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Linear flow induced in fluid particle suspension by an infinite differentially rotating disk

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Abstract. The steady, axisymmetric laminar flow of a homogeneous incompressible fluid with suspended particles occupying the half-infinite space over a differentially rotating rigid plane boundary is analyzed in this paper. The effect of suspended particles is described by two parameters f and τ. The mass concentration parameter f is a measure of the concentration of suspended dust particles. The interaction parameter τ is a measure of the rate at which the velocity of dust particles adjusts to changes in the fluid velocity and depends upon the size of the individual particles. Due to Ekman suction, the particle density remains no longer a constant in the boundary layer but varies with the axial coordinate ξ. Flow characteristics and density variations are studied as functions of f, τ and ξ. Possible limiting cases for $\tau \ll 1$ and $\tau \gg 1$ which correspond to the case of fine dust and coarse dust respectively are derived and discussed.

Keywords. Fluid particle suspension; rotation; linear Ekman layer.

1. Introduction

Saffman [7] who initiated the work on dusty fluids also studied the effect of dust particles on the stability of laminar flow of a dusty gas. Due to academic interest and possible applications in atmospheric, engineering and physiological fields, several authors (Michael and Miller [5], Marble [4], Datta and Mishra [1]) have studied the dynamics of dusty fluids. Zung [8] examined the flow induced in fluid-particle suspension by an infinite rotating disk. Assuming similarity solutions he obtained numerical solutions for the nonlinear problem and presented the velocity and density fields as functions of the axial distance η and the parameters of the problem. Gupta [3] studied the linear Ekman layer over a rigid rotating boundary, when the fluid motion is driven by the movement of the boundary with a uniform velocity parallel to itself. In that problem the vertical velocities of the fluid and dust particles are zero throughout and the density of the particles is a constant.

In the present paper we discuss the linear flow induced in fluid-particle suspension by an infinite differentially rotating disk. The present work differs from that of Zung [8] because this is a linear problem while that of Zung [8] is a nonlinear problem. Our

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A list of symbols appears at the end of the paper.

work also differs from that of Gupta [3] for, in our problem, the axial velocities of the fluid and dust particles do not vanish, and the density of the dust particles is not a constant due to differential rotation of the disk. Due to Ekman suction the number density and hence the density of the dust particles is not a constant in the boundary layer region but is a function of the axial coordinate.

As in reference [7], in the present analysis also the effect of dust is described by two parameters f and τ which are respectively the measures of concentration of dust and the rate at which the velocity of a dust particle adjusts to changes in the fluid velocity. Solutions presented as functions of f and τ are also valid for the limiting case of $\tau \ll 1$ (fine dust) and $\tau \gg 1$ (coarse dust). Some of the qualitatively interesting results of the present analysis are (1) the boundary layer thicknesses for fluid and particle cloud are approximately equal. (2) In the case of fine dust, i.e., when $\tau \ll 1$, the velocities of the dust particles coincide with those of the fluid wherein the fluid density ρ is replaced by $\rho(1+f)$. This result is in agreement with Saffman's observation that for fine dust the particles move along streamlines with the velocity of the fluid. In this case, the density of the dusty cloud, as may be expected, is a constant. (3) In the case of coarse dust, i.e. when $\tau \gg 1$,

$$u_p = 0 + O\left(\frac{1}{\tau}\right),$$

$$v_p = 0 + O\left(\frac{1}{\tau}\right),$$

and

$$\rho_p w_p = \text{constant},$$

where ρ_p is the density and u_p, v_p, w_p are the radial, azimuthal and axial components of velocity of the dust particles. (4) The net mass flux of the fluid is perpendicular to the direction of the stress vector when $\tau \to 0$ and when $\tau \to \infty$ while it is not so for other values of τ.

2. Formulation and solution

An incompressible fluid containing small particles of a single size of radius a is occupying the half-infinite space, $z > 0$. The boundary plane at $z = 0$ and the fluid above it are rotating at a constant angular velocity Ω_0. The bounding plane is then given an extra angular velocity $\Delta\Omega$.

Introducing the non-dimensional variables

$$(u, v, w) = (u^*, v^*, w^*)/(\varepsilon\Omega_0 L),$$

$$(u_p, v_p, w_p) = (u_p^*, v_p^*, w_p^*)/(\varepsilon\Omega_0 L),$$

$$(r, z) = (r^*, z^*)/L,$$

and

$$p = p^*/(\rho\varepsilon\Omega_0^2 L^2),$$

using a cylindrical coordinate system rotating about the z-axis and taking account of rotational symmetry, the non-dimensional governing equations may be written as (refer to [7] and [8])

$$-2v = \frac{-\partial p}{\partial r} + E\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right] + \frac{f}{\tau}(u_p - u), \tag{1}$$

$$2u = E\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right] + \frac{f}{\tau}(v_p - v), \tag{2}$$

$$0 = \frac{-\partial p}{\partial z} + E\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{f}{\tau}(w_p - w), \tag{3}$$

$$-2v_p = \frac{1}{\tau}(u - u_p), \tag{4}$$

$$2u_p = \frac{1}{\tau}(v - v_p), \tag{5}$$

$$0 = \frac{1}{\tau}(w - w_p), \tag{6}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{7}$$

$$\frac{\partial}{\partial r}(r\rho_p u_p) + \frac{\partial}{\partial z}(r\rho_p w_p) = 0. \tag{8}$$

Here u, v, w are components of velocity in the radial, azimuthal and axial directions and p is the reduced pressure. The non-dimensional parameters occurring in the equations are

$E = \dfrac{\nu}{\Omega_0 L^2}$, the Ekman number;

$f = \dfrac{mN_0}{\rho}$, the mass concentration parameter;

$\tau = \dfrac{m\Omega_0}{K}$, the interaction parameter,

where m and N_0 are the mass of single dust particle and the number density of the dust, and ρ, μ, ν are the density, viscosity and kinematic viscosity of the fluid respectively. Further, $K = 6\pi a\mu$ for particles of radius a by Stokes' Drag formula. The subscript p denotes quantities associated with dust particles (here we are following a notation similar to that in [7]. The parameters f and τ correspond to the parameters k and $1/\beta$ of [8]).

The boundary conditions relevant to the problem are

$$u = 0, \quad v = r, w = 0 \quad \text{for } z = 0, \tag{9}$$

and

$$u = 0, \quad v = 0, \quad p = p_\infty,$$
$$u_p = 0, \quad v_p = 0, \quad w_p = w, \quad \rho_p = f\rho \quad \text{for } z = \infty. \tag{10}$$

Introducing the stretched coordinate $\xi(E^{-1/2}z)$ and writing any field variable F as

$$F = F^{(i)} + \bar{F},$$

where superscript i denotes the interior component and bar the boundary layer correction. The governing equations for the Ekman layer region are as follows:

$$-2\bar{v} = \frac{\partial^2 \bar{u}}{\partial \xi^2} + \frac{f}{\tau}(\bar{u}_p - \bar{u}), \tag{11}$$

$$2\bar{u} = \frac{\partial^2 \bar{v}}{\partial \xi^2} + \frac{f}{\tau}(\bar{v}_p - \bar{v}), \tag{12}$$

$$-2\bar{v}_p = \frac{1}{\tau}(\bar{u} - \bar{u}_p), \tag{13}$$

$$2\bar{u}_p = \frac{1}{\tau}(\bar{v} - \bar{v}_p), \tag{14}$$

$$0 = \frac{1}{\tau}(\bar{w} - \bar{w}_p), \tag{15}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + E^{-1/2}\frac{\partial}{\partial \xi}\bar{w} = 0, \tag{16}$$

$$\frac{\rho_p}{r}\frac{\partial}{\partial r}(r\bar{u}_p) + E^{-1/2}\frac{\partial}{\partial \xi}(\rho_p \bar{w}_p) = 0. \tag{17}$$

The pressure field is continuous and so the boundary layer correction for p to the lowest order is zero. Hence the pressure p does not appear in the equations.

Equations (11)–(16) are solved subject to relevant boundary conditions and the solutions to the lowest order are given below:

$$\bar{u} = r\exp(-T_1\xi)\cdot\sin T_2\xi, \tag{18}$$

$$\bar{v} = r\exp(-T_1\xi)\cdot\cos T_2\xi, \tag{19}$$

$$\bar{w} = \frac{2E^{1/2}}{T_1^2 + T_2^2}\exp(-T_1\xi)\cdot[T_1\sin T_2\xi + T_2\cdot\cos T_2\xi], \tag{20}$$

$$\bar{u}_p = \frac{1}{1+4\tau^2}(\bar{u} + 2\tau\bar{v}), \tag{21}$$

$$\bar{v}_p = \frac{1}{1+4\tau^2}(\bar{v} - 2\tau\bar{u}), \tag{22}$$

$$\bar{w}_p = \bar{w}, \tag{23}$$

where

$$\alpha_1 = \frac{4\tau f}{1+4\tau^2},$$

$$\alpha_2 = 2\left(1 + \frac{f}{1+4\tau^2}\right),$$

$$T_1 = \{1/2[(\alpha_1^2 + \alpha_2^2)^{1/2} + \alpha_1]\}^{1/2},$$

and

$$T_2 = \{1/2[(\alpha_1^2 + \alpha_2^2)^{1/2} - \alpha_1]\}^{1/2}. \tag{24}$$

Since

$$w = w^{(i)} + \bar{w} = 0 \text{ at } \xi = 0,$$

the interior component of axial velocity is

$$w^{(i)} = -2E^{1/2}\frac{T_2}{T_1^2 + T_2^2}. \tag{25}$$

We solve (8) for the density of the dust particles to get

$$\bar{\rho}_p(\xi) = \frac{\rho_p}{f\rho} = \exp\left[\int_{\xi}^{\xi_\infty} \frac{E^{1/2}}{\bar{w}_p + w^{(i)}}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}_p) + E^{-1/2}\frac{\partial}{\partial \xi}(\bar{w}_p)\right\}\mathrm{d}\xi\right]. \tag{26}$$

In view of the relations

$$w = \bar{w} + w^{(i)} = 0 \text{ at } \xi = 0$$

and

$$\bar{w}_p = \bar{w},$$

it seems that $\bar{w}_p + w^{(i)}$ will be zero at $\xi = 0$ and hence $\xi = 0$ is a singular point of the integral in (26). However it may be noted that the equation $\bar{w}_p = \bar{w}$ is true only to the lowest order. That is, $\bar{w}_p + w^{(i)} = 0 +$ higher order terms at $\xi = 0$, where the higher order term results due to buoyancy force acting on the dust particles. In fact, the vertical velocity of the dust particles need not vanish at the boundary (see Zung [8]) and care has been taken to reflect this feature in our work.

Using the known expressions for the velocity components, the integral on the right side of relation (26) is evaluated numerically using Simpson's $-1/3$ rule to get the values of the density $\bar{\rho}_p$ at different values of ξ. We have also employed the Runge-Kutta fourth order method to integrate (8) numerically for $\bar{\rho}_p$ taking the initial condition as $\bar{\rho}_p(\xi = \xi_\infty) = 1{\cdot}0$, where ξ_∞ is taken as 10. The numerical results are found to be in good agreement with those obtained by using Simpson's rule.

When $\tau \ll 1$, from (13)–(15) we get to the lowest order

$$\bar{u}_p = \bar{u}, \quad \bar{v}_p = \bar{v}, \quad \bar{w}_p = \bar{w},$$

and from (16) and (17) it follows

$$\bar{\rho}_p = \frac{\rho_p}{f\rho} = 1,$$

thereby implying that the dust particles move along with the fluid particles. Taking the limit as $\tau \to 0$, (18)–(20) give the well-known Ekman layer solutions (Refer to pages 30–31 of Greenspan [2]) except that ρ is replaced by $\rho(1+f)$.

When $\tau \gg 1$, we rewrite (11) and (12) as (also see reference [7])

$$-2\bar{v} = \frac{\partial^2 \bar{u}}{\partial \xi^2} - s\bar{u}, \tag{27}$$

$$-2\bar{u} = \frac{\partial^2 \bar{v}}{\partial \xi^2} - s\bar{v}, \tag{28}$$

where $s = f/\tau$. Solving (27), (28) and (13)–(17) subject to appropriate boundary conditions, we get the same solutions (18)–(23) but with T_1 and T_2 defined as

$$T_1 = \left\{\frac{\sqrt{s^2+4}+s}{2}\right\}^{1/2},$$

$$T_2 = \left\{\frac{\sqrt{s^2+4}-s}{2}\right\}^{1/2}.$$

These expressions for T_1 and T_2 are essentially the same as those that can be obtained from (24) in the limit of large τ, thereby establishing the fact that the effect of dust is equivalent to an extra frictional force proportional to the velocity, as pointed out in [7]. The physical explanation is that coarse dust does not move with the fluid when the flow is perturbed but carries on with the velocity of the basic flow. The disturbance has therefore to flow around the particles. Further, from the solutions (21)–(23) and equation (17) we observe $\bar{u}_p$ and $\bar{v}_p$ to be of order $1/\tau$ and the product $\rho_p \bar{w}_p$ to be a constant. These observations are in complete agreement with the numerical computations of u_p, v_p, w_p and ρ_p for large values of τ.

3. Discussion of the results

The velocity profiles of the dust particles are presented in figures 1, 2 and 3. The radial velocity (u_p) and azimuthal velocity (v_p) of the particles do not vanish on the disk. These velocities tend to zero outside the boundary layer. The axial velocity of the particles (w_p) also takes non-zero values on the disk and approach a constant value outside the boundary layer. For a fixed value of τ as f takes increasing values the magnitudes of the velocity components are found to diminish. This fact can be explained as follows. When τ is fixed and f increases, size of the particles is fixed but the number of particles N_0 increases. Then the drag force, $\mathbf{K}N_0(\mathbf{U}-\mathbf{V})$ (see Saffman [7]) increases

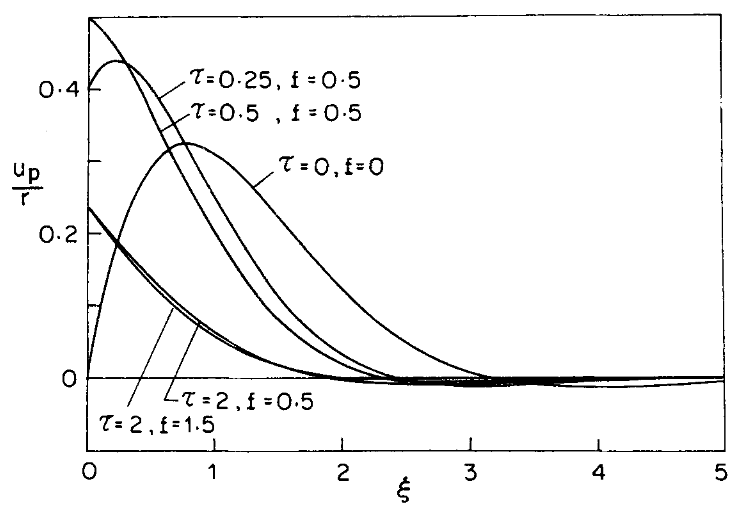

Figure 1. Dust particle velocity in radial direction, u_p/r.

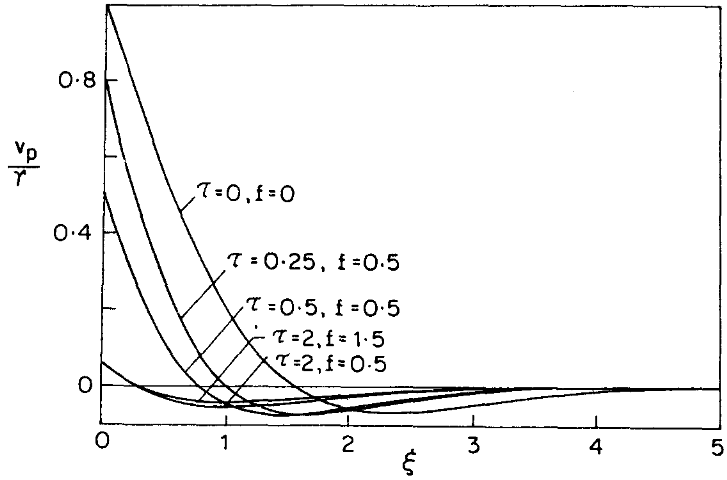

Figure 2. Dust particle velocity in azimuthal direction, v_p/r (curves 1 to 5 as in figure 1).

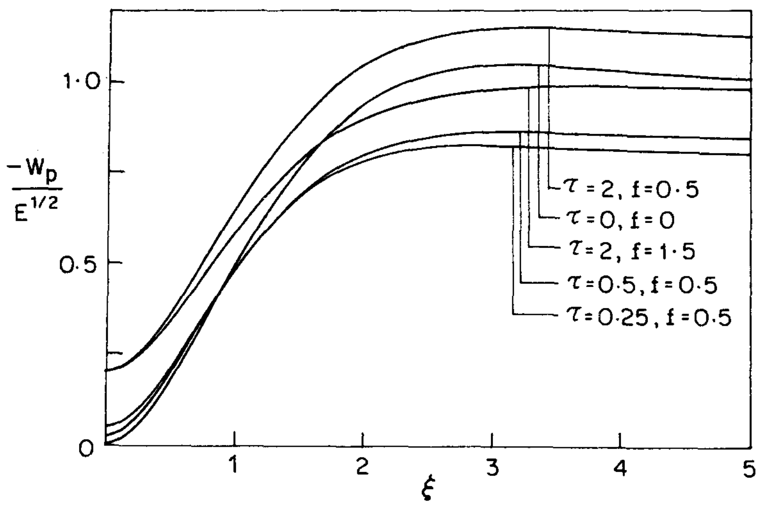

Figure 3. Dust particle velocity in axial direction, $w_p/E^{1/2}$ (curves 1 to 5 as in figure 1).

and hence the velocity decreases. Further, as f increases, the density of the suspension (fluid + dust) increases. Hence $\mu/(\rho\Omega_0 L^2)$ decreases, i.e. the Ekman layer thickness decreases. Hence w_p decreases with f. The boundary layer thickness goes like $1/(1+f)^{1/2}$.

Using (18) and (19) in (21) we get

$$\frac{\bar{u}_p}{r}=\frac{2\tau}{1+4\tau^2} \quad \text{at } \xi=0.$$

Hence we conclude that at $\xi=0$, u_p tends to zero as $\tau\to 0$ and attains its maximum value at $\tau=0{\cdot}5$. In fact, as $\tau\to 0$ we have $u_p=u$ as mentioned earlier; the radial flow fields for the fluid and dust particles as well have to increase from zero to a maximum value and then decrease to zero again (see figure 1) in view of the boundary conditions. This behaviour for u_p cannot abruptly change as τ starts taking non-zero values and persists up to $\tau = 0{\cdot}5$. However the increasing portion of u_p gradually decreases as τ increases up to 0·5. This may be attributed to the fact that u_p takes increasing values at the plate with τ for $\tau < 0{\cdot}5$. For values of $\tau > 0{\cdot}5$, the particle size has increased beyond the critical value and so u_p gradually decreases at $\xi = 0$. However it will still have its maximum value at $\xi = 0$ due to centrifugal force. We may note from (4)–(6) that as $\tau \to 0$, u_p, v_p, w_p become equal to u, v, w respectively since the dust particles move with fluid. So the shapes of the curves for u_p, v_p, w_p when $\tau \to 0$ are the same as those of u, v, w of the conventional problem. The curves of particle velocities closely resemble those of the nonlinear dusty fluid problem analyzed by Zung [8] except for the fact that in the nonlinear problem u_p, v_p do not have oscillatory nature.

The plots of the particle density $\bar{\rho}_p = \rho_p/f\rho$ and the divergence of the velocity of the particles are presented in figures 4 and 5 respectively. The density $\bar{\rho}_p$ takes different values on the disk but tend to a constant value, unity, far away from the disk. The magnitude of $\bar{\rho}_p$ near the disk increases with increasing values of τ but decreases with increasing values of f. As already pointed out, for a fixed value of f, as τ increases, particle size increases and as a result, even at $\xi = 0$ the radial velocities of the particles decrease while the vertical velocities increase. Hence we expect the particle density to increase at $\xi = 0$ with τ. Further the continuity equation for the particles (8) may be rewritten as

$$\nabla\cdot\mathbf{V} = \frac{-w_p}{E^{1/2}}\frac{1}{\bar{\rho}_p}\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{\rho}_p.$$

So, the sign of $\nabla\cdot\mathbf{V}$ (divergence of the velocity of the particles) depends upon the sign of $(\mathrm{d}/\mathrm{d}\xi)\bar{\rho}_p$. From figure 5 we may note that the divergence is changing its sign. Hence we can expect the density to increase and decrease. Since the divergence is positive initially up to a certain distance from the disk, particle density should increase with

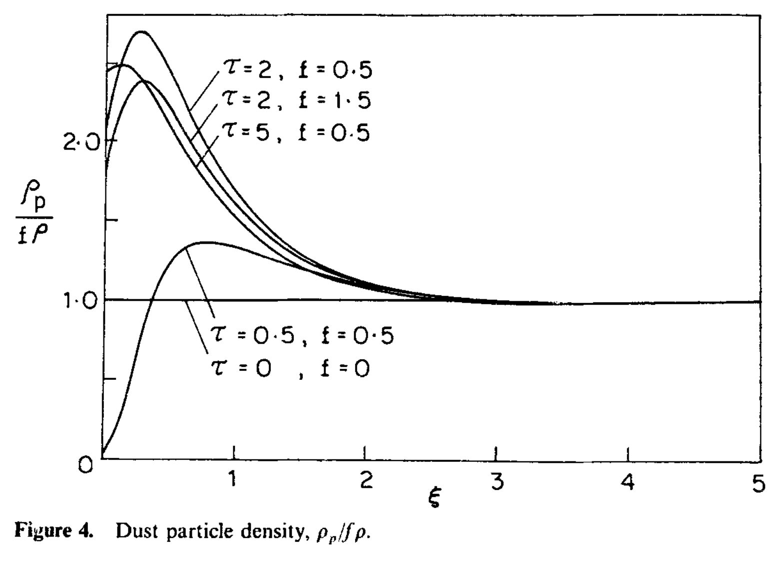

Figure 4. Dust particle density, $\rho_p/f\rho$.

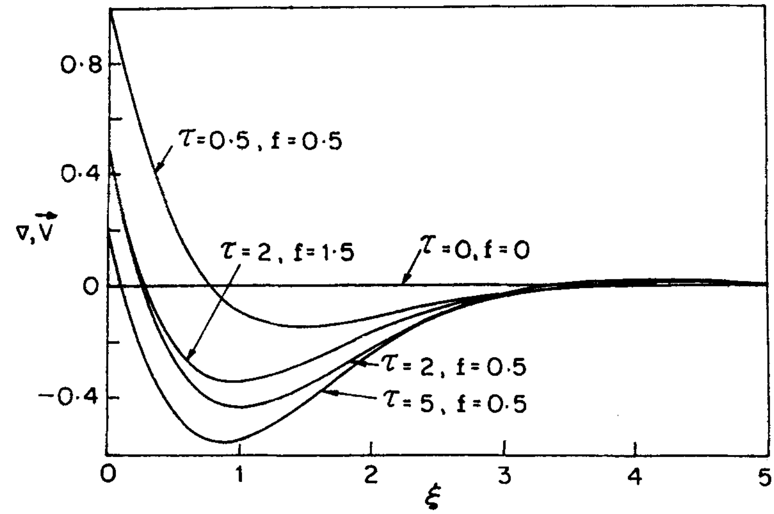

Figure 5. Divergence of dust particle velocity, ∇·V (curves 1 to 5 as in figure 4).

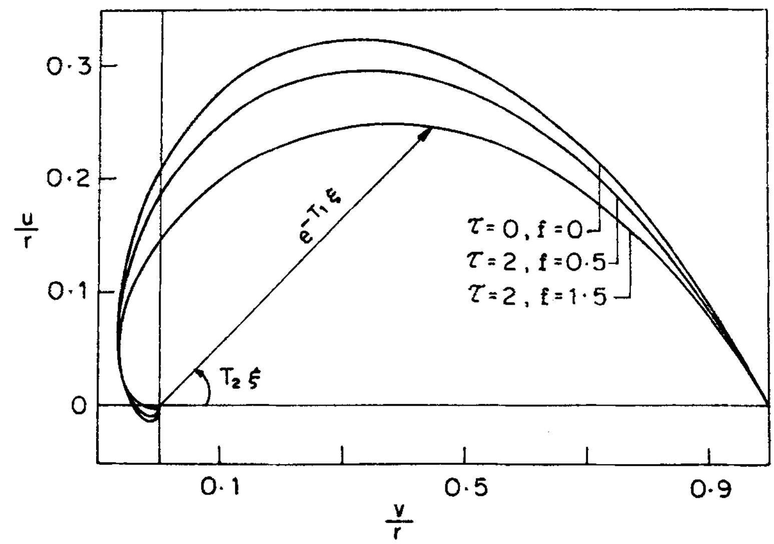

Figure 6. Ekman spiral for fluid particles.

ξ. Later, since the divergence becomes negative, $\bar{\rho}_p$ should start decreasing with ξ. Because of this behaviour $\bar{\rho}_p$ can attain values greater than unity in the boundary layer!

The velocity vectors of the fluid and dust particles at different heights above the rigid boundary are shown in figures 6 and 7. As ξ increases the velocity vector rotates uniformly in the anticlockwise direction and the magnitude falls off exponentially. The general features of the spiral distribution of dust particle velocity near the boundary (figure 7) are much the same as for the fluid velocity (figure 6). The velocity vector for the dust particles at the boundary makes an angle of $\cos^{-1}[1/(1+4\tau^2)^{1/2}]$ with the velocity vector for the fluid and this becomes zero when τ approaches zero.

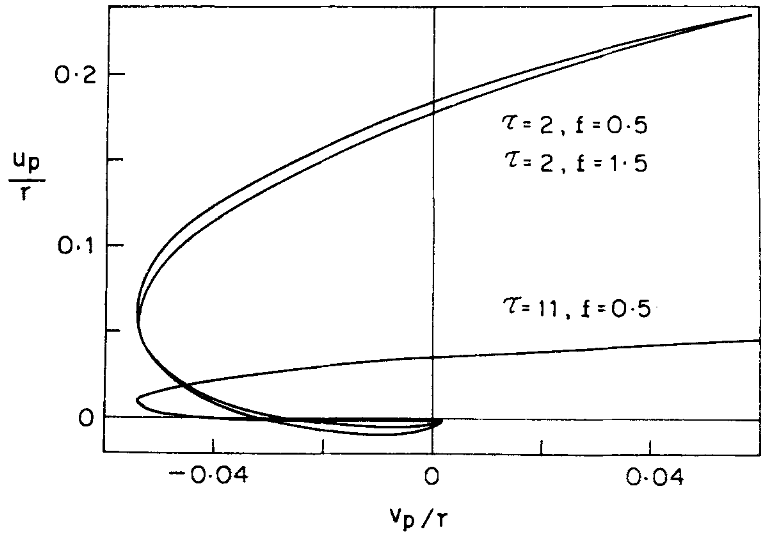

Figure 7. Ekman spiral for dust particles.

4. Net mass flux

The net mass flux of the fluid can be obtained by integrating the velocity vector from $z = 0$ to $z = \delta$ (δ is the boundary layer thickness) as

$$\mathbf{M} = \int_0^{\delta} \mathbf{U}\,\mathrm{d}z$$

$$= E^{1/2}\int_0^{\infty} \mathbf{U}\,\mathrm{d}\xi$$

$$= \frac{rE^{1/2}}{T_1^2 + T_2^2}(T_2\hat{r} + T_1\hat{\theta}). \tag{29}$$

The stress vector at the disk can be obtained to be

$$\mathbf{T} = rE^{1/2}(-T_2\hat{r} + T_1\hat{\theta}). \tag{30}$$

From (29) and (30) it follows that

$$\mathbf{M}\cdot\mathbf{T} = \frac{r^2}{T_1^2 + T_2^2}(-T_2^2 + T_1^2). \tag{31}$$

So, in the presence of dust the net mass flux is not perpendicular to the direction of the stress vector. However, for fine dust, in the limit as $\tau \to 0$, we get from (31)

$$\mathbf{M}\cdot\mathbf{T} = 0,$$

implying that the net mass flux is perpendicular and to the right of the stress vector. This is in agreement with the well established fact that for a conventional dust free fluid the total mass flow in excess of the prescribed geostrophic flow is perpendicular and to the right of the frictional stress on the fluid (refer to page 182 of Pedlosky [6]). For coarse dust, when $\tau \gg 1$, from (31) we have

$$\mathbf{M}\cdot\mathbf{T} = \frac{r^2 s}{(s^2+4)^{1/2}},$$

where

$$s = \frac{f}{\tau}.$$

5. Turning moment

The results obtained and the calculations done are applicable to an infinite disk only. However, we may utilize the results for a finite disk provided that its radius R is large compared to the boundary layer thickness. The turning moment for a disk of radius R is

$$M_t = -2\pi \int_0^R r^2 T_{z\theta}\, \mathrm{d}r,$$

where the radial distance r is in dimensional form and the circumferential component of the shearing stress $T_{z\theta}$ is

$$T_{z\theta} = \mu\left(\frac{\partial v}{\partial z}\right)_0 = \varepsilon \rho r v^{1/2} \Omega_0^{3/2} \cdot T_1.$$

Here T_1 is as defined in (24). The coefficient of turning moment (i.e. the dimensionless moment coefficient) is

$$C_M = \frac{M_t}{\frac{1}{2}\rho\Omega_0^2 R^5} = \frac{\varepsilon\pi T_1}{\sqrt{Re}}.$$

Here Re $(= R^2\Omega_0/\nu)$ is the Reynolds number based on the radius and tip velocity.

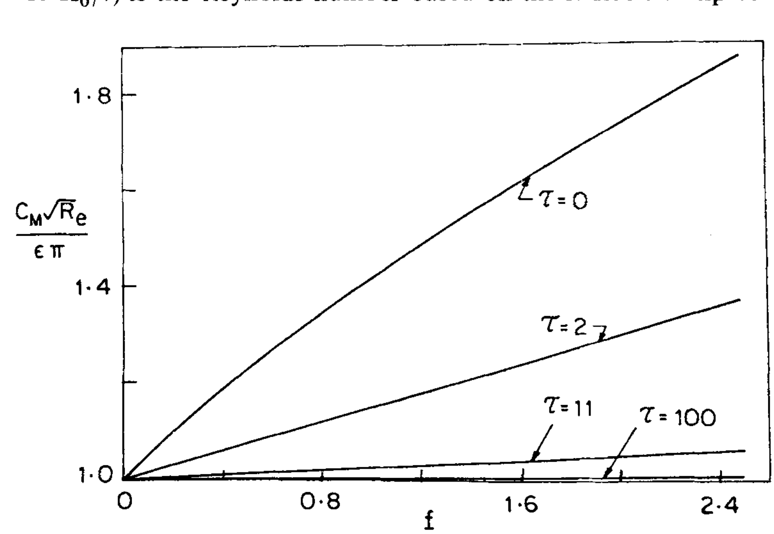

Figure 8. Coefficient of turning moment, $C_M\sqrt{Re}/\varepsilon\pi$.

Figure 8 shows the plots of the coefficient of turning moment, $C_M\sqrt{Re}/\varepsilon\pi$. C_M increases with increasing values of f and this is in agreement with the fact that an increase in f causes a decrease in boundary layer thickness. C_M decreases with increasing values of τ.

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Symbols

C_M	:	coefficient of turning moment
E	:	Ekman number
$\mathbf{M}$	:	mass flux
M_t	:	turning moment
N_0	:	number density of dust particles
$\mathbf{U}=(u, v, w)$	:	non-dimensional velocity of fluid
$\mathbf{V}=(u_p, v_p, w_p)$	:	non-dimensional velocity of the dust particles
Z	:	axial coordinate
f	:	mass concentration parameter
p	:	reduced pressure
r	:	radial coordinate
ε	:	Rossby number
τ	:	interaction parameter
Ω_0	:	angular velocity
ρ_p	:	density of dust particles in the suspension.

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