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Continuous dependence for integrodifferential equations with infinite delay

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Abstract. Continuous dependence for integrodifferential equation with infinite delay

$$\dot{x} = h(t,x) + \int_{-\infty}^{t} q(t,s,x(s))\mathrm{d}s + F(t,x(t),Sx(t))\ t \geqslant 0$$

$x(t) = \phi(t)$

where $Sx(t) = \int_{-\infty}^{t} k(t,s,x(s))\mathrm{d}s$ is studied under the assumption of existence of unique solution.

Keywords. Continuous dependence; infinite delay; integrodifferential equation.

1. Introduction

In this paper we consider the continuous dependence of solutions on initial functions for the integro-differential equation

$$\dot{x} = h(t,x) + \int_{-\infty}^{t} q(t,s,x(s))\mathrm{d}s + F(t,x(t),Sx(t)), \quad t \geqslant 0$$

$$x(t) = \phi(t), \quad -\infty < t \leqslant 0 \tag{1}$$

where $h \in C[J \times R^n, R^n]$, $q \in C[J \times J \times R^n, R^n]$, $F \in C[J \times R^n \times R^n, R^n]$, $Sx(t) = \int_{-\infty}^{t} k(t,s,x(s))\mathrm{d}s$ with $k \in C[J \times J \times R^n, R^n]$ and $J = [0, T]$. Continuous dependence of solutions to functional differential equations with infinite delay was discussed by Hale and Kato [3] and Hino [4]. The stability properties of (1) without delay was considered by Elaydi and Sivasundaram [2].

For a positive continuous nonincreasing function g defined on the interval $I = (-\infty, 0]$ with $g(0) = 1$, let X_g be the space of all continuous functions ϕ: $I \to R^n$ for which

$$|\phi|_g = \sup\{|\phi(t)|/g(t): t \in I\} < \infty,$$

where $|\cdot|$ denotes any norm in R^n. Then X_g is a Banach space with respect to the norm $|\cdot|_g$. This norm was introduced by Burton [1] to study qualitative theory for some functional differential equations.

The equivalent form of (1) can be written as

$$\dot{x}(t) = h(t,x) + \int_{-\infty}^{0} q(t,s,\phi(s))\,ds + \int_{0}^{t} q(t,s,x(s))\,ds$$
$$+ F(t,x(t)), \int_{-\infty}^{0} k(t,s,\phi(s))\,ds + \int_{0}^{t} k(t,s,x(s))\,ds, t \geqslant 0$$
$$x(t) = \phi(t) \text{ for } t \leqslant 0 \tag{2}$$

where $\phi \in X_g$ is an initial function.

2. Basic assumptions

For simplicity we put

$$Q(t,\phi) = \int_{-\infty}^{0} q(t,s,\phi(s))\,ds,$$

$K(t,\phi) = \int_{-\infty}^{0} k(t,s,\phi(s))ds$ and we assume the following hypotheses

(i) The initial value problem (1) has a unique solution $x(t,\phi)$ defined on J for each $\phi \in X_g$

(ii) The improper Riemann integrals

$$Q(t,\phi) = \lim_{c\to\infty} \int_{-c}^{0} q(t,s,\phi(s))\,ds$$

and

$$K(t,\phi) = \lim_{c\to\infty} \int_{-c}^{0} k(t,s,\phi(s))\,ds$$

exist and they are continuous in $t \in J$ for each $\phi \in X_g$.

(iii) For any $r > 0$ there exist positive integrable functions $m_r(t)$ and $n_r(t)$ on J such that $|Q(t,\eta)| \leqslant m_r(t)$ and $|K(t,\eta)| \leqslant n_r(t)$ if $t \in J$ and $\eta \in B(r)$. Here $B(r)$ is a closed ball $\{\eta \in X_g : |\eta| \leqslant r\}$, $r > 0$ and it is clear that $\eta \in B(r)$ if and only if $|\eta(t)| \leqslant rg(t)$ for all $t \in I$.

(iv) There exist integrable functions $a(t)$ and $b(t)$ such that $|F(t,x,Sx)| \leqslant a(t) + b(t)(|x| + |Sx|)$.

Lemma 1. [5] *Suppose $Q(t,\phi)$ and $K(t,\phi)$ exist and are continuous in $t \in J$ for each $\phi \in X_g$. If a sequence $\{\phi_n\} \subset X_g$ is bounded (with respect to the norm $|\cdot|_g$) and if it converges to $\phi \in X_g$ a.e on I, then $\{Q(t,\phi_n)\}$ converges to $Q(t,\phi)$ and $\{K(t,\phi_n)\}$ converges to $K(t,\phi)$ at every point $t \in J$.*

3. Continuous dependence of solutions

Theorem 1. *Suppose the hypotheses (i) to (iv) hold. If a bounded sequence $\{\phi_n\}$ in X_g converges to $\phi \in X_g$ a.e on I and if $\lim_{n\to\infty} \phi_n(0) = \phi(0)$, then $\{x(t,\phi_n)\}$ converges to $x(t,\phi)$ uniformly on J.*

Proof. Let us denote the solutions $x(t) = x(t, \phi)$ and $x_n(t) = x(t, \phi_n)$. Let $r > 0$ be a bound for $\{\phi_n\}$ and ϕ, and let N and L be positive numbers satisfying $|x(t)| < N$ for $0 \leqslant t \leqslant T$ and

$$L > N + \int_0^T (m_r(s) + a(s))\,\mathrm{d}s + \int_0^T b(s) n_r(s)\,\mathrm{d}s + (N + DT) \int_0^T b(s)\,\mathrm{d}s.$$

Furthermore, choose a $\tau \in (0, T]$ satisfying

$$\tau(H + CT) \leqslant L - N - \int_0^T (m_r(s) + a(s))\,\mathrm{d}s - \int_0^T b(s) n_r(s)\,\mathrm{d}s - (N + DT) \int_0^T b(s)\,\mathrm{d}s,$$

where

$$H = \max\{|h(t,x)| : 0 \leqslant t \leqslant T, |x| \leqslant L\},$$
$$C = \max\{|q(t,s,x)| : 0 \leqslant s \leqslant t \leqslant T, |x| \leqslant L\},$$
$$D = \max\{|k(t,s,x)| : 0 \leqslant s \leqslant t \leqslant T, |x| \leqslant L\}.$$

Since $x_n(0) = \phi_n(0) \to \phi(0) = x(0)$ as $n \to \infty$, it follows that $|x_n(0)| < N$ for large n.

Now we shall show that $|x_n(t)| < L$ for $0 \leqslant t \leqslant \tau$ if $|x_n(0)| < N$. Suppose the contrary, then there exists a $t_0 \in [0, \tau]$ such that $|x_n(t_0)| = L$ and $|x_n(t)| < L$ for $0 \leqslant t \leqslant t_0$.

By (iii), (iv) and (2) we obtain that

$$|\dot{x}_n(t)| \leqslant H + m_r(t) + CT + a(t) + b(t)(N + DT + n_r(t)), \quad \text{for } 0 \leqslant t \leqslant t_0.$$

Therefore it follows that

$$\begin{aligned}
|x_n(t)| &\leqslant x_n(0) + (H + CT)t + \int_0^t (m_r(s) + a(s))\,\mathrm{d}s \\
&\quad + \int_0^t b(s) n_r(s)\,\mathrm{d}s + (N + DT) \int_0^t b(s)\,\mathrm{d}s \\
&< N + (H + CT)\tau + \int_0^T (m_r(s) + a(s))\,\mathrm{d}s \\
&\quad + \int_0^T b(s) n_r(s)\,\mathrm{d}s + (N + DT) \int_0^T b(s)\,\mathrm{d}s \\
&\leqslant L.
\end{aligned}$$

That is $|x_n(t)| < L$ which contradicts $|x_n(t_0)| = L$. Thus we get the assertion.

By using (iii), (iv) and (2) again, we obtain that

$$|\dot{x}_n(t)| \leqslant H + m_r(t) + CT + a(t) + b(t)(N + DT + n_r(t))$$

on $[0, \tau]$ for large n, and hence $\{x_n\}$ is equicontinuous there. Therefore $\{x_n\}$ includes a subsequence $\{x_{n_j}\}$ which is uniformly convergent on $[0, \tau]$. On the other hand x_n

satisfies,

$$x_n(t) = x_n(0) + \int_0^t h(s, x_n(s))\mathrm{d}s + \int_0^t Q(s, \phi_n)\mathrm{d}s + \int_0^t \int_0^s q(s, u, x_n(u))\mathrm{d}u\mathrm{d}s$$
$$+ \int_0^t F(s, x_n(u)), K(s, \phi_n) + \int_0^t k(s, u, x_n(u))\mathrm{d}u\mathrm{d}s$$

for $0 \leqslant t \leqslant \tau$. Hence it follows from Lemma (i), (iii) and (iv) that $\{x_{n_j}\}$ converges to a solution of (2). Since x is the unique solution of (2), the sequence $\{x_n\}$ itself converges to x uniformly on $[0, \tau]$.

When $\tau < T$, by the same argument as in the above, we can show that $\{x_n\}$ converges to x uniformly on $[\tau, 2\tau] \cap J$. Repeating this process, we arrive at the conclusion of the theorem.

References

[1] Burton T A, *Stability and periodic solutions of ordinary and functional differential equations.* (New York: Academic Press) (1985)

[2] Elaydi S and Sivasundaram S, A unified approach to stability in integrodifferential equations via Liapunov functions, *J. Math. Anal. Appl.* **144** (1989) 503–531

[3] Hale J K and Kato J, Phase space and retarded equations with infinite delay, *Funk. Ekvacioj.* **21** (1978), 11–41

[4] Hino Y, Continuous dependence for some functional differential equations, *Tohoku Math. J.* **23** (1971) 565–571

[5] Kaminago T, Continuous dependence of solutions for integrodifferential equations with infinite delay, *J. Math. Anal. Appl.* **129** (1988) 307–314