On the \( |\vec{N}, p_n|_k \) summability factors for infinite series

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Abstract. In this paper a theorem on \( |\vec{N}, p_n|_k \) summability factors of infinite series, which generalizes a theorem of Bor [2], has been proved.

Keywords. Summability factors; infinite series.

1. Introduction

Let \( \Sigma a_n \) be a given infinite series with the sequences of partial sums \( (s_n) \). Let \( (p_n) \) be a sequence of positive real constants such that

\[ \sum_{n=0}^{\infty} p_n = \infty \quad \text{as} \quad n \to \infty, \quad (p_{-i} = p_{-1} = 0, i \geq 1). \] (1)

The sequence-to-sequence transformation

\[ t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \] (2)

defines the sequence \( (t_n) \) of the \( (\vec{N}, p_n) \) mean of the sequence \( (s_n) \), generated by the sequence of coefficients \( (p_n) \). The series \( \Sigma a_n \) is summable \( |\vec{N}, p_n|_k, k \geq 1 \), if (see [1])

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^k |t_n - t_{n-1}|^k < \infty. \] (3)

In the special case when \( p_n = 1 \) for all values of \( n \) (resp. \( k = 1 \)), then \( |\vec{N}, p_n|_k \) summability is the same as \( |C, 1|_k \) (resp. \( |\vec{N}, p_n| \)) summability. The series \( \Sigma a_n \) is said to be bounded \([\vec{N}, p_n]_k, k \geq 1\), if (see [1])

\[ \sum_{n=1}^{\infty} p_n |s_n|^k = O(P_n) \quad \text{as} \quad n \to \infty. \] (4)

If we take \( k = 1 \), then \([\vec{N}, p_n]_k \) boundedness is the same as \([\vec{N}, p_n] \) boundedness.
2. Quite recently Bor [2] proved the following theorem.

**Theorem A.** Let \( \Sigma a_n \) be bounded \([\bar{N}, p_n]\). Let \((p_n)\) be a positive sequence such that \( P_n \to \infty \) as \( n \to \infty \) and

\[
\frac{1}{n} = O(p_n). \tag{5}
\]

Suppose there are sequences \((\beta_n)\) and \((\lambda_n)\) such that

\[
|\Delta \lambda_n| \leq \beta_n \tag{6}
\]

\[
\beta_n \to 0 \text{ as } n \to \infty \tag{7}
\]

\[
\sum_{n=1}^{\infty} nP_n |\Delta \beta_n| < \infty \tag{8}
\]

\[
P_n |\lambda_n| = O(1) \text{ as } n \to \infty. \tag{9}
\]

Then the series \( \Sigma a_n \lambda_n \) is summable \([\bar{N}, p_n]\).

3. The object of the present paper is to generalize Theorem A for \([\bar{N}, p_{n_k}]\), with \( k \geq 1 \), by proving the following theorem.

**Theorem.** Let \( \Sigma a_n \) be bounded \([\bar{N}, p_{n_k}]\). If the sequences \((p_n)\), \((\beta_n)\) and \((\lambda_n)\) such that conditions (5)–(9) of Theorem A are satisfied, then the series \( \Sigma a_n \lambda_n \) is summable \([\bar{N}, p_{n_k}]\), \( k \geq 1 \).

It should be noted that if we take \( k = 1 \) in this theorem, then we get Theorem A.

4. We need the following lemma for the proof of our theorem.

**Lemma 2.** Under the conditions on \((P_n)\) and \((\beta_n)\) as taken in statement of the theorem, the following conditions hold, when (8) is satisfied.

\[
nP_n \beta_n = O(1) \text{ as } n \to \infty \tag{10}
\]

\[
\sum_{n=1}^{\infty} P_n \beta_n < \infty. \tag{11}
\]

5. **Proof of the theorem**

Let \((T_n)\) be the sequence of the \([\bar{N}, p_n]\) mean of the series \( \Sigma a_n \lambda_n \). Then, by definition, we have

\[
T_n = \frac{1}{P_{n \beta} = 0} \sum_{r=0}^{n} p_r \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_{n \beta} = 0} \sum_{r=0}^{n} (P_n - P_{n-1}) a_r \lambda_r. \tag{12}
\]
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Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \Delta(p_{v-1}\lambda_v) s_v + \frac{1}{P_n} s_n \lambda_n \tag{13}$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(p_{v-1}\lambda_v) s_v + \frac{1}{P_n} p_n s_n \lambda_n$$

$$= \frac{1}{P_n} p_n s_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \Delta \lambda_v s_v$$

$$= T_{n,1} + T_{n,2} + T_{n,3}, \text{ say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (p_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3. \tag{14}$$

Since $|\lambda_n| = O(1/P_n) = O(1)$, by (9), we have

$$\sum_{n=1}^{m} (p_n/p_n)^{k-1} |T_{n,1}|^k = \sum_{n=1}^{m} \frac{1}{P_n} |p_n s_n|^k = \sum_{n=1}^{m} \frac{1}{P_n} |\lambda_n|^k - \frac{p_n}{P_n} s_n \lambda_n$$

$$= O(1) \sum_{n=1}^{m} |\lambda_n|^k - \frac{p_n}{P_n} s_n \lambda_n$$

$$= O(1) |\lambda_m| \sum_{n=1}^{m} p_n s_n |^k$$

$$+ O(1) |\lambda_m| |P_m - O(1) |\lambda_m| P_m = O(1) \sum_{n=1}^{m} \beta_n p_n$$

$$+ O(1) |\lambda_m| P_m = O(1)$$

as $m \to \infty$, by virtue of the hypotheses of the theorem and Lemma.

Now, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} (p_n/p_n)^{k-1} |T_{n,2}|^k = \sum_{n=2}^{m+1} \left( \frac{p_n}{P_n} \right) \left( \sum_{v=1}^{n} p_v \lambda_v s_v \right)$$

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n} \left( \sum_{v=1}^{n} p_v |\lambda_v| |s_v| \right) \left( \sum_{v=1}^{n} p_v \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |s_v|^k \sum_{v=n+1}^{m+1} \frac{P_n}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |s_v|^k \sum_{v=n+1}^{m+1} \frac{P_n}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v| p_v |s_v|^k.$$
As in $T_{n,1}$, we have that

$$
\sum_{n=2}^{m+1} (P_n/P_1)^k - 1 |T_{n,1}|^k = O(1) \sum_{v=1}^{m} |\lambda_v| p_v s_v |^k = O(1) \text{ as } m \to \infty.
$$

Finally, using the fact that $1/v = O(p_v)$, by (5), we have that

$$
\sum_{n=2}^{m+1} (P_n/P_1)^k - 1 |T_{n,3}|^k = \sum_{n=2}^{m+1} \frac{P_n}{P_1} \left( \sum_{v=1}^{n-1} P_v |\lambda_v| s_v |^k \right)
$$

$$
\leq \sum_{n=2}^{m+1} \frac{P_n}{P_1} \left( \sum_{v=1}^{n-1} P_v |\lambda_v| s_v |^k \right)
$$

$$
\leq \sum_{n=2}^{m+1} \frac{P_n}{P_1} \left( \sum_{v=1}^{n-1} P_v |\lambda_v| s_v |^k \right)
$$

$$
= O(1) \sum_{v=1}^{m} \frac{P_v}{P_1} |\lambda_v| s_v |^k
$$

$$
= O(1) \sum_{v=1}^{m} \beta_v |s_v |^k
$$

$$
= O(1) \sum_{v=1}^{m} \Delta(\beta_v) P_v + O(1) m \beta_m P_m
$$

$$
= O(1) \sum_{v=1}^{m} \Delta(\beta_v) P_v + O(1) m \beta_m P_m
$$

by virtue of the hypotheses of the theorem and Lemma.

Therefore, we get

$$
\sum_{n=1}^{m} (P_n/P_1)^k - 1 |T_{n,r}|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3.
$$

This completes the proof of the theorem.

References
