the divisors $\mathbf{P}_{E_d}(G|_{E_d})$ or in B. We may choose a deformation A_d of $A_0 \cup B_d$ in $\mathbf{Hilb}(X; Y)$. The above construction ensures that deformations along the different A_d's give the same collection of rigid rational curves in X. Now we can apply the argument following Lemma 2 to conclude that the classes e_d (with notation as in § 2) generate a subgroup of infinite rank in the intermediate Jacobian $J(X)$ of X.

As a final point, note that $H^3(Y, \mathbf{Z}) = 0$ and thus by well known arguments (as in [6]), the abelian part of $J(X)$ is zero. This implies that this infinite rank subgroup is actually contained in the Griffiths group. □

Remark. Constructions similar to the one above will also allow us to conclude the theorem in the case where Y is $\mathbf{P}^4$, thereby giving the original results of C H Clemens.

Appendix A. Bertini type results

Let X be a smooth variety and Y be a smooth subvariety of codimension r and L be a line bundle such that $\mathscr{I}_{Y/X} \otimes L$ is generated by global sections. We wish to understand the singularities of the general global section of $\mathscr{I}_{Y/X} \otimes L$.

Let $\mathscr{F}$ be the coherent sheaf on X defined by the exact sequence

$$0 \to L^{-1} \to \Gamma(X, \mathscr{I}_{Y/X} \otimes L)^* \otimes \mathcal{O}_X \to \mathscr{F} \to 0.$$

Then $\mathbf{P}_X(\mathscr{F}) \subset X \times \mathbf{P}(\Gamma(X, \mathscr{I}_{Y/X} \otimes L)^*)$ is the incidence locus between sections of $\mathscr{I}_{Y/X}$ and their zero sets.

For any $y \in Y$ we can pick sections $f_1, \ldots, f_r \in \Gamma(X, \mathscr{I}_{Y/X} \otimes L)$ which define Y in a neighbourhood of y. The remaining sections can then be expressed as linear combinations of the f_i's,

$$g_i \doteq \sum_{j=1}^{r} a_{i,j} f_j \text{ where } i = 1, \ldots, l.$$

Thus, the first homomorphism of the above sequence can be rewritten in a neighbourhood of y as

$$\mathcal{O}_{X,y} \to (\oplus_{j=1}^{r} \mathcal{O}_{X,y} F_j) \oplus (\oplus_{i=1}^{l} \mathcal{O}_{X,y} G_i)$$
$$1 \mapsto \sum_{j=1}^{r} f_j F_j + \sum_{i=1}^{l} g_i G_i$$

where $F_1, \ldots, F_r, G_1, \ldots, G_l$ is the basis of $\Gamma(X, \mathscr{I}_{Y/X} \otimes L)^*$ which is dual to $f_1, \ldots, f_r, g_1, \ldots, g_l$. Put $H_j = F_j + \Sigma_{i=1}^{l} a_{i,j} G_i$ so that the above homomorphism can be written as

$$\mathcal{O}_{X,y} \to (\oplus_{j=1}^{r} \mathcal{O}_{X,y} H_j) \oplus (\oplus_{i=1}^{l} \mathcal{O}_{X,y} G_i)$$
$$1 \mapsto \sum_{j=1}^{r} f_j H_j$$

where $f_1, \ldots, f_r$ define Y, a smooth subvariety of codimension r in a neighbourhood of y and may thus be thought of as "coordinates".

Thus, we have

$$\mathbf{P}_X(\mathscr{F}) = \operatorname{Proj}\left(\frac{\mathcal{O}_{X,y}[H_1, \ldots, H_r, G_1, \ldots, G_l]}{(\Sigma_{j=1}^{r} f_j H_j)} \right)$$

which can thus be expressed as the union of the affine open pieces of two types

1. The regular pieces are, for each j between 1 and r

$$\operatorname{Spec}\left(\frac{\mathcal{O}_{X,y}[H_1/H_j, \ldots, H_r/H_j, G_1/H_j, \ldots, G_l/H_j]}{(f_j + \Sigma_{k \neq j} f_k H_k/H_j)} \right)$$

2. The singular pieces are, for each i between 1 and l

$$\operatorname{Spec}\left(\frac{\mathcal{O}_{X,y}[H_1/G_i, \ldots, H_r/G_i, G_1/G_i, \ldots, G_l/G_i]}{(\Sigma_{j=1}^{r} f_j H_j/G_i)} \right)$$

which is an ordinary double singularity along the locus defined by the vanishing of $f_1, \ldots, f_r$ and $H_1/G_i, \ldots, H_r/G_i$.

Thus the singular locus of $\mathbf{P}_X(\mathscr{F})$ is smooth of dimension $\dim X - r + l - 1$. The dimension of $\mathbf{P}(\Gamma(X, \mathscr{I}_{Y/X} \otimes L)^*)$ is $r + l - 1$. Therefore, if $\dim X < 2r$ then the general element of $\Gamma(X, \mathscr{I}_{Y/X} \otimes L)$ defines a smooth divisor in X containing Y. If $\dim X = 2r$, then the general element as above has finitely many ordinary double points along Y and is smooth outside.

Acknowledgements

I would like to thank the MPI, Bonn for support during January–June 1990 when much of this work was done. I would also like to thank the DFG for support during the Bayreuth Conference on Algebraic Varieties held in April 1990.

This work has been a long time in completion and has derived much help from a lot of people. At the Tata Institute of Fundamental Research, a general method for constructing rigid rational curves on complete intersections was worked out with V Srinivas and N Mohan Kumar; this method finds application in §4 and was the starting point for this investigation. At the Max-Planck-Institut für Mathematik, I had many illuminating discussions with C Schoen and A Todorov on the subject of Calabi-Yau manifolds and I am grateful for the insights they provided. Discussions with B Hunt and P M H Wilson during the Algebraic Geometry Seminar at Bayreuth in April 1990 also helped to build a general picture for these spaces. Finally, some key points in the arguments of C H Clemens were explained to me by S Bloch, M V Nori and N Nygaard at the University of Chicago.

References

[1] Beauville A, Surfaces Algébriques Complexes, *Astérisque* **54** (1978)

[2] Clemens C H, Homological equivalence modulo algebraic equivalence is not finitely generated, *Publ. Math. IHES.* **58** (1983) 19–38

[3] Clemens C H, Néron models for families of intermediate jacobians, *Publ. Math. IHES.* **58** (1983) 5–18

[4] Clemens C H, e-mail message to N. Mohan Kumar (1989)

[5] Clemens C H, Taniguchi Symposium (1988)

[6] Griffiths P A, On the periods of certain rational integrals II, *Ann. Math.* **90** (1969) 496–541

[7] Igusa J, On some problems in abstract algebraic geometry, *Proc. Nat. Acad. Sci. USA* **41** (1955) 964–967

[8] Nagata M, On Rational Surfaces II, *Mem. Coll. Sci. Univ. Kyoto Ser. A Math.* **33** (1960/61) 271–293