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An extension of bilateral generating functions of modified Laguerre polynomials

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Abstract. In this note a theorem concerning the extension of bilateral generating functions of the modified Laguerre polynomials is derived. Some applications of the theorem are also pointed out.

Keywords. Modified Laguerre polynomials; bilateral generating functions.

1. Introduction

Recently the present authors [5] have studied the following theorem on bilateral generating functions of modified Laguerre polynomials, defined by Goyal [3]:

If there exists an unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_{a,b,m,n}(x) w^n \tag{1}$$

then we have

$$(1 - wb)^{-m} \exp\left(\frac{-wax}{1 - wb}\right) G\left(\frac{x}{1 - wb}, \frac{wz}{1 - wb}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(z) L_{a,b,m,n}(x) \cdot w^n \tag{2}$$

where

$$\sigma_n(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k. \tag{3}$$

The aim of this note is to state and prove a theorem in connection with the extension of bilateral generating functions for the modified Laguerre polynomials. In fact, the following theorem is obtained as the main result of our investigation:

Theorem. *If there exists an unilateral generating relation of the form*

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_{a,b,m,n+k}(x) \cdot w^n, \tag{4}$$

where k is a non-negative integer, then we have

$$(1-wb)^{-m-k}\exp\left(\frac{-wax}{1-wb}\right)G\left(\frac{x}{1-wb},\frac{wz}{1-wb}\right)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\,L_{a,b,m,n+k}(x)\cdot w^n \tag{5}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n}a_p\binom{n+k}{p+k}z^p. \tag{6}$$

The importance of the above theorem lies in the fact that a large number of generating relations can be obtained from (5) by attributing different suitable values to a_n in the relation (4).

Some interesting applications and special cases have also been shown.

Proof of the theorem. We shall first consider the following linear partial differential operator

$$R=bxy\frac{\partial}{\partial x}+by^2\frac{\partial}{\partial y}+y\{b(m+k)-ax\} \tag{7}$$

such that

$$R[L_{a,b,m,n+k}(x)y^n]=(n+k+1)\cdot L_{a,b,m,n+k+1}(x)\cdot y^{n+1}. \tag{8}$$

The extended form of the group generated by R is given by

$$\exp(wR)[f(x,y)]=(1-wby)^{-m-k}\exp\left(\frac{-waxy}{1-wby}\right)f\left(\frac{x}{1-wby},\frac{y}{1-wby}\right) \tag{9}$$

Consider the following formula

$$G(x,w)=\sum_{n=0}^{\infty}a_n\cdot L_{a,b,m,n+k}(x)\cdot w^n. \tag{10}$$

Replacing w by wyz in (10) we get

$$G(x,wyz)=\sum_{n=0}^{\infty}a_n\{L_{a,b,m,n+k}(x)\cdot y^n\}(wz)^n. \tag{11}$$

Operating $\exp(wR)$ on both sides of (11) we get

$$\exp(wR)\cdot G(x,wyz)=\exp(wR)\left[\sum_{n=0}^{\infty}a_n\{L_{a,b,m,n+k}(x)\cdot y^n\}(wz)^n\right]. \tag{12}$$

The left member of (12) on recalling (9) becomes

$$(1-wby)^{-m-k}\exp\left(\frac{-waxy}{1-wby}\right)G\left(\frac{x}{1-wby},\frac{wyz}{1-wby}\right). \tag{13}$$

Also the right member of (12) with the help of (8) may be written as

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\cdot(wz)^n\cdot\frac{w^p}{p!}\cdot R^p\{L_{a,b,m,n+k}(x)y^n\}$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\cdot\frac{(wy)^{n+p}}{p!}\cdot z^n\cdot(n+k+1)_p\cdot L_{a,b,m,n+k+p}(x)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p}\cdot\frac{(wy)^n}{p!}\cdot(n-p+k+1)_p\cdot z^{n-p}\cdot L_{a,b,m,n+k}(x)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{p=0}^{n} a_{n-p}\cdot\frac{(n-p+k+1)_p}{p!}\cdot z^{n-p}\right)(wy)^n\cdot L_{a,b,m,n+k}(x)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{a,b,m,n+k}(x)\cdot(wy)^n$$

where

$$\sigma_n(z)=\sum_{p=0}^{n} a_p\cdot\binom{n+k}{p+k}\cdot z^p. \tag{14}$$

Equating (13) and (14) and then putting $y=1$ we get

$$(1-wb)^{-m-k}\cdot\exp\left(\frac{-wax}{1-wb}\right)\cdot G\left(\frac{x}{1-wb},\frac{wz}{1-wb}\right)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{a,b,m,n+k}(x)\cdot w^n$$

where

$$\sigma_n(z)=\sum_{p=0}^{n} a_p\binom{n+k}{p+k}\cdot z^p. \tag{15}$$

This completes the proof of the theorem.

Putting $k=0$, we get the following result on bilateral generating relation as a corollary to our main theorem:

COROLLARY 1

If there exists an unilateral generating relation of the form

$$G(x,w)=\sum_{n=0}^{\infty} a_n\cdot L_{a,b,m,n}(x)\cdot w^n \tag{16}$$

then, we have

$$(1-wb)^{-m}\cdot\exp\left(\frac{-wax}{1-wb}\right)\cdot G\left(\frac{x}{1-wb},\frac{wz}{1-wb}\right)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{a,b,m,n}(x)\cdot w^n \tag{17}$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \cdot \binom{n}{p} \cdot z^p \tag{18}$$

which is found derived in [5].

2. Applications

As an application we consider the following generating relation [6]:

$$\sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} \cdot L_{a,b,m,n+k}(x) \cdot w^n$$
$$= (1-wb)^{-m-k} \cdot \exp\left(\frac{-axy}{1-by}\right) \cdot L_{a,b,m,k}\left(\frac{x}{1-wb}\right). \tag{19}$$

Putting $a_n = (k+1)_n/n!$ in our theorem we get the following generalization of (19),

$$(1-wb-wbz)^{-m-k} \cdot \exp\left[\frac{-wax(1+z)}{1-wb-wbz}\right]$$
$$= \sum_{n=0}^{\infty} \sigma_n(z) \cdot L_{a,b,m,n+k}(x) \cdot w^n \tag{20}$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} \binom{p+k}{p} \cdot \binom{n+k}{p+k} \cdot z^p. \tag{21}$$

3. Special cases

(a) On specializing the parameters as $a = b = 1$ and $m = (1+\alpha)$ in our main theorem we get the following results on Laguerre polynomials:

Theorem. *If there exists an unilateral generating relation of the form*

$$G(x,w) = \sum_{n=0}^{\infty} a_n \cdot L_{n+k}^{(\alpha)}(x) \cdot w^n \tag{22}$$

where k is a non-negative integer, then we have

$$(1-w)^{-1-\alpha-k} \cdot \exp\left(\frac{-wx}{1-w}\right) \cdot G\left(\frac{x}{1-w}, \frac{wz}{1-w}\right)$$
$$= \sum_{n=0}^{\infty} \sigma_n(z) \cdot L_{n+k}^{(\alpha)}(x) \cdot w^n \tag{23}$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \cdot \binom{n+k}{p+k} \cdot z^p \tag{24}$$

which is found derived in [4].

Putting $k = 0$ in (22)–(24), we get the following result:

COROLLARY 2

If there exists an unilateral generating relation of the form

$$G(x,w)=\sum_{n=0}^{\infty} a_n\cdot L_n^{(\alpha)}(x)\cdot w^n \tag{25}$$

then we have

$$(1-w)^{-1-\alpha}\cdot\exp\left(\frac{-wx}{1-w}\right)\cdot G\left(\frac{x}{1-w},\frac{wz}{1-w}\right)=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_n^{(\alpha)}(x)w^n. \tag{26}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n} a_p\cdot\binom{n}{p}\cdot z^p \tag{27}$$

which is found derived by [1] *and* [2].

(b) Again assigning $a=b=1$ and $m=(1+\alpha)$ in (20) we get

$$(1-w-wz)^{-1-\alpha-k}\cdot\exp\left[\frac{-wx(1+z)}{1-w-wz}\right]=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{n+k}^{(\alpha)}(x)\cdot w^n \tag{28}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n}\binom{p+k}{p}\cdot\binom{n+k}{p+k}\cdot z^p \tag{29}$$

Putting $k=0$ in (25) and (26) we get the following result:

$$(1-w-wz)^{-1-\alpha}\exp\left[\frac{-wx(1+z)}{1-w-wz}\right]=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_n^{(\alpha)}(x)\cdot w^n \tag{30}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n}\binom{n}{p}\cdot z^p \tag{31}$$

which is the result obtained in [5].

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