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A note on the multidimensional Weyl fractional operator

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Abstract. The purpose of the present paper is to establish a connection theorem involving the multidimensional Weyl fractional operator and the classical multidimensional Laplace transform. This provides an extension of a result due to Raina and Koul [6].

Keywords. Multidimensional Weyl operator; multidimensional Laplace transform; differintegral operators.

1. Introduction

In the theory of familiar Weyl fractional calculus, Raina and Koul [6] established a connection theorem involving the Laplace transform of $t^q f(t)$ for arbitrary (real) q. With a view to generalizing this Raina–Koul result [6, p. 180, eq. (6)], we try to establish its multidimensional extension (in a slightly variant form). We invoke in our analysis the multidimensional Weyl fractional operator defined and introduced quite recently by Srivastava and Raina [10].

2. Preliminaries and definitions

In the literature there are numerous examples of operators of fractional differintegrals (that is, fractional derivatives and fractional integrals) in a wide variety of fields (see, for example, [3], [4], [7], [8], [9]). Much of the theory of fractional calculus is based upon the familiar differintegral operator ${}_cD_z^\mu$ defined by ([2] and [7])

$$ {}_cD_z^\mu f(z) = \frac{1}{\Gamma(m-\mu)} \frac{\mathrm{d}^m}{\mathrm{d}z^m} \int_c^z (z-t)^{m-\mu-1} f(t)\,\mathrm{d}t $$

$$ (m > \mathrm{Re}(\mu); \quad m \in N_0 = N \cup \{0\}; \quad N = \{1,2,3,\ldots\}). \tag{1} $$

For $c = 0$, eq. (1) defines the classical Riemann–Liouville fractional derivative (or integral) of order μ (or $-\mu$). On the other hand, when $c \to \infty$, (1) may be identified with the definition of the familiar Weyl fractional derivative (or integral) of order μ (or $-\mu$) (see Erdélyi *et al* [1, Vol. II, Chapter 13] for details). By repeated applications of the operator ${}_cD_z^\mu$ to a given function of several variables (see Raina [5]), a corresponding multidimensional fractional operator can be defined in a natural way. Indeed, Srivastava and Raina [10] developed the corresponding multidimensional fractional derivative (or integral) operator. In particular, the corresponding multidimensional extension of Weyl operator of fractional calculus is defined (see [10] for details)

$$W_{\mu_1,\ldots,\mu_n}f(z_1,\ldots,z_n)$$
$$=\frac{(-1)^{m_1+\cdots+m_n}}{\Gamma(m_1-\mu_1)\cdots\Gamma(m_n-\mu_n)}\frac{\partial^{m_1+\cdots+m_n}}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}\int_{z_1}^{\infty}\cdots\int_{z_n}^{\infty}f(t_1,\ldots,t_n)$$
$$\times(t_1-z_1)^{m_1-\mu_1-1}\cdots(t_n-z_n)^{m_n-\mu_n-1}\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$
$$(m_j>\mathrm{Re}(\mu_j);\quad m_j\in N_0;\quad j=1,\ldots,n). \tag{2}$$

Further, if $f(t_1,\ldots,t_n)$ is piecewise continuous for each $t_j\in[0,\infty)$, $j=1,\ldots,n$; and if

$$|f(t_1,\ldots,t_n)|\leqslant M\exp(p_1t_1+\cdots+p_nt_n), \tag{3}$$

for all $t_j\geqslant T_j$ $(j=1,\ldots,n)$, M and T_j being positive constants, then the n-dimensional Laplace transform of $f(t_1,\ldots,t_n)$ is defined by

$$L\{f(t_1,\ldots,t_n:s_1,\ldots,s_n)\}=F(s_1,\ldots,s_n)$$
$$=\int_0^{\infty}\cdots\int_0^{\infty}\exp(-s_1t_1-\cdots-s_nt_n)f(t_1,\ldots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n, \tag{4}$$

where for convergence, $\mathrm{Re}(s_j-p_j)>0$; $j=1,\ldots,n$.

3. The main result

We propose to establish a result connecting the multidimensional Laplace transform (4) with the multidimensional fractional operator (2). Our result is contained in the following:

Theorem. *Let the Laplace transform of a function* $f(t_1,\ldots,t_n)$ *be defined by* (4). *Then*

$$W_{q_1,\ldots,q_n}F(s_1,\ldots,s_n)=L\{t_1^{q_1}\cdots t_n^{q_n}f(t_1,\ldots,t_n):s_1,\ldots,s_n\}, \tag{5}$$

holds true for all (real) values of $q_i(i=1,\ldots,n)$, *provided that the Weyl operator exists.*

Proof. In view of the Remarks (1 and 2) stated in pages 358 and 359, respectively, in [10], concerning the interpretations of the Weyl operator in the two cases $\mathrm{Re}(q_j)<0$, and $\mathrm{Re}(q_j)\geqslant 0$, for $j=1,\ldots,n$, the assertion (5) follows straightforwardly.

When $\mu_j\to m_j(m_j\in N_0; j=1,\ldots,n)$ in (2), and noting the relationship [10, p. 360, eq. (1.12)]:

$$W_{m_1,\ldots,m_n}f(z_1,\ldots,z_n)=\frac{\partial^{m_1+\cdots+m_n}}{\partial z_1^{m_1}\ldots\partial z_n^{m_n}}f(z_1,\ldots,z_n),\quad m_j\in N_0;\quad j=1,\ldots,n; \tag{6}$$

then in this case, our result (5) gives the multidimensional extension of a result recorded in [8, p. 117, eq (7.19)] (and also in [11]).

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