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On the ratio of the maximum term and the maximum modulus of the sum of two entire functions

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Abstract. Some results on the sum of two entire functions pertaining to the ratio of the maximum term and the maximum modulus are proved.

Keywords. Entire functions; maximum term; maximum modulus; rank.

1. Introduction

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function. As usual, let $M(r,f)$, $\mu(r,f)$, and $v(r,f)$ denote the maximum modulus, the maximum term, and the rank of the maximum term of $f(z)$ respectively. Let $M_k(0 \leqslant k \leqslant 1)$ denote the class of all entire functions for which

$$\limsup_{r\to\infty} \frac{\mu(r,f)}{M(r,f)} = k.$$

The set of all entire functions of finite order is closed under addition. Now the question is, given any two transcendental entire functions belonging to M_k, does the sum, if it is transcendental also belong to M_k? (Polynomials are excluded, since polynomials are in M_1).

We shall answer this question in the negative. Actually, we shall show that, given a function of a class (even M_1), there is always another function of the same class such that their sum is not in the class. If the polynomials are also included, then the answer to the above question is immediate when $k \neq 1$. We shall also show that the sum of an entire function and its derivative, and the function are in the same class for certain entire functions, when r is allowed to tend to infinity over a set of infinite measure.

A result in this direction about the sum of two entire functions can be found in (Theorem 4, [2]).

2. Results and Proofs

We shall state the results mentioned in § 1 in precise terms, and prove them.

Theorem 1. *Given an entire function $f_1(z) \in M_k (0 \leqslant k < 1)$ there exists another entire function $f_2(z) \in M_k$ such that $f_1(z) + f_2(z)$ is a transcendental entire function belonging to M_1.*

Theorem 2. *Given a transcendental entire function $f_1(z) \in M_1$ there exists another entire function $f_2(z) \in M_1$ such that $f_1(z) + f_2(z) \in M_k$, $k \neq 1$.*

We shall prove something more than merely that $f(z)$ and $f(z) + f^{(1)}(z)$ belong to the same class for certain entire functions, where in the limit superior, $r \to \infty$ over a set of infinite measure.

Theorem 3. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function for which*

$$\liminf_{r \to \infty} \left| \frac{a_n^2}{a_{n-1} a_{n+1}} \right| > 1,$$

then

$$\frac{\mu(r, f)}{M(r, f)} \sim \frac{\mu(r, f + f^{(1)})}{M(r, f + f^{(1)})} \tag{1}$$

as $r \to \infty$ over a set of infinite measure.

Proof of Theorem 1. Let

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Let

$$b_n = \begin{cases} [|a_n|^{-1}] + 1 & \text{if } a_n \neq 0 \\ 2 & \text{if } a_n = 0, \end{cases}$$

and

$$c_n = b_0 b_1 \ldots b_n,$$

where $[|a_n|^{-1}]$ denotes the integral part of $|a_n|^{-1}$. Define

$$\tilde{f}_2(z) = \sum_{n=0}^{\infty} \frac{z^n}{c_0! c_1! \ldots c_n!}.$$

For all r satisfying the inequalities $c_n! \leqslant r < c_{n+1}!$, we have

$$M(r, \tilde{f}_2) = \sum_{i=0}^{\infty} \frac{r^i}{c_0! c_1! \ldots c_i!} < \mu(r, \tilde{f}_2) \left(\ldots + \frac{c_{n-1}!}{c_{n+1}!} + 3 + \frac{c_{n+1}!}{c_{n+2}!} + \ldots \right)$$
$$\leqslant \mu(r, \tilde{f}_2)(3 + B(n)), \tag{2}$$

where $B(r) \to 0$ as $r \to \infty$. And for $r = c_n!(c_{n+1}!/c_n!)^{1/2}$, we have

$$M(r, \tilde{f}_2) \leqslant \mu(r, \tilde{f}_2)(1 + c(r)), \tag{3}$$

where $c(r)\to 0$ as $r\to\infty$. From (3), it follows that $\tilde{f}_2(z)\in M_1$, and from (2), we have

$$\liminf_{r\to\infty}\frac{\mu(r,\tilde{f}_2)}{M(r,\tilde{f}_2)}\geqslant\frac{1}{3}.$$

Again for all r satisfying the inequalities $c_n!\leqslant r<c_{n+1}!$, we have

$$\frac{\mu(r,f_1)}{\mu(r,\tilde{f}_2)}=\frac{|a_n|r^n}{(c_0!c_1!\dots c_n!)^{-1}r^n}$$
$$>\frac{c_0!c_1!\dots c_n!}{c_n}.$$

This inequality yields

$$\liminf_{r\to\infty}\frac{\mu(r,f_1)}{\mu(r,\tilde{f}_2)}=\infty.$$

So

$$\liminf_{r\to\infty}\frac{\mu(r,f_1)}{M(r,\tilde{f}_2)}\geqslant\liminf_{r\to\infty}\frac{\mu(r,f_1)}{\mu(r,\tilde{f}_2)}\liminf_{r\to\infty}\frac{\mu(r,\tilde{f}_2)}{M(r,\tilde{f}_2)}=\infty.$$

Hence

$$M(r,f_1)\sim M(r,f_1+\tilde{f}_2)$$

as $r\to\infty$. Since

$$\mu(r,\tilde{f}_2)=o(\mu(r,f_1))$$

as $r\to\infty$, we have

$$\mu(r,f_1)\sim\mu(r,f_1+\tilde{f}_2)$$

as $r\to\infty$. Hence $f_1(z)+\tilde{f}_2(z)\in M_k$, $k\neq 1$. Clearly $\tilde{f}_2(z)\in M_1$. So, as $-f_1(z)\in M_k$, $k\neq 1$, the theorem is proved, and $f_1+\tilde{f}_2$ is our f_2. □

Proof of Theorem 2. Let

$$f_1(z)=\sum_{n=0}^{\infty}a_nz^n.$$

Since $f_1(z)\in M_1$ there is an unbounded set $S\subset\mathbb{R}_+$, the set of positive real numbers, such that

$$\frac{\mu(r,f_1)}{M(r,f_1)}\to 1$$

as $r\to\infty$ over S. Let $\{r_t\}_{t=1}^{\infty}$ be an unbounded sequence of S. For $t=1,2,\dots$, let

$$\nu(r_t,f_1)=n_t,$$

and

$$[|a_{\nu(r_t,f_1)}|^{-1}]=N_t.$$

Note that $N_t\to\infty$ as $t\to\infty$, since $f_1(z)$ is an entire function. Without any loss of generality, we can assume the following about the sequence $\{r_t\}_{t=1}^{\infty}$. For all $t=1,2,\ldots,N_t \geqslant \max(r_t, 2n_t)$; and $n_{t+1} \geqslant n_t + [\log n_t] + 3$, $N_{t+1} \geqslant N_t^2$. Let

$$T = \{n_t + h_t; t = 1,2,\ldots, h_t = 1,2,\ldots,[\log n_t]\},$$

$$b_{n_t+h_t} = \frac{1}{N_t + h_t!},$$

and

$$c_n = \begin{cases} b_n & \text{if } n \in T \\ 0 & \text{if } \mathbb{N}\, U\{0\} \setminus T, \end{cases}$$

where $\mathbb{N}$ denotes the set of all natural numbers. Define

$$f_2(z) = \sum_{n=0}^{\infty} (c_n - a_n) z^n.$$

Clearly $f_2(z)$ is an entire function $f_2(z)$. Since $c_{\nu(r_t, f_1)} = 0$, we see that

$$\mu(r_t, f_2) \sim \mu(r_t, f_1) \tag{4}$$

as $r_t \to \infty$. Also, if $j < t$

$$\frac{N_t r_t^{n_j+h_j-n_t}}{N_j + h_j!} \leqslant \frac{N_t^{n_j+h_j-n_t+1}}{N_j + h_j!} \leqslant \frac{N_t^{-2}}{N_j + h_j!} < \frac{1}{N_t^2}, \tag{5}$$

and if $j > t$

$$\frac{N_t r_t^{n_j+h_j-n_t}}{N_j + h_j!} \leqslant \frac{N_t^{n_j+h_j-n_t+1}}{N_j + h_j!} < \frac{1}{N_j^2}. \tag{5'}$$

From (4), (5) and (5′), and the fact that $f_1(z) \in M_1$, we get

$$M(r_t, f_1) \sim \mu(r_t, f_1) \sim \mu(r_t, f_2) \sim M(r_t, f_2)$$

as $r_t \to \infty$. Hence $f_2(z) \in M_1$. We have

$$f_1(z) + f_2(z) = f_3(z),$$

where

$$f_3(z) = \sum_{t=1}^{\infty} \sum_{h_t=1}^{[\log n_t]} \frac{z^{n_t+h_t}}{N_t + h_t!}.$$

Let

$$\mu(r, f_3) = \frac{r^{n_j+h_j}}{N_j + h_j!}.$$

If $h_j \neq 1$, then

$$\frac{r^{n_j+h_j}}{N_j + h_j!} \geqslant \frac{r^{n_j+h_j-1}}{N_j + h_j - 1!}, \textit{i.e.,}\ r \geqslant N_j + h_j.$$

If $h_j = 1$, then

$$\frac{r^{n_j+1}}{N_j+1!} \geqslant \frac{r^{n_{j-1}+1}}{N_{j-1}+1!}, \text{ i.e. from Starling's formula}$$

$$r \geqslant \left(\frac{N_j+1!}{N_{j-1}+1!}\right)(n_j - n_{j-1})^{-1} > N_j \text{ for large } j.$$

Also

$$\frac{r^{n_j+h_j}}{N_j+h_j!}\frac{N_j+1!}{r^{n_j+1}} = r^{h_j-1}\frac{N_j+1!}{N_j+h_j!} > \frac{N_j^{h_j-1}}{(N_j+2)(N_j+3)\cdots(N_j+h_j)} > \frac{1}{2}.$$

So

$$M^2(r,f_3) \geqslant \int_0^{2\pi} |f_3(re^{i\theta})|^2 \mathrm{d}\theta = \sum_{t=1}^{\infty}\sum_{h_t=1}^{[\log n_t]} \frac{r^{2(n_t+h_t)}}{(N_t+h_t!)^2}$$

$$> \sum_{h_j=1}^{[\log n_j]} \frac{r^{2(n_j+h_j)}}{(N_j+h_j!)^2}$$

$$> \mu^2(r,f_3)p(r),$$

where $z = re^{i\theta}$, and $p(r) \to \infty$ as $r \to \infty$. That is,

$$\liminf_{r\to\infty} \frac{M(r,f_3)}{\mu(r,f_3)} = \infty.$$

So $f_3(z) \in M_0$. Hence the theorem is proved. □

For the proof of Theorem 3, we need the following

Lemma 1 (pp. 4–10, [1]). *If* $M(r,f^{(j)})$ *denotes the maximum modulus of* $f^{(j)}(z)$, *the jth derivative of* $f(z)$, *then for* $j = 1,2,\ldots$

$$\left(\frac{\nu(r,f)}{r}\right)^j M(r,f)(1-\varepsilon_j(r)) \leqslant M(r,f^{(j)}) \leqslant \left(\frac{\nu(r,f)}{r}\right)^j,$$

where $0 < \varepsilon_j(r) \to 0$ *as* $r \to \infty$ *outside a set S of finite logarithemic measure.*

Proof of Theorem 3. Since (1) holds, there is a positive integer n_1 such that $\{|a_{n-1}/a_n|\}_{n=n_1}^{\infty}$ is a strictly increasing sequence. Let

$$f(z) + f^{(1)}(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = a_n + (n+1)a_{n+1}.$$

Let

$$a_n = |a_n|e^{i\alpha_n}, n = 0,1,\ldots, R_n = |a_{n-1}/a_n|,$$

$$R'_n = |b_{n-1}/b_n|,$$

and

$$\alpha(n, f+f^{(1)}) = \left(\frac{1+(n-1/R_{n-1})^2 + 2(n-1/R_{n-1})\cos(\alpha_{n-2}-\alpha_{n-1})}{1+(n/R_n)^2 + 2(n/R_n)\cos(\alpha_{n-1}-\alpha_n)}\right)^{\frac{1}{2}}.$$

Then

$$R'_n = R_{n+1}\alpha(n+1, f+f^{(1)}).$$

Since $\alpha(n+1, f+f^{(1)}) \to 1$ as $n \to \infty$, there exists a positive integer n_2 such that $\{R'_n\}_{n=n_2}^{\infty}$ is a strictly increasing sequence. Let

$$E_n = [R_n, R_{n+1}] \text{ and } F_n = [R'_n, R'_{n+1}].$$

For $r \in E_n$, $n \geqslant n_1$, we have

$$\nu(r,f) = n \text{ and } \mu(r,f) = |a_n| r^n.$$

For $r \in F_n$, $n \geqslant n_2$, we have

$$\nu(r, f+f^{(1)}) = n \text{ and } \mu(r, f+f^{(1)}) = |b_n| r^n.$$

Hence for $r \in E_n$, $n \geqslant n_1$

$$\mu(r, f+f^{(1)}) \geqslant \mu(r,f)\left(1 - \frac{\nu(r,f)+1}{R_{n+1}}\right), \tag{6}$$

and for $r \in F_n$, $n \geqslant n_2$

$$\mu(r, f+f^{(1)}) \leqslant \mu(r,f)\left(1 + \frac{\nu(r, f+f^{(1)})+1}{R_{n+1}}\right). \tag{7}$$

Let $n_0 = \text{Max}\,(n_1, n_2)$. Obviously

$$\text{meas}\, \mathbb{R}_+ \setminus \bigcup_{n=n_0}^{\infty} ((E_n \setminus E_n \cap F_n) \cup (F_n \setminus E_n \cap F_n)) = \infty.$$

For

$$r \notin S_1 = S \bigcup_{n=n_0}^{\infty} ((E_n \setminus E_n \cap F_n) \cup (F_n \setminus E_n \cap F_n)),$$

we have

$$\nu(r,f) = \nu(r, f+f^{(1)}) = n.$$

So from Lemma 1, (6) and (7)

$$\frac{\mu(r,f)}{M(r,f)} \sim \frac{\mu(r, f+f^{(1)})}{M(r, f+f^{(1)})}$$

as $r \to \infty$ outside S_1. Hence the theorem is proved. □

References

[1] Hans Wittich, *Neure Untersuchungen Uber Eindeutige Analytische Funktonen* (Berlin: Springer-Verlag) (1955)

[2] Shah S M, The behaviour of entire functions and a conjecture of P Erdös, *Am. Math. Mon.* **68** (1961) 419–425