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Summability of Laguerre series at the point $x = 0$

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Abstract. In this paper, the authors prove a theorem on matrix summability of Laguerre series at the point $x = 0$. Various results on Casàro, Nörlund and generalized Nörlund summability method have been deduced.

Keywords. Matrix summability; Laguerre series.

1. Introduction

The triangular matrix $[A] = \{\lambda_{n,k}\}$, where $n = 0, 1, 2, 3, \ldots$ and $k = 0, 1, 2, 3, \ldots$ and $\lambda_{n,k} = 0$ for $k > n$, is regular (in the sense of defining a regular sequence to sequence transformation) if

$$\lim_{n\to\infty} \lambda_{n,k} = 0, \quad \text{for every fixed } k \tag{1}$$

$$\sum_{k=0}^{n} |\lambda_{n,k}| \leqslant M, \quad \text{independent of } n \tag{2}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{n,k} = 1. \tag{3}$$

A series $\Sigma_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{S_n\}$ is said to be summable $[A]$ to the sum S, if

$$\sum_{k=0}^{n} \lambda_{n,k} S_k \to S, \quad \text{as } n \to \infty.$$

In the case

$$\lambda_{n,k} = \frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}, \quad \beta \geqslant 0$$

where $A_n^{\beta-1}$ is determined by the identity

$$(1-x)^{-\beta} = \sum_{n=0}^{\infty} A_n^{\beta-1} x^n, \quad |x| < 1,$$

the method $[A]$ reduces to the well-known Cesàro method (C,β). For

$$\lambda_{n,k}=\frac{p_{n-k}}{P_n},\quad P_n=p_0+p_1+\cdots+p_n(\leqslant 0),$$

the method $[A]$ reduces to the Nörlund method (N,p_n). Also for $\lambda_{n,k}=p_{n-k}q_k/R_n$, where $R_n=q_0p_n+q_1p_{n-1}+q_2p_{n-2}+\cdots+q_np_0(\neq 0)$, the method $[A]$ reduces to the well known generalized Nörlund method (N,p,q).

The Fourier Laguerre expansion of a function $f(x)\in L[0,\infty]$ is given by

$$f(x)\sim\sum_{n=0}^{\infty}a_n L_n^{(\alpha)}(x) \tag{4}$$

where

$$a_n=\left\{\Gamma(\alpha+1)\binom{n+\alpha}{\alpha}\right\}^{-1}\int_0^{\infty}e^{-y}y^{\alpha}f(y)\,L_n^{(\alpha)}(y)\,\mathrm{d}y \tag{5}$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of other $\alpha>-1$, defined by the generating function

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)w^n=(1-w)^{-\alpha-1}\exp\left(-\frac{xw}{1-w}\right)$$

and existence of the integral (5) is presumed in the sense of Lebesgue. Let us write

$$\phi(y)=\{\Gamma(\alpha+1)\}^{-1}e^{-y}y^{\alpha}\{f(y)-f(0)\}$$

M denotes a constant, which may not be the same at each of its occurrence.

2. Main result

In this short note we establish the following theorem:

Theorem. *Let the non-negative real sequence $\{\lambda_{n,k}\}$, be non-decreasing with respect to k. If for some suitable constants c and w (c and w are defined in lemma 3),*

$$\int_{c/n}^{w}\frac{|\phi(y)|\,\mathrm{d}y}{y^{(2\alpha+3)/4}}=o(n^{-(2\alpha+1)/4}),\quad \text{as } n\to\infty \tag{6}$$

and

$$\int_1^{\infty}e^{y/2}y^{-1/4}|\phi(y)|\,\mathrm{d}y<\infty, \tag{7}$$

then for $-1<\alpha<-1/2$, the series (4) *is summable $[A]\ (=[\lambda_{n,k}])$ at $x=0$ to the sum $f(0)$ provided that*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\lambda_{n,k}=1 \tag{8}$$

Note. Since $\{\lambda_{n,k}\}$ is non-negative and non-decreasing in k, we have

$$(n-k)\lambda_{n,k} \leqslant \sum_{m=k+1}^{n} \lambda_{n,m} \leqslant 1. \tag{9}$$

Thus, for each fixed k,

$$\lim_{n\to\infty} \lambda_{n,k} = 0. \tag{10}$$

Hence from (8) and (10) it follows that $[A]$ is a regular method.

Lemma. For the proof of the theorem we need the following lemmas.

Lemma 1. *If $\mu > -1$, then as $n \to \infty$,*

$$\sum_{k=1}^{n} \lambda_{n,k} k^{\mu} = 0(n^{\mu}).$$

Proof. If $\mu \geqslant 0$ proof is obvious, so let $0 > \mu > -1$. Also let $V = [n/2]$, where $[\phi]$ denotes the integral part of ϕ.

Thus,

$$\begin{aligned}\sum_{k=1}^{n} \lambda_{n,k} k^{\mu} &= \sum_{k=0}^{n-1} \lambda_{n,n-k}(n-k)^{\mu}\\ &= \left(\sum_{k=0}^{V} + \sum_{k=V+1}^{n-1}\right)\lambda_{n,n-k}(n-k)^{\mu}\\ &= \Sigma_1 + \Sigma_2, \quad \text{say.}\end{aligned}$$

Now

$$\begin{aligned}|\Sigma_1| &\leqslant (n-V)^{\mu} \sum_{k=0}^{V} \lambda_{n,n-k}\\ &= O(n^{\mu})\end{aligned}$$

and

$$\begin{aligned}|\Sigma_2| &\leqslant \lambda_{n,n-V} \sum_{k=V+1}^{n-1} (n-k)^{\mu}\\ &= O(\lambda_{n,n-V} n^{\mu+1})\\ &= O(n^{\mu}), \quad \text{by (9).}\end{aligned}$$

which proves the lemma.

Lemma 2. *From* (6) *we have*

$$\int_0^t |\phi(y)|\,\mathrm{d}y = o(t^{\alpha+1}), \quad \text{as } t \to 0.$$

Proof. Working on the lines of Khare and Tripathi [1] we can prove the lemma.

Lemma 3. ([2] p. 175). *If α is real, c and w are fixed positive constants, then as $n \to \infty$.*

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-1/4} O(n^{\alpha/2-1/4}), & \text{if } \frac{c}{n} \leqslant x \leqslant w \\ O(n^{\alpha}), & \text{if } 0 \leqslant x \leqslant \frac{c}{n} \end{cases}$$

Lemma 4. ([2], p. 239). *If α and λ are real constants, $a > 0$, $0 < \eta < 4$, then for $n \to \infty$*

$$\max \exp(-x/2) x^{\lambda} |L_n^{(\alpha)}(x)| \sim n^{Q}$$

where

$$Q = \begin{cases} \max\left(\lambda - \frac{1}{2}, \frac{\alpha}{2} - \frac{1}{4}\right), & \text{if } a \leqslant x \leqslant (4-\eta)n \\ \max\left(\lambda - \frac{1}{3}, \frac{\alpha}{2} - \frac{1}{4}\right), & \text{if } x \geqslant a \end{cases}$$

the maxima being taken in the intervals pointed out in the right hand members.

Proof of the theorem. The nth partial sum of the series (4) at $x = 0$ is given by (see [2])

$$S_n(0) - f(0) = \int_0^{\infty} \phi(y)\, L_n^{(\alpha+1)}(y) \mathrm{d}y.$$

Therefore

$$I = \sum_{k=0}^{n} \lambda_{n,k}(S_k(0) - f(0)) = \sum_{k=0}^{n} \lambda_{n,k} \int_0^{\infty} \phi(y)\, L_k^{(\alpha+1)}(y) \mathrm{d}y.$$

In order to prove the theorem, we must show that

$$I = o(1), \quad \text{as } n \to \infty.$$

We choose a positive integer $N(> w^{-1})$. We may assume $\lambda_{n,k} = 0$ for $k = 0, 1, 2, 3, \ldots, N-1$.

Now, for $n > N$, we have

$$I = \sum_{k=N}^{n} \lambda_{n,k} \int_0^{\infty} \phi(y)\, L_k^{(\alpha+1)}(y) \mathrm{d}y.$$

Let us write

$$I = \int_0^{c/n} + \int_{c/n}^{w} + \int_w^{n} + \int_n^{\infty}, \quad \text{where } c \text{ and } w \text{ are defined in lemma 3.}$$

$$= I_1 + I_2 + I_3 + I_4, \quad \text{say.}$$

Using lemma 3, we have

$$|I_1| \leqslant M \sum_{k=N}^{n} |\lambda_{n,k}| k^{\alpha+1} \int_0^{c/n} |\phi(y)| \mathrm{d}y$$

$$\leqslant M n^{\alpha+1} \mathrm{o}(n^{-\alpha-1}), \quad \text{by lemmas 1 and 2.}$$
$$= \mathrm{o}(1), \quad \text{as } n \to \infty.$$

Again using lemma 3, we have

$$|I_2| \leqslant M \sum_{k=N}^{n} |\lambda_{n,k}| k^{(2\alpha+1)/4} \int_{c/n}^{w} \frac{|\phi(y)|\,\mathrm{d}y}{y^{(2\alpha+3)/4}}$$
$$\leqslant M n^{(2\alpha+1)/4} \mathrm{o}(n^{-(2\alpha+1)/4}) \quad \text{using (6) and lemma 1}$$
$$= \mathrm{o}(1), \quad \text{as } n \to \infty.$$

Now I_3 can be written as

$$|I_3| \leqslant \sum_{k=N}^{n} |\lambda_{n,k}| \int_{w}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| e^{-y/2} \cdot y^{(2\alpha+3)/4} |L_k^{(\alpha+1)}(y)|\,\mathrm{d}y.$$

Hence, by Lemma 4,

$$|I_3| \leqslant M \sum_{k=N}^{n} |\lambda_{n,k}| k^{(2\alpha+1)/4} \int_{w}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)|\,\mathrm{d}y$$
$$\leqslant M n^{(2\alpha+1)/4} \int_{w}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)|\,\mathrm{d}y.$$

Since $(2\alpha+3)/4 > 1/4$, we have

$$|I_3| \leqslant M n^{(2\alpha+1)/4} \int_{w}^{n} e^{y/2} y^{-1/4} |\phi(y)|\,\mathrm{d}y$$
$$= O(n^{(2\alpha+1)/4}), \quad \text{from (7)},$$
$$= \mathrm{o}(1), \quad \text{as } n \to \infty.$$

Finally, consider I_4

$$|I_4| \leqslant \sum_{k=N}^{n} |\lambda_{n,k}| \int_{n}^{\infty} e^{(y/2)} y^{-(3\alpha+5)/6} |\phi(y)| e^{(-y/2)} y^{(3\alpha+5)/6}$$
$$\times |L_k^{(\alpha+1)}(y)|\,\mathrm{d}y.$$

Using lemma 4, we get

$$|I_4| \leqslant M \sum_{k=N}^{n} |\lambda_{n,k}| k^{(\alpha+1)/2} \int_{n}^{\infty} \frac{e^{(y/2)} y^{-1/4} |\phi(y)|\,\mathrm{d}y}{y^{(6\alpha+7)/12}}$$
$$\leqslant M n^{(\alpha+1)/2} n^{-(6\alpha+7)/12} \int_{n}^{\infty} e^{(y/2)} y^{-1/4} |\phi(y)|\,\mathrm{d}y$$
$$= O(n^{(1/2)-(7/12)}), \quad \text{using (7)}$$
$$= \mathrm{o}(1), \quad \text{as } n \to \infty.$$

This completes proof of the theorem.

Corollaries

COROLLARY 1

Let $\beta > 0$. If (6) and (7) hold good, then for $-1 < \alpha < -1/2$, the series (4) is summable (C, β) at $x = 0$, to the sum $f(0)$.

COROLLARY 2

Let the regular Nörlund method (N, p_n) be defined by a non-negative, non-increasing sequence $\{p_n\}$. If (6) and (7) hold good, then for $-1 < \alpha < -1/2$, the series (4) is summable (N, p_n) at $x = 0$ to the sum $f(0)$.

COROLLARY 3

Let the regular generalised Nörlund method (N, p, q) be defined by a non-negative, non-increasing sequence $\{p_n\}$, and a non-negative non-decreasing sequence $\{q_n\}$. If

$$G(t) \equiv \int_0^t |\phi(y)| \mathrm{d}y = o(t^{\alpha+1}) \quad \text{as } t \to 0, \tag{11}$$

and (7) holds good, then for $-1 < \alpha < -1/2$ the series (4) is summable (N, p, q) at $x = 0$ to the sum $f(0)$.

COROLLARY 4

Let the regular generalised Nörlund method (N, p, q) be defined as in Corollary 3. If (6) and (7) hold good, then for $-1 < \alpha < -1/2$, the series (4) is summable (N, p, q) at $x = 0$, to the sum $f(0)$.

Proof of corollaries. If we put $\lambda_{n,k} = p_{n-k} q_k / R_n$, in our theorem, we get corollary 4. Putting $q_n = 1$ for all n, in corollary 4, we get corollary 2. If we set $p_n = \binom{n+\beta-1}{\beta-1}$ $\beta > 0$, in corollary 2, we get corollary 1. To prove corollary 3, it is sufficient to show that condition (11) implies condition (6). From condition (11), given $\varepsilon > 0$, we can choose δ such that

$$|G(t)| \leqslant \varepsilon t^{\alpha+1}, \quad 0 < t \leqslant \delta.$$

Now

$$J = \int_{c/n}^{\delta} \frac{|\phi(y)| \mathrm{d}y}{y^{(2\alpha+3)/4}} = M \left[\frac{G(y)}{y^{(2\alpha+3)/4}} \right]_{c/n}^{\delta} + M \int_{c/n}^{\delta} \frac{G(y) \mathrm{d}y}{y^{(2\alpha+7)/4}}$$

where M is a constant, may be different at each occurrence. Hence

$$J \leqslant M\varepsilon + M\varepsilon n^{-(2\alpha+1)/4} + M\varepsilon \int_{c/n}^{\delta} y^{(2\alpha-3)/4} \mathrm{d}y$$

or

$$\int_{c/n}^{\delta} \frac{|\phi(y)| \mathrm{d}y}{y^{(2\alpha+3)/4}} = o(n^{-(2\alpha+1)/4}), \quad \text{as } n \to \infty.$$

Also

$$\int_{\delta}^{w} \frac{|\phi(y)|\,dy}{y^{(2\alpha+3)/4}} \text{ is a constant.}$$

Hence

$$\int_{c/n}^{w} \frac{|\phi(y)|\,dy}{y^{(2\alpha+3)/4}} = o(n^{-(2\alpha+1)/4}), \quad \text{as } n \to \infty.$$

This completes the proof of corollary 3.

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