Proc. Indian Acad. Sci. (Math. Sci.), Vol. 101, No. 1, April 1991, pp. 37–41.

Integrals involving Fox's *H*-function

SHASHI KANT and C L KOUL
Department of Mathematics, M. R. Engineering College, Jaipur 302017, India

MS received 15 January 1990; revised 18 August 1990

Abstract. We evaluate four integrals involving Fox's H-functions and a general class of polynomials $S_n^m[x]$, introduced earlier by Srivastava.

Keywords. H-function; generalized polynomials; extended Jacobi polynomials.

1. Introduction and definitions

Recently Kalla *et al* [1] and Kalla [2] established a number of integrals involving Jacobi polynomials and generalized Jacobi functions. The aim of this paper is to evaluate four integrals involving Fox's H-function and a general class of polynomials $S_n^m[x]$. The technique followed is essentially that of Kalla [1], [2]. Srivastava [4] studied the general class of polynomials $S_n^m[x]$, defined as

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots \tag{1}$$

where m is an arbitrary positive integer, the coefficients $A_{n,k} (n, k \geqslant 0)$ are arbitrary constants, real or complex, and

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}. \tag{2}$$

By suitably specialising the coefficients $A_{n,k}$, the polynomials $S_n^m[x]$ can be reduced to the classical orthogonal polynomials (see Srivastava and Singh [5] for details). The Fox's H-function is defined and represented as follows [6]:

$$\begin{aligned} H(x) = H_{P,Q}^{M,N}\left[x \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] \\ = \frac{1}{2\pi i} \int_{\mathscr{L}} \theta(s) x^s \, \mathrm{d}s, \end{aligned} \tag{3}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j s) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s)}. \tag{4}$$

By summing up the residues at the simple poles of the integrand of (3), the following expression for $H[x]$ was derived by Braaksma [see 6]:

$$H[x]=\sum_{h=1}^{M}\sum_{r=0}^{\infty}\left\{\frac{(-1)^r}{r!}\frac{\phi(\lambda)}{\beta_h}x^{\lambda}\right\}, \tag{5}$$

where

$$\lambda=\frac{b_h+r}{\beta_h},\quad r=0,1,2,\ldots \tag{6}$$

and

$$\phi(t)=\frac{\theta(t)}{\Gamma(b_h-\beta_h t)}, \tag{7}$$

provided that the series on the right side of (5) is absolutely convergent.

2. Result required

The following integral is required to establish the main integrals:

$$\begin{aligned}&\int_{-1}^{1}(1-x)^a(1+x)^b S_n^m\left[c\left(\frac{1-x}{2}\right)^{\delta_1}\left(\frac{1+x}{2}\right)^{\delta_2}\right]\\&\quad\times H_{P,Q}^{M,N}\left[z\left(\frac{1-x}{2}\right)^{\mu}\left(\frac{1+x}{2}\right)^{\nu}\middle|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right]\mathrm{d}x\\&=2^{a+b+1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}H_{P+2,Q+1}^{M,N+2}\left[z\middle|\begin{matrix}(-a-\delta_1 k,\mu),\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.,\\&\qquad\left.\begin{matrix}(-b-\delta_2 k,\nu),(a_j,\alpha_j)_{1,P}\\(-1-a-b-\delta_1 k-\delta_2 k,\mu+\nu)\end{matrix}\right]c^k,\end{aligned} \tag{8}$$

$$=2^{a+b+1}\sum_{k=0}^{[n/m]}\sum_{h=1}^{M}\sum_{r=0}^{\infty}\left\{\frac{(-n)_{mk}}{k!}\frac{(-1)^r}{r!}f(\lambda)A_{n,k}c^k\frac{z^{\lambda}}{\beta_h}\right\} \tag{9}$$

with

$$\lambda=\frac{b_h+r}{\beta_h},\quad r=0,1,2,\ldots$$

and

$$f(t)=\phi(t)B(1+a+\delta_1 k+\mu t,1+b+\delta_2 k+\nu t), \tag{10}$$

provided that the following conditions are satisfied:

(i) $A>0$, $\delta>0$, $|\arg(z)|<(1/2)A\pi$
where

$$A=\sum_{j=1}^{N}(\alpha_j)-\sum_{j=N+1}^{P}(\alpha_j)+\sum_{j=1}^{M}(\beta_j)-\sum_{j=M+1}^{Q}(\beta_j) \tag{11}$$

and

$$\delta=\sum_{j=1}^{Q}(\beta_j)-\sum_{j=1}^{P}(\alpha_j); \tag{12}$$

(ii) $a,b,\delta_1,\delta_2,\mu,\nu$ are all positive, and

(iii) $\mu \min_{1\leq j\leq M} [\mathrm{Re}(b_j/\beta_j)] + 1 > 0,$

$\nu \min_{1\leq j\leq M} [\mathrm{Re}(b_j/\beta_j)] + 1 > 0.$

The result in (8) is easily established when we replace the H-function by its Mellin-Barnes contour integral from (3), interchange the order of integrations (which is justified due to absolute convergence of the integrals involved in the process), replace $S_n^m[x]$ by its series representation with the help of (1) and then integrate term by term with the help of the result [3], viz.

$$\int_{-1}^{1} (1-x)^{a-1}(1+x)^{b-1}\,\mathrm{d}x = 2^{a+b-1}B(a,b), \mathrm{Re}(a)>0, \mathrm{Re}(b)>0. \tag{13}$$

(See also Theorem 1 of [5] for a multivariable H-function integral analogous to (8) above.)

3. Main integrals

If $\psi(z)$ denotes the logarithmic derivative of the gamma function $\Gamma(z)$, i.e., $\psi(z) = \Gamma'(z)/\Gamma(z)$, then with $\lambda = (b_h + r)/(\beta_h)$, $r = 0, 1, 2, \ldots$ and $f(t)$ given by (10), we have, for $A > 0$, $\delta > 0$, $|\arg z| < 1/2\, A\pi(A$ and δ being given by (11) and (12), $a, b, \delta_1, \delta_2, \mu, \nu > 0$ and

$$\mu \min_{1\leq j\leq M} [\mathrm{Re}(b_j/\beta_j)] + 1 > 0, \quad \nu \min_{1\leq j\leq M} [\mathrm{Re}(b_j/\beta_j)] + 1 > 0$$

and

$$F(x) = (1-x)^a(1+x)^b S_n^m\left[c\left(\frac{1-x}{2}\right)^{\delta_1}\left(\frac{1+x}{2}\right)^{\delta_2}\right] \times H_{P,Q}^{M,N}\left[z\left(\frac{1-x}{2}\right)^{\mu}\left(\frac{1+x}{2}\right)^{\nu}\right], \tag{14}$$

$$\int_{-1}^{1} F(x)\log(1-x)\,\mathrm{d}x = 2^{b+1}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}}{k!}A_{n,k}c^k\frac{\partial}{\partial a}\left\{2^a H_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}(-a-\delta_1 k,\mu),(-b-\delta_2 k,\nu),(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q},(-1-a-b-\delta_1 k-\delta_2 k,\mu+\nu)\end{matrix}\right.\right]\right\}\right],$$

$$= 2^{a+b+1}\sum_{k=0}^{[n/m]}\sum_{h=1}^{M}\sum_{r=0}^{\infty}\left[\frac{(-n)_{mk}}{k!}\frac{(-1)^r f(\lambda)}{r!\ \beta_h}A_{n,k}\{\log 2 + \psi(1+a+\delta_1 k+\mu\lambda) - \psi(2+a+b+(\delta_1+\delta_2)k+(\mu+\nu)\lambda)\}c^k z^\lambda\right]; \tag{15}$$

$$\int_{-1}^{1} F(x)\log(1+x)\,\mathrm{d}x = 2^{a+1}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}}{k!}A_{n,k}c^k\frac{\partial}{\partial b}\left\{2^b H_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}(-a-\delta_1 k,\mu),(-b-\delta_2 k,\nu),(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q},(-1-a-b-\delta_1 k-\delta_2 k,\mu+\nu)\end{matrix}\right.\right]\right\}\right]$$

$$= 2^{a+b+1} \sum_{k=0}^{[n/m]} \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left[\frac{(-n)_{mk}}{k!} \frac{(-1)^r f(\lambda)}{r!\ \beta_h} A_{n,k} \{\log 2 + \psi(1 + b + \delta_2 k + \nu\lambda) - \psi(a + b + 2 + (\delta_1 + \delta_2)k + (\mu + \nu)\lambda)\} c^k z^\lambda \right]; \tag{16}$$

$$\int_{-1}^{1} F(x) \log(1 - x^2) dx = 2^{a+b+1} \sum_{k=0}^{[n/m]} \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left[\frac{(-n)_{mk}}{k!} \frac{(-1)^r f(\lambda)}{r!\ \beta_h} A_{n,k} \times \{\log 4 + \psi(1 + a + \delta_1 k + \mu\lambda) + \psi(1 + b + \delta_2 k + \nu\lambda) - 2\psi(a + b + 2 + (\delta_1 + \delta_2)k + (\mu + \nu)\lambda\} c^k z^\lambda \right]; \tag{17}$$

$$\int_{-1}^{1} F(x) \log\left(\frac{1-x}{1+x}\right) dx = 2^{a+b+1} \sum_{k=0}^{[n/m]} \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left[\frac{(-n)_{mk}}{k!} \frac{(-1)^r f(\lambda)}{r!\ \beta_h} A_{n,k} \times \{\psi(1 + a + \delta_1 k + \mu\lambda) - \psi(1 + b + \delta_2 k + \nu\lambda)\} c^k z^\lambda \right]. \tag{18}$$

Outline of Proof: The results in (15) and (16) are established by taking the partial derivatives of both the sides of (9) with respect to a and b respectively. The integrals in (17) and (18) are obtained by first adding (15) and (16) and then subtracting them.

4. Particular cases

1. If, in (17) we take

$$A_{n,k} = \frac{(\gamma + n)_k \prod_{j=1}^{p} (\varepsilon_j)_k}{\prod_{j=1}^{q} (\eta_j)_k} = D_{n,k}, \text{ say} \tag{19}$$

and $m = 1$, it reduces to the following integral involving extended Jacobi polynomials [5]:

With $$G(x) = (1 - x)^a (1 + x)^b H_{P,Q}^{M,N}\left[z \left(\frac{1-x}{2}\right)^\mu \left(\frac{1+x}{2}\right)^\nu \right], \tag{20}$$

$$\int_{-1}^{1} G(x) \log(1 - x^2)\, {}_{p+2}F_q\left[\begin{matrix} -n, \gamma + n, (\varepsilon_p) \\ (\eta_q) \end{matrix}; c\left(\frac{1-x}{2}\right)^{\delta_1} \left(\frac{1+x}{2}\right)^{\delta_2} \right] dx$$

$$= 2^{a+b+1} \sum_{k=0}^{n} \frac{(-n)_k}{k!} D_{n,k} c^k E_1(M, a, b, k), \tag{21}$$

where

$$E_1(M, a, b, k) = \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left\{ \frac{(-1)^r f(\lambda)}{r!\ \beta_h} L_1(a, b, k, r) z^\lambda \right\} \tag{22}$$

with

$$L_1(a, b, k, r) = \{\log 4 + \psi(1 + a + \delta_1 k + \mu\lambda) + \psi(1 + b + \delta_2 k + \nu\lambda) - 2\psi(a + b + 2 + (\delta_1 + \delta_2)k + (\mu + \nu)\lambda\},$$

$\lambda=(b_h+r)/\beta_h$, $r=0,1,2,\ldots$ and $f(\lambda)$ is given by (10); provided that the conditions (i), (ii) and (iii) given with the integral (8) are satisfied.

2. If, in (18), we take

$$A_{n,k}=\frac{\prod_{j=1}^{l-1}\left(\frac{\gamma+n+j}{l}\right)_k\prod_{j=1}^{p}(\varepsilon_j)_k}{\prod_{j=1}^{q}(\eta_j)_k}=B_{n,k},\quad \text{say} \tag{23}$$

and $c=m^{-m}$ so that $S_n^m[x]$ reduces to the generalized extended Jacobi polynomial [5].

$${}_{l+m+p}F_q\left[\begin{matrix}\Delta(m;-n),\Delta(l;\gamma+n),\ (\varepsilon_p)\\ (\eta_q)\end{matrix};\ x\right],$$

then (18) gives the following integral:

With $G(x)$ given by (20), we have

$$\int_{-1}^{1}G(x)\log\left(\frac{1-x}{1+x}\right){}_{l+m+p}F_q\left[\begin{matrix}\Delta(m;-n),\Delta(l;\gamma+n),\ (\varepsilon_p)\\ (\eta_q)\end{matrix};\right.$$
$$\left.\times\left(\frac{1-x}{2}\right)^{\delta_1}\left(\frac{1+x}{2}\right)^{\delta_2}\right]\mathrm{d}x$$
$$=2^{a+b+1}(n)!\sum_{k=0}^{[n/m]}\frac{(-1/m)^{mk}}{k!(n-mk)!}B_{n,k}E_2(M,a,b,k) \tag{24}$$

where

$$E_2(M,a,b,k)=\sum_{h=1}^{M}\sum_{r=0}^{\infty}\left\{\frac{(-1)^r f(\lambda)}{r!\quad\beta_h}L_2(a,b,k,r)z^{\lambda}\right\} \tag{25}$$

with $L_2(a,b,k,r)=\psi(1+a+\delta_1k+\mu\lambda)-\psi(1+b+\delta_2k+\nu\lambda)$, $\lambda=b_h+r/\beta_h$, $r=0,1,2\ldots$ and $f(\lambda)$ is given by (10); provided that the conditions (i), (ii) and (iii) given with the integral (8) are satisfied.

A number of other particular cases can be obtained from the main integrals but these are not recorded here for lack of space.

Acknowledgements

The authors are grateful to the referee for his valuable suggestions.

References

[1] Kalla S L, Conde S and Luke Y L, Integrals of Jacobi functions, *Math. Comp.* **38** (1982) 207–214
[2] Kalla S L, Integrals of generalized Jacobi functions, *Proc. Nat. Acad. Sci. India* **A58** (1988) 123–128
[3] Rainville E D, *Special Functions* (New York: Macmillan) (1963) 31
[4] Srivastava H M, A contour integral involving Fox's *H*-function, *Indian J. Math.* **14** (1972) 1–6
[5] Srivastava H M and Singh N P, The integration of certain products of the multivariable *H*-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* (2) **32** (1983) 157–187
[6] Srivastava H M, Gupta K C and Goyal S P, The *H*-Functions of One and Two Variables with Applications (New Delhi: South Asian Publishers) (1982), 10, 12