On integrability of power series

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Abstract. This paper deals with the integrability of a power series. Our results generalize certain results of Ram, and Askey and Karlin.

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1. Introduction

Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) in \([0, 1)\) and let \( S_n = \sum_{k=0}^{n} a_k \). In what follows we assume that \( \Phi(x) \) is a positive increasing and convex function defined on \([0, \infty)\).

In 1970 Askey and Karlin [1] proved, among others, the following results.

**Theorem A.**

\[
\int_{0}^{1} \Phi(|f(x)|) \, dx \leq \sum_{n=0}^{\infty} \left[ (n+1)(n+2) \right]^{-1} \Phi(|S_n|),
\]

\[
\int_{0}^{1} (1-x)^{\beta} \Phi(|f(x)|) \, dx \leq \sum_{n=0}^{\infty} \Phi(|S_n|) \frac{\Gamma(\beta+2)\Gamma(n+1)}{\Gamma(n+\beta+3)}
\]

\[
\leq K_{\beta} \sum_{n=0}^{\infty} \Phi(|S_n|) (n+1)^{-\beta-2}
\]

for \( \beta > -2 \).

These results were subsequently generalized by Ram [2] in the following form.

**Theorem B.** Suppose \( \psi \) is a non-negative, non-decreasing function and integrable \( L([0, 1]) \). Then

\[
\int_{0}^{1} \psi(x) \Phi(|f(x)|) \, dx \leq K \sum_{n=0}^{\infty} \Phi(|S_n|) \alpha_n,
\]

where \( \alpha_n = \int_{1}^{1/n} (1-x) \psi(x) \, dx \).

For \( \psi(x) = 1 \), we get Theorem A (1) while for \( \psi(x) = (1-x)^{\beta} \), \( 0 \geq \beta > -2 \) we get (2). The case \( \beta > 0 \) is not included in Theorem B. In order to include this case he also proved another theorem.
Theorem C. Suppose there is an integer \( p \geq 1 \) such that \( \psi, \psi', \ldots, \psi^{(p-1)} \) are absolutely continuous in \([0, 1)\) and that

\[
\psi(1) = \psi'(1) = \cdots = \psi^{(p-1)}(1) = 0.
\]

Furthermore, suppose that \( \psi^{(p)} \) has a constant sign and \(|\psi^{(p)}|\) is non-decreasing in the set \( \{x \in (0, 1)/|\psi^{(p)}(x)| \text{ exists}\} \). Then

\[
\int_0^1 \psi(x)\Phi(\|f(x)\|) \, dx \leq K \sum_{n=0}^\infty \Phi(|S_n|) n^{-p-1} \left| \psi^{(p-1)}(1 - \frac{1}{n}) \right|.
\]

(4)

Writing \( \psi(x) = (1 - x)\beta, \beta > 0 \) and \( p = -[ - \beta] \) we get the remaining case of (2).

The aim of this note is to show that it is possible to generalize Theorem B in such a manner that it alone includes both (1) and (2) of Theorem A. Our theorem is as follows:

Theorem 1. Let \( \psi(x) \) be a non-negative function such that \( (1-x)\psi(x) \in L(0, 1) \) and \( (1-x)^{-\delta}\psi(x) \) is non-decreasing for some \( \delta > 0 \) in \((0, 1)\). Then

\[
\int_0^1 \psi(x)\Phi(\|f(x)\|) \, dx \leq K \sum_{n=0}^\infty \Phi(|S_n|) z_n,
\]

where \( z_n = \int_0^{1 - 1/n} (1 - x)\psi(x) \, dx \).

It is clear that if \( \psi(x) \) is non-decreasing, then \( (1-x)^{-\delta}\psi(x) \) is also non-decreasing but the converse need not be true. Thus Theorem B is a corollary of Theorem 1. With \( \psi(x) \equiv 1 \), Theorem 1 includes (1) and with \( \psi(x) = (1-x)\beta, -2 < \beta \leq 0; \psi(x) = (1-x)^\beta, \beta > 0 \) (choosing \( \delta > \beta \)) we deduce (2).

2. For the proof of Theorem 1 we need the following lemma.

Lemma. If \( (1-x)\psi(x) \in L(0, 1) \), \( \psi(x) \) is non-negative and \( (1-x)^{-\delta}\psi(x) \) is non-decreasing for some \( \delta > 0 \) in \((0, 1)\) then

\[
\int_0^1 x^n(1-x)\psi(x) \, dx = \int_{1-1/n}^1 (1-x)\psi(x) \, dx.
\]

Proof of the lemma.

\[
\int_0^1 x^n(1-x)\psi(x) \, dx = \left( \int_0^{1-1/n} + \int_{1-1/n}^1 \right) x^n(1-x)\psi(x) \, dx
\]

\[
= L_1 + L_2, \text{ say,}
\]

we have

\[
L_2 \leq \int_{1-1/n}^1 (1-x)\psi(x) \, dx
\]

and

\[
L_1 = \int_0^{1-1/n} x^n(1-x)^{1+\delta}(1-x)^{-\delta}\psi(x) \, dx
\]
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\begin{align*}
\leq n^k \psi(1 - 1/n) \int_0^{1 - 1/n} x^k (1 - x)^{1+\delta} \, dx \\
\leq Kn^k \psi(1 - 1/n) n^{-2-\delta} \\
= Kn^{-2} \psi(1 - 1/n).
\end{align*}

Since
\[ \int_{1-1/n}^1 (1-x) \psi(x) \, dx \geq n^k \psi(1 - 1/n) \int_{1-1/n}^1 (1-x)^{1+\delta} \, dx = \frac{n^{-2} \psi(1 - 1/n)}{2 + \delta}, \]

it follows that
\[ L_1 \leq K \int_{1-1/n}^1 (1-x) \psi(x) \, dx. \]

Thus
\[ \int_0^1 x^k (1-x) \psi(x) \, dx \leq K \int_{1-1/n}^1 (1-x) \psi(x) \, dx. \]

For the converse part
\[ \int_0^1 x^k (1-x) \psi(x) \, dx \geq (1 - 1/n)^k \int_{1-1/n}^1 (1-x) \psi(x) \, dx \]
\[ \sim e^{-1} \int_{1-1/n}^1 (1-x) \psi(x) \, dx. \]

This proves our lemma.

3. Proof of Theorem 1. By Abel's partial summation
\[ f(x) = \sum_{0}^{\infty} S_n x^n (1-x). \]

Using Jensen's inequality, in view of the fact that
\[ \sum_{0}^{\infty} x^n (1-x) = 1, \]

we have
\[ \Phi(|f(x)|) \leq \sum_{n=0}^{\infty} \Phi(|S_n|) x^n (1-x) \]

and consequently by virtue of our lemma
\[ \int_{0}^{1} \psi(x) \Phi(|f(x)|) \, dx \leq \sum_{n=0}^{\infty} \Phi(|S_n|) \int_{0}^{1} x^n (1-x) \psi(x) \, dx \]
\[ \leq K \sum_{n=0}^{\infty} \Phi(|S_n|) \int_{1-1/n}^{1} (1-x) \psi(x) \, dx \]
\[ = K \sum_{n=0}^{\infty} \Phi(|S_n|) a_n. \]
4. In view of the identities
\[
\sum_{n=0}^{\infty} a_n x^n = (1 - x)^{-x-1},
\]
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n = (1 - x)^{a+1} \sum_{n=0}^{\infty} S_n x^n\]
\[
S_n = \sum_{k=0}^{n} A_{n-k} A_{k}, S_k = \sum_{k=0}^{n} A_{n-k} a_k,\]
\[
\sigma_n^2 = \frac{S_n}{A_n^2}, \quad x > -1,
\]
we have
\[
|f(x)| \leq \sum_{n=0}^{\infty} |S_n| x^2 (1 - x)^{a+1} = \sum_{n=0}^{\infty} |\sigma_n^2| A_n^2 x^2 (1 - x)^{a+1}.
\]

Applying Jensen's inequality
\[
\Phi(|f(x)|) \leq \sum_{n=0}^{\infty} \Phi(|\sigma_n^2|) A_n^2 x^2 (1 - x)^{a+1}
\]
and hence
\[
\int_0^1 \psi(x) \Phi(|f(x)|) \, dx \leq \sum_{n=0}^{\infty} \Phi(|\sigma_n^2|) A_n^2 \int_0^1 x^2 \psi(x) \, dx
\]
\[
\leq K \sum_{n=0}^{\infty} \Phi(|\sigma_n^2|) A_n^2 \beta_n,
\]
where
\[
\beta_n = \int_{1-1/n}^1 \psi(x) \, dx, \quad \psi(x) \in L(0, 1).
\]

Thus we have established the following result.

**Theorem 2.** Suppose \( \psi \) is non-negative and integrable \( L(0, 1) \). If \((1 - x)^{-\delta} \psi(x)\) is non-decreasing for some \( \delta > 0 \), then

\[
\int_0^1 \psi(x) \Phi(|f(x)|) \, dx \leq K \sum_{n=0}^{\infty} \Phi(|\sigma_n^2|) A_n^2 \beta_n,
\]

where
\[
\beta_n = \int_{1-1/n}^1 \psi(x) \, dx, \quad \delta > -1.
\]

Choosing \( \delta = 1 \) and \( \psi(x) = 1 \) we get the following result due to Askey and Karlin ([1], Theorem 3)

\[
\int_0^1 \Phi(|f(x)|) \, dx \leq K \sum_{n=0}^{\infty} \Phi(|\sigma_n^2|).
\]

(7)

Using (6) we have also

\[
\int_0^1 \psi(x) \Phi(|f(x)|) \, dx \leq \sum_{n=0}^{\infty} \Phi(|\sigma_n^2|) A_n^2 \beta_n,
\]

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where $\gamma_s = \int_{-1}^{1} (1-x)^{s+1} \psi(x) dx$, provided $(1-x)^{s+1} \psi(x) \in L(0,1)$ for $s > -1$ and $(1-x)^{-\delta} \psi(x)$ is non-decreasing for some $\delta > 0$. In particular if $\psi(x) = (1-x)^{-2}$, $s > 0$ we get

$$\int_0^1 \frac{\Phi(|f(x)|) dx}{(1-x)^s} \leq K \sum_{n=0}^{\infty} \Phi(|\sigma_n^s|), \quad s > 0$$

(9)

This gives another estimate for the left side expression in (7).

Writing $\psi(x) = 1$ in (8) we have

$$\int_0^1 \Phi(|f(x)|) dx \leq \sum_{n=0}^{\infty} \Phi(|\sigma_n^s|) A_n^2 n^{-s-2}$$

$$\leq K \sum_{n=0}^{\infty} \Phi(|\sigma_n^s|) n^{-2}.$$ 

Taking $\Phi(t) = t^p$, $1 \leq p < \infty$ we get

$$\int_0^1 |f(x)|^p dx \leq K \sum_{n=0}^{\infty} |S_n^s|^p n^{-2}$$

$$\leq K \sum_{n=0}^{\infty} |S_n^s|^p n^{-2-p}, \quad s > -1$$

which is Theorem 4 in [1], where they assume that $s$ is any non-negative integer.

Let

$$f(x) = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{(1+x)^2},$$

then $|S_n| \sim n/2$ and $S_{2n}^1 \sim n/2$, $S_{2n+1}^1 = 0$.

It is clear that

$$\sum_{n=0}^{\infty} |S_n|^p n^{-2} = \infty,$$

but

$$\sum_{n=0}^{\infty} |S_n^1|^p n^{-2-p} < \infty.$$

This shows that replacing $S_n$ by $S_n^s$ is more effective in the study of such integrability problems for a power series.

References
