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Bilateral generating functions for Jacobi polynomials

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Abstract. In this paper, we have obtained three theorems on generating functions. We derive from these theorems a large number of bilateral generating functions for Jacobi polynomials. Certain interesting expansions of triple hypergeometric series are also obtained from one of the theorems.

Keywords. Bilateral generating functions; orthogonal polynomials; generating relations; Jacobian polynomials; triple hypergeometric series.

1. Introduction

With the usual notation, the general triple hypergeometric series is defined by Srivastava [7] as follows:

$$F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\, x,y,z\right] \tag{1}$$

$$= \sum_{r,s,t=0}^{\infty} \frac{[(a)]_{r+s+t}[(b)]_{r+s}[(b')]_{s+t}[(b'')]_{r+t}}{[(e)]_{r+s+t}[(g)]_{r+s}[(g')]_{s+t}[(g'')]_{r+t}}$$

$$\times \frac{[(c)]_r[(c')]_s[(c'')]_t}{[(h)]_r[(h')]_s[(h'')]_t}\frac{x^r y^s z^t}{r!s!t!}$$

where

$$(\alpha)n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\begin{cases}1, & \text{if } n=0,\\ (\alpha)(\alpha+1)\ldots(\alpha+n-1), & \forall n\in\{1,2,3,\ldots\},\end{cases} \tag{2}$$

and, for the sake of brevity, (a) is taken to abbreviate the sequence of A parameters $a_1, a_2, \ldots, a_A$ and so on, and

$$[(a)]_n = \prod_{n=0}^{A} (a_j)n, \text{etc.}$$

Theorem 1. *Let $\{A_n\}$ be a sequence of arbitrary complex numbers, then we have*

$$\sum_{n=0}^{\infty} A_n \frac{[(a)]_n[(b)]_n[(b'')]_n[(c)]_n}{[(e)]_n[(g)]_n[(g'')]_n[(h)]_n}$$

$$\times F^{(3)}\left[\begin{matrix}(a)+n::(b)+n;(b');(b'')+n:(c)+n;(c');(c'');\\(e)+n::(g)+n;(g');(g'')+n:(h)+n,1+\delta+2n;(h');(h'');\end{matrix}\, x,y,z\right]$$

$$= \sum_{r,s,t=0}^{\infty} \frac{[(a)]_{r+s+t}[(b)]_{r+s}[(b')]_{s+t}[(b'')]_{r+t}[(c)]_r[(c')]_s}{[(e)]_{r+s+t}[(g)]_{r+s}[(g')]_{s+t}[(g'')]_{r+t}[(h)]_r[(h')]_s}$$
$$\times \frac{[(c'')]_t}{[(h'')]_t} \frac{x^r y^s z^t}{r!s!t!} \sum_{n=0}^{r} A_n \frac{(-r)_n \Gamma(1+\delta+2n)}{\Gamma(1+\delta+n+r)} x^{-n}(-1)^n. \tag{3}$$

Proof. Denoting the first member of (3) by Ω and using the definition (1), it is readily observed that

$$\Omega = \sum_{n,r,s,t=0}^{\infty} \frac{[(a)]_{r+s+t+n}[(b)]_{r+s+n}[(b')]_{s+t}[(b'')]_{r+t+n}}{[(e)]_{r+s+t+n}[(g)]_{r+s+n}[(g')]_{s+t}[(g'')]_{r+t+n}}$$
$$\times \frac{[(c)]_{r+n}[(c')]_s[(c'')]_t}{[(h)]_{r+n}(1+\delta+2n)_r[(h')]_s[(h'')]_t} A_n \frac{x^r y^s z^t}{r!s!t!}$$
$$= \sum_{r,s,t=0}^{\infty} \sum_{n=0}^{r} \frac{[(a)]_{r+s+t}[(b)]_{r+s}[(b')]_{s+t}[(b'')]_{r+t}}{[(e)]_{r+s+t}[(g)]_{r+s}[(g')]_{s+t}[(g'')]_{r+t}}$$
$$\times \frac{[(c)]_r[(c')]_s[(c'')]_t}{[(h)]_r(1+\delta+2n)_{r-n}[(h')]_s[(h'')]_t} A_n \frac{x^{r-n} y^s z^t}{(r-n)!s!t!}$$

which, in view of (2), is precisely the second member of (3). This evidently completes the proof of Theorem 1 under the assumption that the interchange of the order of summations is permissible by the absolute convergence of the series involved. Now we recall the familiar expansion ([3], p. 212)

$$[\tfrac{1}{2}(1-w)]^r = \Gamma(r+\alpha+1)$$
$$\times \sum_{n=0}^{r} \frac{(-r)n(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+r+\alpha+\beta+2)} P_n^{(\alpha,\beta)}(w), \tag{4}$$

where $P_n^{(\alpha,\beta)}(w)$ is the classical Jacobi polynomial ([3], p. 170). Now replacing δ by $1+\alpha+\beta$, A_n by $(-x)^n/(1+\alpha+\beta+n)_n(1+\alpha)_n P_n^{(\alpha,\beta)}(w)$ and using the expansion (4) we obtain the following bilateral generating function from (3):

$$\sum_{n=0}^{\infty} \frac{[(a)]_n[(b)]_n[(b'')]_n[(c)]_n(-x)^n P_n^{(\alpha,\beta)}(w)}{[(e)]_n[(g)]_n[(g'')]_n[(h)]_n(1+\alpha)_n(1+\alpha+\beta+n)_n}$$
$$\times F^{(3)}\left[\begin{matrix}(a)+n::(b)+n;(b');(b'')+n:(c)+n;(c');(c'');\\ (e)+n::(g)+n;(g');(g'')+n:(h)+n,\\ 2+\alpha+\beta+2n;(h');(h'');\end{matrix} x,y,z\right]$$
$$= F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\ (e)::(g);(g');(g''):(h),1+\alpha;(h');(h'');\end{matrix} \tfrac{1}{2}x(1-w),y,z\right]. \tag{5}$$

The above result (5) may also be obtained from the special case $r=3$ of Theorem 3 of [9].

Application. Since the triple hypergeometric series $F^{(3)}$ is a unification of Lauricella's functions $F_A, F_B, \ldots, F_T$ ([5], p. 114) (see also [11]) and Srivastava's functions H_A, H_B, H_C ([6], pp. 99–100), a number of bilateral generating functions involving these functions are deducible from the result (5). Thus, for example, if we set in (5) $A = C = C' = C'' = H' = H'' = 1$, $B = B' = B'' = E = G = G' = G'' = H = 0$, we shall obtain the result (3.7) of [9]. Similarly on specializing the parameters, the result (3.8) of [9] and other results involving these functions follow from (5). For more general multivariable results on generating functions of this type and other, see the works of Srivastava and Pathan ([9] and [10]; see also [12]).

Expansions. Next, setting

$$A_n = \frac{(\delta)_n(\alpha)_n(\beta)_n}{(\frac{1}{2}\delta)_n[\frac{1}{2}(1+\delta)]_n(1+\delta-\alpha)_n(1+\delta-\beta)_n}(-\tfrac{1}{4}x)^n \tag{6}$$

and recalling the following sum ([2], p. 25)

$${}_5F_4\left[\begin{matrix} -r, \delta, 1+\frac{1}{2}\delta, \alpha, \beta; \\ \frac{1}{2}\delta, 1+\delta-\alpha, 1+\delta-\beta, 1+\delta+r; \end{matrix} 1\right] = \frac{(1+\delta)_r(1+\delta-\alpha-\beta)_r}{(1+\delta-\alpha)_r(1+\delta-\beta)_r} \tag{7}$$

we obtain from Theorem 1:

$$\sum_{n=0}^{\infty} \frac{[(a)]_n[(b)]_n[(b'')]_n[(c)]_n(\alpha)_n(\beta)_n(-x)^n}{[(e)]_n[(g)]_n[(g'')]_n[(h)]_n(\delta+n)_n(1+\delta-\alpha)_n(1+\delta-\beta)_n}$$
$$\times F^{(3)}\left[\begin{matrix} (a)+n::(b)+n;(b');(b'')+n:(c)+n;(c');(c''); \\ (e)+n::(g)+n;(g');(g'')+n:(h)+n, 1+\delta+2n;(h');(h''); \end{matrix} x, y, z\right]$$
$$= F^{(3)}\left[\begin{matrix} (a)::(b);(b');(b''):(c), 1+\delta-\alpha-\beta;(c');(c''); \\ (e)::(g);(g');(g''):(h), 1+\delta-\alpha, 1+\delta-\beta;(h');(h''); \end{matrix} x, y, z\right] \tag{8}$$

Deductions. We give below some interesting expansions from the above result on specializing the parameters suitably:

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(a)_n(-x)^n}{(\delta+n)_n(1+\delta-\alpha)_n}$$
$$\times F_A(a+n, 1+\delta-\beta+n, b_2, b_3; 1+\delta+2n, c_2, c_3; x, y, z)$$
$$= F_A(a, 1+\delta-\alpha-\beta, b_2, b_3; 1+\delta-\alpha, c_2, c_3; x, y, z) \tag{9}$$

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(a_1)_n(-x)^n}{(\delta+n)_n(1+\delta-\alpha)_n}$$
$$\times F_E(a_1+n, a_1+n, a_1+n, 1+\delta-\beta+n, b_2, b_2; 1+\delta+2n, c_2, c_3; x, y, z)$$
$$= F_E(a_1, a_1, a_1, 1+\delta-\alpha-\beta, b_2, b_2; 1+\delta-\alpha, c_2, c_3; x, y, z) \tag{10}$$

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(b_1)_n(-x)^n}{(\delta+n)_n(1+\delta-\alpha)_n}$$
$$\times F_N(1+\delta-\beta+n, a_2, a_3, b_{1+n}, b_2, b_{1+n}; 1+\delta+2n, c_2, c_2; x, y, z)$$
$$= F_N(1+\delta-\alpha-\beta, a_2, a_3, b_1, b_2, b_1; 1+\delta-\alpha, c_2, c_2; x, y, z) \tag{11}$$

Further, setting $z=0$ in (9) we get

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(a)_n(-x)^n}{(\delta+n)_n(1+\delta-\alpha)_n}$$
$$\times F_2(a+n,1+\delta-\beta+n,b_2;1+\delta+2n,c_2;x,y)$$
$$=F_2(a,1+\delta-\beta,b_2;1+\delta-\alpha,c_2;x,y) \tag{12}$$

where F_2 is Appell's double hypergeometric function (111, p. 14).

3. Further generalizations of Theorem 1

In this section we give two generalizations of bilateral generating function (5) which are contained in the following theorems. Just as the generating function (5), Theorem 2 can also be deduced from the special case $r=3$ of Theorem 3 of [9].

Theorem 2. *Let $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$ be sequences of arbitrary complex numbers, then*

$$\sum_{n=0}^{\infty}\frac{(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}P_n^{(\alpha,\beta)}(w)\cdot\sum_{k,m,p=0}^{\infty}A_{n+k+m+p}B_{(m+p,m,p)}$$
$$\times C_{n+k+m}D_{n+k+p}E_{n+k}\frac{x^ky^mz^p}{(2+\alpha+\beta+2n)_k k!m!p!}$$
$$=\sum_{k,m,p=0}^{\infty}A_{k+m+p}B_{(n,+p,m,p)}C_{k+m}D_{k+p}E_k\frac{[\frac{1}{2}(1-w)x]^ky^mz^p}{(1+\alpha)_k k!m!p!} \tag{13}$$

provided each side has a meaning.

Theorem 3. *Let $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$ be sequences of arbitrary complex numbers, then*

$$\sum_{n=0}^{\infty}\frac{(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}P_n^{(\alpha,\beta)}(w)\sum_{k,m,p=0}^{\infty}A_{2n+2k+2m+p}$$
$$\times B_{2n+2k+m+p}C_{2n+2k+m}D_{(m+p,m,p)}E_{n+k}\frac{x^ky^mz^p}{(2+\alpha+\beta+2n)_k k!m!p!}$$
$$=\sum_{k,m,p=0}^{\infty}A_{2k+2m+p}B_{2k+m+p}C_{2k+m}D_{(m+p,m,p)}E_k\frac{[\frac{1}{2}(1-w)x]^ky^mz^p}{(1+\alpha)_k k!m!p!} \tag{14}$$

provided each side has a meaning.

Proof of Theorem 2. Following Srivastava and Pathan ([9], [10]) rather closely, we denote the first member of (13) by Ω, say, and apply the definition (2) it is readily observed that

$$\Omega=\sum_{k,m,p=0}^{\infty}\sum_{n=0}^{k}\frac{(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}P_n^{(\alpha,\beta)}(w)$$

$$\times A_{k+m+p}B_{(m+p,m,p)}C_{k+m}D_{k+p}E_k\frac{x^{k-n}y^m z^p}{(2+\alpha+\beta+2n)_{k-n}(k-n)!m!p!}$$

$$=\sum_{k,m,p=0}^{\infty}A_{k+m+p}B_{(m+p,m,p)}C_{k+m}D_{k+p}E_k\frac{x^k y^m z^p}{k!m!p!}$$

$$\times\Gamma(1+\alpha)\left[\sum_{n=0}^{\infty}\frac{(-k)_n(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(2+\alpha+\beta+n+k)}P_n^{(\alpha,\beta)}(w)\right]$$

which, in view of the expansion (4), gives

$$\Omega=\sum_{k,m,p=0}^{\infty}A_{k+m+p}B_{(m+p,m,p)}C_{k+m}D_{k+p}E_k\frac{[\frac{1}{2}(1-w)x]^k y^m z^p}{(1+\alpha)^k k!m!p!}$$

which is same as the right-hand side of (13). This evidently completes the proof of Theorem 2 under the assumption that various changes of the order of summations are permissible by the absolute convergence of the series involved. Thus in general, Theorem 2 holds for such values of the variables x, y, z and w for which each member of (13) exists. Theorem 2 may also be obtained directly from Theorem 3 of [9] by taking $r=3$.

Proof of Theorem 3 is similar to that of Theorem 2 and we, therefore, omit the details involved.

Deductions. If we assign suitable values to arbitrary sequences $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$ suitably, Theorem 2 can obviously be reduced to the general result (5) and its various special cases.

Similarly, by assigning suitable special values to the arbitrary coefficients $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$, Theorem 3 can be applied to deduce a number of bilateral generating functions involving Exton's triple hypergeometric functions [4].

Thus if we set

$$A_{2k+2m+p}=(a_1)_{2k+2m+p},\quad B_{2k+m+p}=C_{2k+m}=E_k=1$$

and

$$D_{(m+p,m,p)}=[(a_2)_p/(c_1)_{m+p}]$$

we obtain from (14):

$$\sum_{n=0}^{\infty}\frac{(a_1)_{2n}(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}X_1[a_1+2n,a_2;c_1,2+\alpha+\beta+2n;x,y,z]P_n^{(\alpha,\beta)}(w)$$

$$=X_1[a_1,a_2;c_1,1+\alpha;\tfrac{1}{2}(1-w)x,y,z]. \tag{15}$$

Similarly, the following bilateral generating functions are easily obtainable from Theorem 3 on assigning suitable values to the arbitrary coefficients:

$$\sum_{n=0}^{\infty}\frac{(a_1)_{2n}(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}X_2[a_1+2n,a_2;2+\alpha+\beta+2n,c_2,c_3;x,y,z]P_n^{(\alpha,\beta)}(w)$$

$$=X_2[a_1,a_2;1+\alpha,c_2,c_3;\tfrac{1}{2}(1-w)x,y,z] \tag{16}$$

$$\sum_{n=0}^{\infty}\frac{(a_1)_{2n}(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}X_4[a_1+2n,a_2;2+\alpha+\beta+2n,c_2,c_3;x,y,z]P_n^{(\alpha,\beta)}(w)$$

$$=X_4[a_1,a_2;1+\alpha,c_2,c_3;\tfrac{1}{2}(1-w)x,y,z] \tag{17}$$

$$\sum_{n=0}^{\infty} \frac{(a_1)_{2n}(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n} X_7[a_1+2n, a_2, a_3; c_1, 2+\alpha+\beta+2n; x, y, z] P_n^{(\alpha,\beta)}(w)$$

$$= X_7[a_1, a_2, a_3; c_1, 1+\alpha; \tfrac{1}{2}(1-w)x, y, z]. \tag{18}$$

We conclude by observing that many other results may also be obtained for the Exton's triple hypergeometric functions X_8, X_{12}, X_{15}, X_{17} and X_{19}.

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