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Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces

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Abstract. We show that if **G** is a semisimple algebraic group defined over **Q** and Γ is an arithmetic lattice in $G := \mathbf{G}_{\mathbf{R}}$ with respect to the **Q**-structure, then there exists a compact subset C of G/Γ such that, for any unipotent one-parameter subgroup $\{u_t\}$ of G and any $g \in G$, the time spent in C by the $\{u_t\}$-trajectory of $g\Gamma$, during the time interval $[0, T]$, is asymptotic to T, unless $\{g^{-1}u_t g\}$ is contained in a **Q**-parabolic subgroup of G. Some quantitative versions of this are also proved. The results strengthen similar assertions for $SL(n, \mathbf{Z})$, $n \geqslant 2$, proved earlier in [5] and also enable verification of a technical condition introduced in [7] for lattices in $SL(3, \mathbf{R})$, which was used in our proof of Raghunathan's conjecture for a class of unipotent flows, in [8].

Keywords. Homogeneous spaces; unipotent flows; trajectories.

Margulis [10] showed that if $\{u_t\}$ is a unipotent one-parameter subgroup of $G = SL(n, \mathbf{R})$ and $g \in G$ then there exists a compact subset C of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ such that the set $\{t \geqslant 0 | u_t g(SL(n, \mathbf{Z})) \in C\}$ is unbounded. The result played an important role in one of the proofs of the arithmeticity theorem for lattices (cf. [11]). In [3] and [5], motivated by certain problems on orbits and invariant measures of horospherical flows, the first named author improved the result. In [3] it was concluded that for $G = SL(n, \mathbf{R})$ and $\Gamma = SL(n, \mathbf{Z})$, given $\varepsilon > 0$ there exists a compact subset C of G/Γ such that for any unipotent one-parameter subgroup $\{u_t\}$ and any $g \in G$ either

$$\ell(\{t \in [0, T] | u_t g\Gamma \in C\}) \geqslant (1 - \varepsilon)\, T$$

for all large T (ℓ being the Lebesgue measure on **R**) or there exists a proper nonzero subspace W of $\mathbf{R}^n$ which is defined by a system of linear equations with rational coefficients and invariant under $g^{-1}u_t g$ for all $t \in \mathbf{R}$. Using a standard embedding argument one can deduce from this that if G is the group of **R**-elements of an algebraic group defined over **Q** and Γ is an arithmetic lattice in G then for any $\varepsilon > 0$ there exists a compact subset C of G/Γ such that for any unipotent one-parameter subgroup $\{u_t\}$ in G and $g \in G$ either $\ell(\{t \in [0, T] | u_t g\Gamma \in C\}) \geqslant (1 - \varepsilon)\, T$ for all large T or there exists an algebraic subgroup L of G defined over **Q** such that $g^{-1}u_t g \in L$ for all $t \in \mathbf{R}$. The result was used in the description of orbit closures of horospherical subgroups obtained in [6].

This set of ideas was again involved in [7] where we proved that if H is the subgroup of $SL(3,\mathbf{R})$ of all elements leaving invariant a non-degenerate indefinite quadratic form in 3 variables then every H-orbit on $SL(3,\mathbf{R})/SL(3,\mathbf{Z})$ is either closed or dense and used this result to conclude in particular that the set of values $B(\not{p}(\mathbf{Z}^n))$, where B is a nondegenerate indefinite quadratic form in $n\geqslant 3$ variables and $\not{p}(\mathbf{Z}^n)$ is the set of primitive elements in $\mathbf{Z}^n$, is dense in $\mathbf{R}$ whenever B is not a multiple of a rational quadratic form; the latter result strengthened the theorem of the second named author proving a conjecture of Oppenheim (cf. [12]). The proof used a somewhat technical result from [5] yielding a version of the above mentioned result, involving a quantitative condition in the second alternative. It was noted that the proof of the theorem about H-orbits on $SL(3,\mathbf{R})/SL(3,\mathbf{Z})$ would go through for any lattice Γ, in the place of $SL(3,\mathbf{Z})$, if it satisfied a condition which was called Condition $(*)$ (cf. [7] Remark 1·8). While the result from [5] alluded to above is sufficient to conclude that $SL(3,\mathbf{Z})$ satisfies Condition $(*)$, it does not yield such a result for other lattices in $SL(3,\mathbf{R})$. This is because, though any lattice in $SL(3,\mathbf{R})$ is arithmetic, the embedding argument used earlier is not adequate, since the subgroup L in the second alternative there is in general not insured to be contained in a parabolic subgroup. This called for an intrinsic approach to proving analogues of the results in [5] on asymptotic behaviour of the trajectories, in the case of a general arithmetic lattice. It is the purpose of this paper to carry this out. In particular we shall verify the Condition $(*)$ for any lattice Γ in $SL(3,\mathbf{R})$. It may be mentioned that the condition is also used in our more recent paper [8] where we describe the orbit-closures of any generic unipotent one-parameter subgroup on $SL(3,\mathbf{R})/\Gamma$, Γ any lattice, verifying a conjecture of Raghunathan for the case. We now introduce some notation and state the results.

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbf{Q}$ and let $G=\mathbf{G}_{\mathbf{R}}$, the group of $\mathbf{R}$-elements of $\mathbf{G}$. Let r be the $\mathbf{Q}$-rank of $\mathbf{G}$. We suppose that $r\geqslant 1$. Let $\mathbf{S}$ be a maximal $\mathbf{Q}$-split torus in $\mathbf{G}$. We fix an order on the system of $\mathbf{Q}$-roots on $\mathbf{S}$ for $\mathbf{G}$ and denote by $\{\alpha_1,\ldots,\alpha_r\}$ the corresponding system of simple $\mathbf{Q}$-roots (cf. [2]). For $i=1,\ldots,r$ let $\mathbf{P}_i$ be the standard maximal $\mathbf{Q}$-parabolic subgroup corresponding to the set of simple roots other than α_i. For each i the root α_i, which is a character on $\mathbf{S}$, extends uniquely to a character on $\mathbf{P}_i$; the extension will also be denoted by α_i. For $1\leqslant i\leqslant r$, let $\mathbf{U}_i$ be the unipotent radical of $\mathbf{P}_i$ and $\mathscr{U}_i$ be the Lie algebra of $\mathbf{U}_i$. Then there is a positive integer m_i, such that for any $x\in\mathbf{P}_i$ $\det\,(Ad\,x)|\mathscr{U}_i=\alpha_i^{m_i}(x)$; equivalently m_i can be defined to be the sum $\Sigma n_\lambda\lambda_i$ taken over all positive roots λ, where for each λ, n_λ is the dimension of the root subspace corresponding to λ and λ_i is the coefficient of α_i in the expansion of λ in terms of $\alpha_1,\ldots,\alpha_r$.

Let S and P_i, $i=1,\ldots,r$, denote the subgroups of G consisting of $\mathbf{R}$-elements of $\mathbf{S}$ and $\mathbf{P}_i$ respectively. We fix a maximal compact subgroup K of G such that S is invariant under the Cartan involution of G associated to K (cf. [13]). We now define for each $i=1,\ldots,r$ a function d_i on G as follows. Let $1\leqslant i\leqslant r$ be given. We recall that $G=KP_i$. We observe also that $K\cap P_i$ is a compact subgroup of P_i and hence $|\alpha_i(x)|=1$ for all $x\in K\cap P_i$. In view of this, for $g\in G$ expressed as $g=kx$ with $k\in K$ and $x\in P_i$, the number $|\alpha_i(x)|$ depends only on g and not on the choices of $k\in K$ and $x\in P_i$; we define $d_i(g)$ to be $|\alpha_i(x)|^{m_i}$.

The functions d_i, $1\leqslant i\leqslant r$ play a role in the present proofs similar to that of the function d in [5] on the class of discrete subgroups of $\mathbf{R}^n$, $n\geqslant 2$. The two are related as follows. Let $\mathbf{G}=\mathbf{SL}(n)$, $n\geqslant 2$, equipped with the usual $\mathbf{Q}$-structure. Let $\mathbf{S}$ be the maximal $\mathbf{Q}$-split torus consisting of diagonal matrices and let $\alpha_1,\ldots,\alpha_{n-1}$ be the usual system of simple **Q**-roots defined by $\alpha_i(\mathrm{diag}\,(a_1,\ldots,a_n)) = a_{i+1}/a_i$. Let $e_1,\ldots,e_n$ be the standard basis of $\mathbf{R}^n$ and for $i = 1,\ldots,n-1$ let Δ_i be the (discrete) subgroup generated by $\{e_1,\ldots,e_i\}$. Let K be the subgroup of $G = SL(n,\mathbf{R})$ consisting of orthogonal matrices. Then for $1 \leqslant i \leqslant n-1$ and $g \in G$, $d_i(g)$ as above can be seen to be the same as $d^2(g\Delta_i)$ with d as in [5]; since both the functions are K-invariant it is enough to check their equality for g in P_i.

We now state the main technical result of the paper. It gives a sufficient condition in terms of d_i, $i = 1,\ldots,r$, for the Lebesgue measure of the set of return times, within an interval, to a certain compact set to be large. The Lebesgue measure on **R** will be denoted by ℓ.

Theorem 1. *Let the notation be as above. Further let $\Gamma \subset G$ be an arithmetic lattice in G with respect to the Q-structure on $\mathbf{G}$. Then there exists a finite subset F of $\mathbf{G}_Q$ such that the following holds: for any $\varepsilon > 0$ and $\theta > 0$ there exists a compact subset C of G/Γ such that for any unipotent one-parameter subgroup $\{u_t\}$ in G, any $g \in G$ and any $T \geqslant 0$ either*

$$\ell(\{t \in [T, \sigma T] \mid u_t g\Gamma \in C\}) \geqslant (1-\varepsilon)(\sigma - 1)\, T$$

for all $\sigma > 1$ such that $(1-\sigma^{-1})^r > 1-\varepsilon$, or there exist $i \in \{1,\ldots,r\}$, $\gamma \in \Gamma$ and $f \in F$ such that

$$d_i(u_t g\gamma f) < \theta \quad \forall t \in [0, T].$$

Remarks 1. The set F is so chosen to be the set of inverses of a set of 'cusp elements' for the standard fundamental domain for Γ in G (cf. [1], Theorem 13.1) with respect to the triple (K, P, S) with K and S as above and P the standard minimal parabolic subgroup corresponding to the system $\{\alpha_1,\ldots,\alpha_r\}$ of simple **Q**-roots. (See § 1 for details about the set; it can be chosen to be any $F \subset \mathbf{G}_{\mathrm{Q}}$ such that $\Lambda(\phi) = \Gamma F$ in the notation as in (1.2).
2. In the case of $G = SL(n,\mathbf{R})$ and $\Gamma = SL(n,\mathbf{Z}), n \geqslant 2$, the second condition in the conclusion of the theorem can be seen to be equivalent to the condition that there exists a nonzero subgroup Δ of $\mathbf{Z}^n$ such that $d^2(u_t g\Delta) < \theta$ for all $t \in [0, T]$.

The proof of Theorem 1 will be completed in § 3. In § 4 we shall deduce various consequences of Theorem 1, which we now describe. For this purpose, for each $i = 1,\ldots,r$ let $Q_i = \{x \in P_i \mid \alpha_i(x) = 1\}$.

Theorem 2. *Let the notation be as before. Also let F be a finite subset of $\mathbf{G}_Q$ for which the contention of Theorem 1 holds. Then for any $\varepsilon > 0$ and $\theta > 0$ there exists a compact subset C of G/Γ such that for any unipotent one-parameter subgroup $\{u_t\}$ of G and $g \in G$ either*

$$\ell(\{t \in [0, T] \mid u_t g\Gamma \in C\}) \geqslant (1-\varepsilon)\, T$$

for all large T or there exist $i \in \{1,\ldots,r\}$ and $\lambda \in \Gamma F$ such that $g^{-1}u_t g \in \lambda Q_i \lambda^{-1}$ for all $t \in \boldsymbol{R}$ and $d_i(g\lambda) < \theta$.

Theorem 1 can also be applied to get compact subsets intersected by *all* orbits of certain subgroups. Let P_0 be the standard minimal **Q**-parabolic subgroup corresponding to the system of **Q**-roots as above; namely $P_0 = \cap_{i=1}^r P_i$. We note that P_0 contains a conjugate of any unipotent subgroup of G and hence the following result applies to any unipotent subgroup, rather than a subgroup of P_0, after appropriate modifications; the compact set for a conjugate would be different, however.

A subgroup V of P_0 is said to be in *general position* (relative to S and the order on the roots) if for any $i\in\{1,\ldots,r\}$ and $x\in G$, $xVx^{-1}\subset P_i$ if and only if $x\in P_i$.

Theorem 3. *Let the notation be as above. Then there exists a compact subset C of G/Γ such that the following holds: If V is a connected Lie subgroup of P_0 which consists of unipotent elements and is in general position and $\{x_k\}$ is a sequence in P_0 such that $d_i(x_k)\to\infty$ for all $i=1,\ldots,r$, then for any $g\in G$, $C\cap Vx_kg\Gamma/\Gamma$ is nonempty for all large k. In particular, if R is the subgroup generated by V and $\{x_k|k=1,2\ldots\}$ then every R-orbit on G/Γ intersects C.*

As stated before, one of our aims here is also to verify a technical condition on lattices in $SL(3,\mathbf{R})$ introduced in [7]; namely Condition (*) recalled below. In [7] it was noted that the arguments in the proof of Theorem 2 there went through for any lattice satisfying Condition (*) in the place of $SL(3,\mathbf{Z})$; for the lattice $SL(3,\mathbf{Z})$ the condition was verified using the results in [5]. We had mentioned that the condition in fact holds for all lattices but did not go into the proof, as our primary interest in that paper lay in the lattice $SL(3,\mathbf{Z})$. The condition is also used in the more recent paper [8] where we obtain a full description of orbit closures of generic unipotent one-parameter subgroups on $SL(3,\mathbf{R})/\Gamma$, Γ any lattice in $SL(3,\mathbf{R})$, verifying a conjecture of Raghunathan for the case.

For each $t\in\mathbf{R}$ let

$$v_1(t)=\begin{pmatrix}1 & t & \frac{1}{2}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}$$

and let V_1 be the subgroup $\{v_1(t)|t\in\mathbf{R}\}$. A lattice Γ in $SL(3,\mathbf{R})$ is said to satisfy Condition (*) if there exists a compact subset C of G/Γ such that for any $g\in G$ the following conditions hold:

a) the sets $\{t\geqslant 0|v_1(t)g\Gamma\in C\}$ and $\{t\leqslant 0|v_1(t)g\Gamma\in C\}$ are both unbounded unless there exists a proper parabolic subgroup P of $SL(3,\mathbf{R})$ such that if L is the closed subgroup generated by all unipotent elements in P then $g^{-1}V_1g\subset L$, $L\Gamma$ is closed and $L\cap\Gamma$ is a lattice in L and

b) if $\{f(t)\}_{t\geqslant 0}$ is a curve in $N(V_1)$ (the normalizer of V_1) such that $|\det f(t)|W|\to\infty$ as $t\to\infty$ for every proper nonzero $N(V_1)$-invariant subspace W of $\mathbf{R}^3$ then $C\cap V_1f(t)g\Gamma/\Gamma$ is nonempty for all large t.

Theorem 4. *Any lattice in $SL(3,R)$ satisfies Condition (*).*

1. On compactness of some subsets of G/Γ

We follow the notation as before. Further for $i=1,\ldots,r$ let

$$\mathbf{Q}_i=\{x\in\mathbf{P}_i|\alpha_i(x)=1\}\text{ and }\mathbf{S}_i=\{x\in\mathbf{S}|\alpha_j(x)=1\,\forall j\neq i\}.$$

Then each $\mathbf{S}_i$ is a one-dimensional $\mathbf{Q}$-split torus and $\mathbf{P}_i=\mathbf{S}_i\mathbf{Q}_i$ for all i.

Now let I be any (possibly empty) subset of $\{1,\ldots,r\}$. We define

$$\mathbf{P}_I = \cap_{i\in I}\mathbf{P}_i, \quad \mathbf{Q}_I = \cap_{i\in I}\mathbf{Q}_i \text{ and } \mathbf{S}_I = \Pi_{i\in I}\mathbf{S}_i.$$

Then $\mathbf{P}_i$ is the standard parabolic $\mathbf{Q}$-subgroup corresponding to the subset of $\{\alpha_1,\ldots,\alpha_r\}$ complementary to I (in particular $\mathbf{P}_\phi = \mathbf{G}$), $\mathbf{Q}_I$ is a normal algebraic $\mathbf{Q}$-subgroup of $\mathbf{P}_I$, $\mathbf{S}_I$ is a $\mathbf{Q}$-split torus and $\mathbf{P}_I = \mathbf{S}_I\mathbf{Q}_I$. Let $\mathbf{U}_I$ be the unipotent radical of $\mathbf{P}_I$ (and also $\mathbf{Q}_I$) and let $\mathbf{H}_I$ be the centraliser of $\mathbf{S}_I$ in $\mathbf{Q}_I$. Then $\mathbf{Q}_I = \mathbf{H}_I\mathbf{U}_I$ (semidirect product). We also note that $\mathbf{H}_I$ and $\mathbf{U}_I$ are defined over $\mathbf{Q}$. We denote by P_I, Q_I, S_I, H_I and U_I the subgroups of G consisting of $\mathbf{R}$-elements of $\mathbf{P}_I$, $\mathbf{Q}_I$, $\mathbf{S}_I$, $\mathbf{H}_I$ and $\mathbf{U}_I$ respectively.

Since $\mathbf{H}_I$ is defined over $\mathbf{Q}$, $\Gamma\cap H_I$ is an arithmetic subgroup of H_I. It is easy to see that there is no nontrivial character on $\mathbf{H}_I$ defined over $\mathbf{Q}$. Therefore $\Gamma\cap H_I$ is a lattice in H_I. If $I=\{1,\ldots,r\}$, H_I is of $\mathbf{Q}$-rank 0 and hence $\Gamma\cap H_I$ is a uniform lattice in H_I; that is, $H_I/\Gamma\cap H_I$ is compact. Since $\mathbf{U}_I$ is a unipotent algebraic subgroup defined over $\mathbf{Q}$, $U_I/\Gamma\cap U_I$ is also compact. Thus in the case $I=\{1,\ldots,r\}$, $Q_I/\Gamma\cap Q_I$ is compact.

Now let I be any (possibly empty) proper subset of $\{1,\ldots,r\}$ and let $J=\{1,\ldots,r\}-I$. We note that $\mathbf{S}_J$ is a maximal $\mathbf{Q}$-split torus in $\mathbf{H}_I$, $\mathbf{P}_J\cap\mathbf{H}_I$ is a minimal $\mathbf{Q}$-parabolic subgroup of $\mathbf{H}_I$ and $\mathbf{U}_J\cap\mathbf{H}_I$ is the unipotent radical of $\mathbf{P}_J\cap\mathbf{H}_I$. We note next that since, by choice, the Cartan involution associated to K leaves S invariant, it also follows that it leaves H_I invariant. This implies that $K\cap H_I$ is a maximal compact subgroup of H_I. Corresponding to the triple $(K\cap H_I, P_J\cap H_I, S_J)$ there exists a $t_I>0$, a compact subset C_I of $U_J\cap H_I$ and a finite subset E_I of $\mathbf{G}_\mathbf{Q}\cap H_I$ such that

$$H_I=(K\cap H_I)\Omega(t_I)C_IE_I(\Gamma\cap H_I),$$

where

$$\Omega(t_I)=\{s\in S_J \mid 0<\alpha_j(s)\leqslant t_I \quad \forall j\in J\}$$

(cf. [1] Theorem 13.1). Since $\mathbf{U}_I$ is a unipotent algebraic $\mathbf{Q}$-group, the arithmetic subgroup $\Gamma\cap U_I$ is a uniform lattice in U_I (that is, $U_I/\Gamma\cap U_I$ is compact) and hence there exists a compact subset D_I of U_I such that $U_I=D_I(\Gamma\cap U_I)$. Then we have

$$\begin{aligned} Q_I=H_IU_I &= (K\cap H_I)\Omega(t_I)C_IE_I(\Gamma\cap H_I)U_I \\ &= (K\cap H_I)\Omega(t_I)C_IU_IE_I(\Gamma\cap H_I) \\ &= (K\cap H_I)\Omega(t_I)C_ID_I(\Gamma\cap U_I)E_I(\Gamma\cap H_I). \end{aligned}$$

It is easy to see that since $E_I\subset\mathbf{G}_\mathbf{Q}\cap H_I$ there exists a finite subset F_I of $\mathbf{G}_\mathbf{Q}\cap Q_I$ such that

$$(\Gamma\cap U_I)E_I(\Gamma\cap H_I)\subset F_I(\Gamma\cap Q_I).$$

Hence we have

$$Q_I=(K\cap H_I)\Omega(t_I)\Psi_IF_I(\Gamma\cap Q_I) \tag{1.1}$$

where $\Psi_I=C_ID_I$ is a compact subset of $Q_I\cap Q_J$. We put

$$\Lambda(I)=(\Gamma\cap Q_I)F_I^{-1}=\{\gamma f \mid \gamma\in\Gamma\cap Q_I, f^{-1}\in F_I\}\subset Q_I. \tag{1.2}$$

The set F involved in the conclusion of Theorem 1 is taken to be any subset of $\mathbf{G}_\mathbf{Q}$ such that $\Lambda(\phi)=\Gamma F$; e.g. $F=F_\phi^{-1}$ in the above notation.

We shall use the facts mentioned above and the notation to deduce compactness of certain sets which we now introduce.

A p-tuple $((i_1, \lambda_1), \ldots, (i_p, \lambda_p))$, where $p \geqslant 1$, $i_1, \ldots, i_p \in \{1, \ldots, r\}$ and $\lambda_1, \ldots, \lambda_p \in \mathbf{G}_\mathbf{Q}$ is called an *admissible* sequence of length p if $i_1, \ldots, i_p$ are distinct and $\lambda_{j-1}^{-1} \lambda_j \in \Lambda(\{i_1, \ldots, i_{j-1}\})$ for all $j = 1, \ldots, p$, λ_0 being taken to be the identity element. The empty sequence is called an admissible sequence of length 0. If ξ and η are two admissible sequences of lengths p and q respectively and $p \leqslant q$ then η is said to extend ξ if the first p terms of η coincide with the corresponding terms of ξ; any admissible sequence extends the empty sequence.

For any admissible sequence ξ of length $p \geqslant 0$ we denote by $\mathscr{C}(\xi)$ the set of all pairs (i, λ), where $1 \leqslant i \leqslant r$ and $\lambda \in \mathbf{G}_\mathbf{Q}$, for which there exists an admissible sequence η of length $p+1$ extending ξ and containing (i, λ) as a (necessarily the last) term; note that if $p = 0$, namely if ξ is the empty sequence, $\mathscr{C}(\xi)$ consists of all (i, λ) where $1 \leqslant i \leqslant r$ and $\lambda \in \Lambda(\phi)$.

For any admissible sequence ξ of length $p \geqslant 0$ we define the *support* of ξ, to be the empty set if $p = 0$ and the set $\{(i_1, \lambda_1), \ldots, (i_p, \lambda_p)\}$ if $\xi = ((i_1, \lambda_1), \ldots, (i_p, \lambda_p))$; the support of ξ will be denoted by supp ξ.

The main result on compact subsets of G/Γ needed in the sequel is the following:

PROPOSITION 1.3

Let ξ be an admissible sequence of length $p \geqslant 0$. Let α, a and b be positive real numbers and let

$$W = \{g \in G \mid d_i(g\lambda) \geqslant \alpha \text{ for all } (i, \lambda) \in \mathscr{C}(\xi) \text{ and } a \leqslant d_i(g\lambda) \leqslant b \text{ for all } (i, \lambda) \in \operatorname{supp} \xi\}.$$

Then $W\Gamma/\Gamma$ is contained in a compact subset of G/Γ.

For proving the proposition we need the following Lemmas.

Lemma 1.4. *Let $i \in \{1, \ldots, r\}$ and let C be a compact subset of G. Then there exists a $c > 0$ such that $d_i(xg) \geqslant c d_i(g) \forall x \in C$ and $g \in G$.*

Proof. Recall that $G = KP_i$. Since CK is a compact subset of G there exists a compact subset D of P_i such that $CK \subset KD$. Since D is compact and α_i is continuous, there exists a $c > 0$ such that $|\alpha_i(y)|^{m_i} \geqslant c$ for all $y \in D$. Now let $x \in C$ and $g \in G$ be given. Then there exist $k \in K$ and $h \in P_i$ such that $g = kh$. Further, by the choice of D, there exist $k' \in K$ and $y \in D$ such that $xk = k'y$. Then $xg = xkh = k'yh$ and hence

$$\begin{aligned} d_i(xg) = d_i(k'yh) &= |\alpha_i(yh)|^{m_i} = |\alpha_i(y)|^{m_i} |\alpha_i(h)|^{m_i} \\ &\geqslant c|\alpha_i(h)|^{m_i} = c d_i(g) \end{aligned}$$

which proves the Lemma.

Lemma 1.5. *Let I be a subset of $\{1, \ldots, r\}$ and let $j \in \{1, \ldots, r\} - I$. Let $0 < a \leqslant b$ be given. Then there exists a compact subset K_0 of Q_I such that if $g \in Q_I$ and $d_j(g) \in [a, b]$ then $g \in K_0 Q_{I \cup \{j\}}$.*

Proof. Since $K\cap H_I$ is a maximal compact subgroup of H_I and $P_j\cap H_I$ is a parabolic subgroup of H_I we have $H_I=(K\cap H_I)(P_j\cap H_I)$. Hence $Q_I=H_I\cdot U_I=(K\cap H_I)\cdot(P_j\cap H_I)U_I=(K\cap H_I)(P_j\cap Q_I)$. It is also easy to see, by comparing the root subgroups on either side, that $P_j\cap Q_I=S_jQ_{I\cup\{j\}}$. Thus $Q_I=(K\cap H_I)S_jQ_{I\cup\{j\}}$. Further, for $g\in Q_I$ expressed as $g=ksh$ with $k\in K\cap H_I$, $s\in S_j$ and $h\in Q_{I\cup\{j\}}$ we have $d_j(g)=|\alpha_j(s)|^{m_j}$. This shows that if $g\in Q_I$ and $d_j(g)\in[a,b]$ then $g\in K_0Q_{I\cup\{j\}}$, where $K_0=(K\cap H_I)\cdot\{s\in S_j\,\|\,\alpha_j(s)|^{m_j}\in[a,b]\}$. Since K_0 is a compact subset of Q_I, this proves the Lemma.

Proof of Proposition 1.3. First let $p=0$, namely let ξ be the empty sequence. Then we see that $W=\{g\in G|d_i(g\lambda)\geqslant\alpha$ for all $i=1,\dots,r$ and $\lambda\in\Lambda(\phi)\}$. Let $g\in W$. By the particular case of (1.1) with $I=\phi$, g (in fact, any element of G) can be expressed as $kw\psi f$ where $k\in K$, $w\in\Omega(t_\phi)$, $\psi\in\Psi_\phi$ and $f\in F_\phi\Gamma=\Lambda(\phi)^{-1}$. Consider such a decomposition and let $\lambda=f^{-1}\in\Lambda(\phi)$. Then we see that for any $i=1,\dots,r$

$$|\alpha_i(w)|^{m_i}=d_i(kw\psi)=d_i(g\lambda)\geqslant\alpha.$$

This shows that

$$W\subseteq K\Omega_0\Psi_\phi F_\phi\Gamma \tag{1.6}$$

where $\Omega_0=\{w\in\Omega(t_\phi)|\ |\alpha_i(w)|^{m_i}\geqslant\alpha\forall i=1,\dots,r\}=\{w\in S|\alpha^{1/m_i}\leqslant|\alpha_i(w)|\leqslant t_\phi\forall_i\}$. Since Ω_0 is a compact subset of S, (1.6) implies that $W\Gamma/\Gamma$ is contained in a compact subset of G/Γ, thus proving the proposition in the case at hand.

Now let ξ be an admissible sequence of length $p\geqslant 1$, say $\xi=((i_1,\lambda_1),\dots,(i_p,\lambda_p))$, where $i_1,\dots,i_p$ are distinct elements of $\{1,\dots,r\}$ and $\lambda_1,\dots,\lambda_p\in\mathbf{G}_\mathbf{Q}$ are such that $\lambda_{j-1}^{-1}\lambda_j\in\Lambda(\{i_1,\dots,i_{j-1}\})$ for all $j=1,\dots,p$, with $\lambda_0=e$, the identity element. For $j=1,\dots,p$ let $I(j)=\{i_1,\dots,i_j\}$. We first show that there exist compact subsets $K_1,\dots,K_p$ of G such that for each $j=1,\dots,p$ and $g\in W$ there exists a $k_j\in K_j$ such that $k_jg\lambda_j\in Q_{I(j)}$. We proceed by induction on j. We choose $K_1=K_0^{-1}$ where K_0 is a compact subset for which the contention of Lemma 1.5 holds for the choices $I=\phi$, $j=i_1$ and a and b as in the hypothesis of the Proposition. Since $d_{i_1}(g\lambda_1)\in[a,b]$ for all $g\in W$, the Lemma implies that for each $g\in W$ there exists a $k_1\in K_1$ such that $k_1g\lambda_1\in Q_{I(1)}$. Now suppose that compact subsets $K_1,\dots,K_j$ have been found, satisfying the condition as above for some $1\leqslant j\leqslant p-1$. By Lemma 1.4 there exists a $c\in(0,1)$ such that $d_{i_{j+1}}(xh)\geqslant cd_{i_{j+1}}(h)$ for all $x\in K_j\cup K_j^{-1}$ and $h\in G$. Let K_0 be the compact subset for which the contention of Lemma 1.5 holds for the choices $I=I(j)$ and $j=i_{j+1}$ and ca and $c^{-1}b$ in the place of a and b. Put $K_{j+1}=K_0^{-1}K_j$. Now let $g\in W$. By our choice there exists a $k_j\in K_j$ such that $k_jg\lambda_j\in Q_{I(j)}$. Since $\lambda_j^{-1}\lambda_{j+1}\in Q_{I(j)}$ we get that $k_jg\lambda_{j+1}\in Q_{I(j)}$. Further, we have

$$ca\leqslant cd_{i_{j+1}}(g\lambda_{j+1})\leqslant d_{i_{j+1}}(k_jg\lambda_{j+1})\leqslant c^{-1}d_{i_{j+1}}(g\lambda_{j+1})\leqslant c^{-1}b.$$

Hence by Lemma 1.5 there exists a $k_0\in K_0$ such that $k_jg\lambda_{j+1}\in k_0Q_{I(j+1)}$. Thus we see that for $k_{j+1}=k_0^{-1}k_j$, $k_{j+1}g\lambda_{j+1}\in Q_{I(j+1)}$ as desired. Thus the inductive construction is complete.

Recall that $d_i(g\lambda)\geqslant\alpha$ for all $g\in W$ and $(i,\lambda)\in\mathscr{C}(\xi)$. Hence by Lemma 1.4 there exists a $\beta>0$ such that $d_i(kg\lambda)\geqslant\beta$ for all $k\in K_p$, $g\in W$ and $(i,\lambda)\in\mathscr{C}(\xi)$. Now let $g\in W$ and $k_p\in K_p$ be such that $k_pg\lambda_p\in Q_{I(p)}$. If $I=\{1,\dots,r\}$, $Q_I/\Gamma\cap Q_I$ is compact, and since $\lambda_p\in\mathbf{G}_\mathbf{Q}$ this implies that $Q_I\lambda_r^{-1}\Gamma/\Gamma$ is compact. In this case the preceding condition implies that $W\Gamma/\Gamma \subset K_r^{-1} Q_I \lambda_r^{-1} \Gamma/\Gamma$, which is a compact subset. Now suppose that I is a proper subset. By (1.1) and (1.2) there exists a $\theta \in \Lambda(I(p))$ such that $k_p g \lambda_p \theta \in K\Omega(t_{I(p)})\Psi_{I(p)}$ say $k_p g \lambda_p \theta = kw\psi$ where $k \in K, w \in \Omega(t_{I(p)})$ and $\psi \in \Psi_{I(p)}$. Let $J = \{1, \ldots, r\} - I(p)$. Observe that for any $j \in J$, $(j, \lambda_p \theta) \in \mathscr{C}(\xi)$ and hence $d_j(k_p g \lambda_p \theta) \geqslant \beta$. Hence we get that

$$|\alpha_j(w)|^{m_j} = d_j(kw\psi) = d_j(k_p g \lambda_p \theta) \geqslant \beta.$$

Let

$$\begin{aligned}\Omega_0 &= \{w \in \Omega(t_{I(p)}) \,\|\, \alpha_j(w)|^{m_j} \geqslant \beta \quad \forall j \in J\} \\ &= \{w \in S_J \,|\, \beta^{1/m_j} \leqslant |\alpha_j(w)| \leqslant t_{I(p)}\}.\end{aligned}$$

Then Ω_0 is a compact subset of S_J and the above argument shows that for any $g \in W$ there exist a $k_p \in K_p$ and a $\theta \in \Lambda(I(p)) = (F_{I(p)}\Gamma)^{-1} = \Gamma F_{I(p)}^{-1}$ such that $k_p g \lambda_p \theta \in K\Omega_0 \Psi_{I(p)}$. Therefore

$$W \subset K_p^{-1} K \Omega_0 \Psi_{I(p)} F_{I(p)} \Gamma \lambda_p^{-1}. \tag{1.7}$$

Since $K_p^{-1} K \Omega_0 \Psi_{I(p)}$ is a compact subset of G and $F_{I(p)} \Gamma \lambda_p^{-1}$ is contained in a finite union of cosets of Γ, (1.7) implies that $W\Gamma/\Gamma$ is contained in a compact of G/Γ. This proves the Proposition.

PROPOSITION 1.8

Let ξ be an admissible sequence of length $p \geqslant 1$; say $\xi = ((i_1, \lambda_1), \ldots, (i_p, \lambda_p))$. Let α, a and b be positive real numbers and let W be the subset of G as in Proposition 1.3 for this data. Let $I = \{i_1, \ldots, i_p\}$. Then

$$W = \{g \in G \,|\, d_i(g\lambda_p \theta) \geqslant \alpha \forall i \notin I \text{ and } \theta \in \Lambda(I) \text{ and } a \leqslant d_i(g\lambda_p) \leqslant b \forall i \in I.\}$$

In particular, the set $W\Gamma/\Gamma$ is determined by I and $\Gamma\lambda_p$, in the sense that if $\xi' = ((i_1, \lambda'_1), \ldots, (i_p, \lambda'_p))$ is an admissible sequence and $\lambda'_p \in \Gamma\lambda_p$, then the corresponding set for ξ' is the same as $W\Gamma/\Gamma$.

Proof. For any $1 \leqslant j \leqslant p$ let $I(j) = \{i_1, \ldots, i_j\}$. Since, by admissibility of $\xi, \lambda_j^{-1}\lambda_{j+1} \in \Lambda(I(j)) \subset Q_{I(j)}$ for all $j = 1, \ldots, p-1$ we get that $\lambda_j^{-1}\lambda_p \in Q_{I(j)}$ for all j. Therefore if $i = i_j$ for some j then $d_i(g\lambda_j) = d_i(g\lambda_p)$. Also clearly $(i, \lambda) \in \mathscr{C}(\xi)$ if and only if $i \notin I$ and $\lambda = \lambda_p \theta$ for some $\theta \in \Lambda(I)$. The first part of the proposition is immediate from these two observations. The remaining part now follows from an obvious substitution argument.

2. More on the functions d_i

We follow the notation as before. For each $i = 1, \ldots, r$ we define a representation ρ_i of G as follows. Let $1 \leqslant i \leqslant r$. Let U_i be the unipotent radical of P_i and let u_i be the dimension of U_i. Let $\mathscr{G}$ be the Lie algebra of G. Let $V_i = \wedge^i \mathscr{G}$, the ith exterior power of $\mathscr{G}$. We define ρ_i as the ith exterior power representation of the adjoint representation of G over $\mathscr{G}$. We equip $\mathscr{G}$ with a AdK-invariant norm. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathscr{G}$ with respect to the norm. For any ℓ, this defines a canonical basis of $\wedge^l \mathscr{G}$, namely $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell} | 1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n\}$. In particular we get a basis for each V_i; we equip V_i with the norm, denoted by $\|\cdot\|$, making the basis into an orthonormal basis. It is straightforward to verify that the norm is $\rho_i(K)$-invariant. Let p_i be an element of norm 1 in the one-dimensional subspace of $V_i = \wedge^i \mathscr{G}$ corresponding to the Lie subalgebra of $\mathscr{G}$ associated to U_i, which is a u_i-dimensional subspace. A straightforward computation shows that

$$\rho_i(x)(p_i) = \alpha_i(x)^{m_i} p_i \quad \forall x \in P_i. \tag{2.1}$$

This implies that $d_i(x) = \| \rho_i(x)(p_i) \|$ for all $x \in P_i$. Since d_i and the norm are K-invariant and $G = KP_i$ we get that

$$d_i(g) = \| \rho_i(g)(p_i) \| \quad \forall g \in G. \tag{2.2}$$

We also note at this point that for $g \in G$, $\rho_i(g)(p_i) = p_i$ if and only if $g \in Q_i$. The 'if' part follows from (2.1). Now let $g \in G$ be such that $\rho_i(g)(p_i) = p_i$. Then the definition of ρ_i shows that the Lie subalgebra of U_i is $Ad\ g$-invariant. Since U_i is a connected Lie subgroup this implies that g normalizes U_i. But P_i is the normalizer of U_i (cf. [2]). Hence $g \in P_i$. But then by (2.1) $\alpha_i(g) = 1$ which means that $g \in Q_i$.

PROPOSITION 2.3

Let $1 \leqslant i \leqslant r$ and let n_i be the dimension of V_i. Let $\{u_t\}$ be a unipotent one-parameter subgroup of G and let $g \in G$. Then $d_i^2(u_t g)$ is a polynomial in t of degree at most $2(n_i - 1)$. Further $d_i(u_t g)$ is constant (that is, independent of t) if and only if $g^{-1} u_t g \in Q_i$ for all $t \in \mathbf{R}$.

Proof. Since $\{u_t\}$ is a unipotent one-parameter subgroup of G, $\{\rho_i(u_t)\}$ is a unipotent one-parameter group of linear transformations of V_i. By Jordan decomposition this implies that for any $v \in V_i$ the expansion of $\{\rho_i(u_t)(v)\}$ with respect to any basis has coefficients which are polynomials in t of degree at most $(n_i - 1)$. Applying this to an orthonormal basis we see that for any $v \in V_i$, $\| \rho_i(u_t)(v) \|^2$ is a polynomial of degree at most $2(n_i - 1)$. Given $g \in G$, choosing $v = \rho_i(g) p_i$ we see that $\| \rho_i(u_t g)(p_i) \|^2$ is a polynomial of degree at most $2(n_i - 1)$ and hence by (2.2) so is $d_i^2(u_t g)$.

Now let $g \in G$ be such that $d_i(u_t g)$ is constant in t. Then by (2.2), $\| \rho_i(u_t g)(p_i) \| = \| \rho_i(u_t) \rho_i(g)(p_i) \|$ is constant. For a unipotent one-parameter group of linear transformations any orbit other than a fixed point is an unbounded subset of the vector space. Therefore under the above condition $\rho_i(u_t)\rho_i(g)(p_i) = \rho_i(g)(p_i)$ for all $t \in \mathbf{R}$. Hence $\rho_i(g^{-1} u_t g)$ fixes p_i for all t. As noted before, this implies that $g^{-1} u_t g \in Q_i$ for all $t \in \mathbf{R}$. This proves the Proposition.

Lemma 2.4. *Let $1 \leqslant i \leqslant r$, $f \in G_{\mathbf{Q}}$ and $g \in G$ be given. Then for any $\delta > 0$ the set $\{\gamma \in \Gamma \mid d_i(g\gamma f) < \delta\}$ is finite.*

Proof. Let $\mathscr{G}$ be equipped with the $\mathbf{Q}$-structure corresponding to the $\mathbf{Q}$-structure on $\mathbf{G}$. Since $\mathbf{U}_i$ is an algebraic subgroup defined over $\mathbf{Q}$, the Lie subalgebra of $\mathscr{G}$ corresponding to U_i is a rational subspace (spanned, over $\mathbf{R}$, by rational elements) of $\mathscr{G}$. The $\mathbf{Q}$-structure on $\mathscr{G}$ induces canonically a $\mathbf{Q}$-structure on $V_i = \wedge^i \mathscr{G}$ and ρ_i is (the restriction of) a rational representation with respect to the $\mathbf{Q}$-structure. Also in view of the preceding assertion p_i is a scalar multiple of a rational element, say $p_i = tq_i$ where $t \in \mathbf{R}$ and q_i is rational. Since $f \in \mathbf{G}_{\mathbf{Q}}$ we get that $\rho_i(f)(q_i)$ is rational.

Since Γ is an arithmetic subgroup, this implies in turn that $\rho_i(\Gamma)\rho_i(f)q_i$ is a discrete subset of V_i. Since

$$\rho_i(g\Gamma f)(p_i) = \rho_i(g)\rho_i(\Gamma)\rho_i(f)(p_i) = t\rho_i(g)\rho_i(\Gamma)\rho(f)(q_i)$$

we get the $\rho_i(g\Gamma f)(p_i)$ is a discrete subset of V_i. In particular for any $\delta > 0$ there exist only finitely many $\gamma\in\Gamma$ such that $\|\rho_i(g\gamma f)p_i\| \geqslant \delta$. In view of (2.2), this implies the Lemma.

Lemma 2.5. *There exists a finite subset $\tilde{F}$ of G_Q such that for any admissible sequence ξ and any $(i,\lambda)\in supp\ \xi$, $\lambda\in\Gamma\tilde{F}$.*

Proof. If $((i_1,\lambda_1),\ldots,(i_p,\lambda_p))$ is an admissible sequence of length $p\geqslant 1$ then for all $j=2,\ldots,p$ we have $\lambda_{j-1}^{-1}\lambda_j\in\Lambda(I(j-1))$, where $I(k)=\{i_1,\ldots,i_k\}$ for all k, and hence $\lambda_j\in\Lambda(\phi)\Lambda(I(1))\ldots\Lambda(I(j-1))$. This shows that for any admissible sequence ξ and any $(i,\lambda)\in\operatorname{supp}\xi$, λ is an element of a set of the form $\Lambda(\phi)\Lambda(I_1)\ldots\Lambda(I_j)$ where $j\in\{1,\ldots,r-1\}$ and $I_1,\ldots,I_j$ are subsets of $\{1,\ldots,r\}$ of cardinalities $1,\ldots,j$ respectively, such that $I_1\subset I_2\subset\cdots\subset I_j$. Since each $\Lambda(I)$, $I\subset\{1,\ldots,r\}$, is a finite union of cosets of the form Γf, $f\in\mathbf{G}_{\mathbf{Q}}$ and Γ is an arithmetic lattice, it follows that each product $\Lambda(\phi)\Lambda(I_1)\ldots\Lambda(I_j)$ as above is a finite union of cosets of the form Γf, $f\in\mathbf{G}_{\mathbf{Q}}$. Hence the preceding assertion implies that there are finitely many such cosets which together contain the supports of all admissible sequences. We can therefore choose a subset $\tilde{F}$ of $\mathbf{G}_{\mathbf{Q}}$ for which the contention of the Lemma holds.

Lemma 2.6. *Let $1\leqslant i\leqslant r$ and let $\{u_t\}$ be a unipotent one-parameter subgroup of G. Then the function v: $\boldsymbol{R}\to\boldsymbol{R}$ defined by*

$$v(t)=\sup\{d_i(u_tg)/d_i(g)\,|\,g\in G\}\ \forall t\in\mathbf{R}$$

is continuous.

Proof. Consider the function φ: $\mathbf{R}\times G\to\mathbf{R}$ defined by $\varphi(t,g)=d_i(u_tg)/d_i(g)$ for all $t\in\mathbf{R}$ and $g\in G$. Since $d_i(hp)=d_i(h)d_i(p)$ for all $h\in G$ and $p\in P_i$ we see that $\varphi(t,gp)=\varphi(t,g)$ for all $t\in\mathbf{R}$, $g\in G$ and $p\in P_i$. Hence we get a well-defined function $\tilde{\varphi}$: $\mathbf{R}\times G/P_i\to\mathbf{R}$ such that $\tilde{\varphi}(t,gP_i)=\varphi(t,g)$ for all $t\in\mathbf{R}$ and $g\in G$. Since φ is continuous so is $\tilde{\varphi}$. Also, clearly

$$v(t)=\sup\{\tilde{\varphi}(t,x)\,|\,x\in G/P_i\}.$$

Since $\tilde{\varphi}$ is continuous and G/P_i is compact, an elementary argument shows that the right hand side is a continuous function. This proves the lemma.

PROPOSITION 2.7

Let $1\leqslant i\leqslant r$, let $\{u_t\}$ be a unipotent one-parameter subgroup of G and let $g\in G$. Let A be a subset of G_Q contained in a finite union of cosets of the form Γf, $f\in\boldsymbol{G}_{\mathbf{Q}}$. Let $\delta>0$ and $t_1,t_2\in\boldsymbol{R}$, $t_1<t_2$, be such that $d_i(u_{t_1}g\lambda)>\delta$ for all $\lambda\in A$ and $d_i(u_{t_2}g\lambda)\leqslant\delta$ for some $\lambda\in A$. Let

$$s=\inf\{t\in[t_1,t_2]\,|\,d_i(u_tg\lambda)\leqslant\delta \text{ for some } \lambda\in A\}.$$

Then there exists a $\lambda\in A$ such that $d_i(u_sg\lambda)=\delta$.

Proof. Let v: $\mathbf{R} \to \mathbf{R}$ be the function as in Lemma 2.6 for i and $\{u_t\}$ as above. By the Lemma there exists a neighbourhood Ω of 0 such that $v(t) < 2$ for all $t \in \Omega$. By the definition of s there exist sequences $\{t_k\}$ in $[t_1, t_2]$ and $\{\lambda_k\}$ in A such that $t_k \to s$ and $d_i(u_{t_k} g \lambda_k) = \delta$ for all k. We may clearly assume that $t_k - s \in \Omega$ for all k. Then $d_i(u_s g \lambda_k) \leqslant v(s - t_k) d_i(u_{t_k} g \lambda_k) \leqslant 2\delta$ for all k. Since A is contained in finitely many cosets of the form Γf, by Lemma 2.4 this implies that $\{\lambda_k | k = 1, 2, \ldots\}$ is a finite set. Passing to a subsequence we may assume that $\lambda_k = \lambda$ for all k, where $\lambda \in A$. Then, since $t_k \to s$ and $d_i(u_{t_k} g \lambda) = \delta$ for all k, we get that $d_i(u_s g \lambda) = \delta$. This proves the Proposition.

3. Proof of Theorem 1

In this section we complete the proof of Theorem 1. We begin by recalling some properties of nonnegative polynomials and fixing some more notation.

For $m \in N$ let $\mathscr{P}_m$ denote the set of all nonnegative valued polynomials of degree at most m. We need the following simple properties of nonnegative polynomials (cf. [9] Lemma A.4 or [5] Lemmas 1.3 and 1.4).

Lemma 3.1. a) *For any $m \in \mathbf{N}$ and $\rho > 0$ there exists a $\alpha > 0$ such that the following holds: If $P \in \mathscr{P}_m$ is such that $P(1) < \alpha$ and $P(s) \geqslant 1$ for some $s \in [0, 1]$ then there exists a $t \in [1, \rho]$ such that $P(t) = \alpha$.*
b) *For any $m \in \mathbf{N}$ and $\sigma > 1$ there exist constants $\beta_1, \beta_2 > 0$ such that the following holds: If $P \in \mathscr{P}_m$, $P(s) \leqslant 1$ for all $s \in [0, 1]$ and $P(1) = 1$ then there exists a $\ell, 0 \leqslant \ell \leqslant m$, such that $\beta_1 \leqslant P(t) \leqslant \beta_2$ for all $t \in [\sigma^{2\ell+1}, \sigma^{2\ell+2}]$.*

For the rest of the argument we fix some constants as follows. Let $\varepsilon > 0$ be arbitrary (we shall later choose this to be as in Theorem 1). Let $\sigma > 1$ be such that $(1 - \sigma^{-1})^r > (1 - \varepsilon)$ where r, as in § 1, is the $\mathbf{Q}$-rank of G. We next choose $\tau > 1$ such that $(\tau^{-1} - \sigma^{-1})^r \geqslant (1 - \varepsilon)$. Let $m = 2 \max \{n_i - 1 | 1 \leqslant i \leqslant r\}$ and let $\rho > 1$ be such that $(\rho - 1) \leqslant (\tau - 1)/\sigma^{2m+2}$. Let $\alpha \in (0, 1)$ be such that the contention of Lemma 3.1 a) holds for these choices of m and ρ. Let $0 < \beta_1 < 1 < \beta_2$ be such that the contention of Lemma 3.1 b) holds for the choices of m and σ as above.

PROPOSITION 3.2

Let $\{u_t\}$ be a unipotent one-parameter subgroup of G and let $g \in G$. Let ξ be an admissible sequence of length $p \geqslant 0$. Let $s \geqslant 0$ and $\chi > 0$ be such that for any $(i, \lambda) \in \mathscr{C}(\xi)$ there exists a $t \in [0, s]$ such that $d_i^2(u_t g \lambda) \geqslant \chi$. Then at least one of the following conditions holds:

i) *there exists a $s' \in (s, \tau s)$ such that for all $(i, \lambda) \in \mathscr{C}(\xi)$ and $t \in [s, s')$*

$$d_i^2(u_t g \lambda) > \chi\alpha/2$$

ii) *there exist s_0, $s_1 \in [s, \tau s]$ such that $(s_1 - s) = \sigma(s_0 - s)$ and the following conditions are satisfied:*

a) *for any $(i, \lambda) \in \mathscr{C}(\xi)$ there exists a $t \in [s, s_0]$ such that $d_i^2(u_t g \lambda) \geqslant \chi\alpha/2$ and* b) *there exists a $(j, \mu) \in \mathscr{C}(\xi)$ such that $\chi\alpha\beta_1 \leqslant d_j^2(u_t g \mu) \leqslant \chi\alpha\beta_2$ for all $t \in [s_0, s_1]$ and $d_j^2(u_y g \mu) \geqslant 2 d_j^2(u_s g \mu)$ for some $y \in [s, s_0]$.*

Proof. Let

$$\mathscr{F} = \{(i, \lambda) \in \mathscr{C}(\xi) | d_i^2(u_s g \lambda) \leqslant \chi\alpha/2\}.$$

First suppose that $\mathscr{F}$ is empty. Consider the set

$$E = \{t \in [s, \tau s) \mid d_i^2(u_t g\lambda) > \chi\alpha/2 \; \forall (i, \lambda) \in \mathscr{C}(\xi)\}.$$

If $E = [s, \tau s)$ then condition i) of the Proposition holds for $s' = \tau s$. Now suppose that E is a proper subset of $[s, \tau s)$. Let $s' = \inf\{t \mid t \in [s, \tau s) - E\}$. Then by Lemma 2.5 and Proposition 2.7, there exists a $(i, \lambda) \in \mathscr{C}(\xi)$ such that $d_i^2(u'_s g\lambda) = \chi\alpha/2$. Hence $s' \in [s, \tau s) - E$. On the other hand, since $\mathscr{F}$ is empty $s \in E$. In particular $s' > s$. Clearly condition i) of the Proposition holds for this s'.

Next suppose that $\mathscr{F}$ is nonempty. By Lemmas 2.4 and 2.5 $\mathscr{F}$ is a finite set. By hypothesis for any $(i, \lambda) \in \mathscr{F} \subset \mathscr{C}(\xi)$ there exists a $t \in [0, s]$ such that $d_i^2(u_t g\lambda) \geqslant \chi$ and hence by Lemma 3.1 a), applied to the polynomial $t \mapsto d_i^2(u_{st} g\lambda)/\chi$, which is of degree $2(n_i - 1) \leqslant m$ (cf. Proposition 2.3), there exists a $t \in [s, \rho s]$ such that $d_i^2(u_t g\lambda) = \chi\alpha$. For each $(i, \lambda) \in \mathscr{C}(\xi)$ let $t(i, \lambda) = \inf\{t \in [s, \rho s] \mid d_i^2(u_t g\lambda) = \chi\alpha\}$ and let $y = \max\{t(i, \lambda) \mid (i, \lambda) \in \mathscr{F}\}$. Let $(j, \mu) \in \mathscr{F}$ be such that $t(j, \mu) = y$. We note that

$$d_j^2(u_y g\mu) = \chi\alpha \geqslant 2 d_j^2(u_s g\mu). \tag{3.3}$$

Now observe that $d_j^2(u_t g\mu) \leqslant \chi\alpha$ for all $t \in [s, y]$ and $d_j^2(u_y g\mu) = \chi\alpha$. Hence by Lemma 3.1 b), applied to the polynomial $t \mapsto d_j^2(u_{s+(y-s)t} g\mu)/\chi\alpha$, there exists a $\ell, 0 \leqslant \ell \leqslant m$, such that

$$\chi\alpha\beta_1 \leqslant d_j^2(u_t g\mu) \leqslant \chi\alpha\beta_2 \ldots \forall t \in [s_0, s_1] \tag{3.4}$$

where

$$s_0 = s + \sigma^{2\ell+1}(y - s) \text{ and } s_1 = s + \sigma^{2\ell+2}(y - s).$$

Observe that $s \leqslant s_0 \leqslant s_1 \leqslant s + \sigma^{2m+2}(\rho - 1)s \leqslant \tau s$. Also clearly $(s_1 - s) = \sigma(s_0 - s)$. We next verify conditions ii) for these choices of s_0 and s_1. We see that for $(i, \lambda) \in \mathscr{C}(\xi)$, $d_i^2(u_s g\lambda) > \chi\alpha/2$ if $(i, \lambda) \notin \mathscr{F}$ and $d_i^2(u_{t(i,\lambda)} g\lambda) = \chi\alpha$ if $(i, \lambda) \in \mathscr{F}$; since $s \leqslant t(i, \lambda) \leqslant y < s_0$, this shows that condition ii) (a) holds. Condition ii) (b) follows from (3.3) and (3.4). This proves the Proposition.

PROPOSITION 3.5

Let $\{u_t\}$ be a unipotent one-parameter subgroup of G. Let ξ be an admissible sequence of length $p \geqslant 0$. Let $g \in G$, $s \geqslant 0$ and $\chi' \geqslant \chi > 0$ be such that for any $(i, \lambda) \in \mathscr{C}(\xi)$ there exists a $t \in [0, s]$ such that $d_i^2(u_t g\lambda) \geqslant \chi$ and for any $(i, \lambda) \in supp\, \xi$, $\chi\beta_1 \leqslant d_i^2(u_t g\lambda) \leqslant \chi'\beta_2$ for all $t \in [s, \sigma s]$. For any admissible sequence ζ extending ξ, say of length q, let

$$X(\zeta) = \{t \in [s, \sigma s] \mid d_i^2(u_t g\lambda) \geqslant \chi(\alpha/2)^{q-p+1} \quad \forall (i, \lambda) \in \mathscr{C}(\zeta) \text{ and}$$
$$(\alpha/2^{q-p})\chi\beta_1 \leqslant d_i^2(u_t g\lambda) \leqslant \chi'\beta_2 \quad \forall (i, \lambda) \in supp\, \xi\}$$

and let

$$X = \cup_\zeta X(\zeta)$$

where the union is taken over all admissible sequences ζ extending ξ. Then

$$\ell(X) \geqslant (\tau^{-1} - \sigma^{-1})^{r-p}(\sigma - 1)s.$$

Proof. We proceed by induction on $(r - p)$. If $p = r$ then $X = X(\xi) = [s, \sigma s]$ and hence the Proposition evidently holds. Now let $0 \leqslant p < r$ and suppose that the Proposition holds for all admissible sequences of length $\geqslant p+1$, for all $g \in G$, $s \geqslant 0$ and $\chi > 0$ satisfying the conditions in the hypothesis, and let an admissible sequence ξ of length $p, g \in G$, $s \geqslant 0$ and $\chi > 0$ be given, satisfying the conditions in the hypothesis. Let X be the set as in the statement of the Proposition, for this data.

We first show that for any $x \in [s, \tau^{-1}\sigma s]$ there exists a $x' \in (x, \tau x]$ for which either $[x, x') \subset X$ or the following conditions are satisfied:

$$\ell(X \cap [x, x']) \geqslant (1-\sigma^{-1})(\tau^{-1}-\sigma^{-1})^{r-p-1}(x'-x) \tag{3.6}$$

and there exist $(j, \mu) \in \mathscr{C}(\xi)$ and $y \in [x, x']$ such that

$$d_j^2(u_y g\mu) \geqslant 2d_j^2(u_x g\mu) \tag{3.7}$$

Let $x \in [s, \tau^{-1}\sigma s]$ be given. We apply Proposition 3.2 with x in the place of s, the requisite conditions being satisfied since $x \geqslant s$. Suppose Condition i), as in the conclusion of that Proposition, holds. Then there exists a $x' \in (x, \tau x)$ such that for all $(i, \lambda) \in \mathscr{C}(\xi)$ and $t \in [x, x')$, $d_i^2(u_t g\lambda) \geqslant \chi\alpha/2$. We also see that $[x, \tau x] \subset [s, \sigma s]$ and hence $\chi\beta_1 \leqslant d_i^2(u_t g\lambda) \leqslant \chi'\beta_2$ for all $(i, \lambda) \in \operatorname{supp} \xi$ and $t \in [x, \tau x]$. The two assertions imply that $[x, x') \subset X(\xi) \subset X$ and hence we are through in this case. Next suppose that Condition ii) (of Proposition 3.2) holds. Thus there exist $s_0, s_1 \in [x, \tau x]$ such that $(s_1 - x) = \sigma(s_0 - x)$ and the following conditions are satisfied: a) for any $(i, \lambda) \in \mathscr{C}(\xi)$ there exists a $t \in [x, s_0]$ such that $d_i^2(u_t g\lambda) \geqslant \chi\alpha/2$ and b) there exists a $(j, \mu) \in \mathscr{C}(\xi)$ such that $\chi\alpha\beta_1 \leqslant d_j^2(u_t g\mu) \leqslant \chi\alpha\beta_2$ for all $t \in [s_0, s_1]$ and $d_j^2(u_y g\mu) \geqslant 2d_j^2(u_x g\mu)$ for some $y \in [x, s_0]$. Let η be the admissible sequence of length $p+1$ extending ξ and containing (j, μ) (as in condition (b)) as the last term. Then we see that the conditions in the hypothesis of the present proposition are satisfied for η, in the place of ξ, with $u_x g, s_0 - x$, and $\chi\alpha/2$ in the place of g, s and χ respectively: condition a) above implies that for any $(i, \lambda) \in \mathscr{C}(\eta)$ there exists a $t \in [x, s_0]$ such that $d_i^2(u_{(t-x)} u_x g\lambda) \geqslant \chi\alpha/2$. For all $(i, \lambda) \in \operatorname{supp} \xi$ we have

$$d_i^2(u_{t-x} u_x g\lambda) = d_i^2(u_t g\lambda) \in [\chi\beta_1, \chi'\beta_2] \subset [\chi(\alpha/2)\beta_1, \chi'\beta_2]$$

for all $t \in [s, \sigma s]$ and, in particular, whenever $(t-x) \in [s_0 - x, \sigma(s_0 - x)]$, since $\sigma(s_0 - x) = s_1 - x$ and $s_0, s_1 \in [x, \tau x] \subset [s, \sigma s]$; also $d_j^2(u_t g\mu) \in [\chi\alpha\beta_1, \chi\alpha\beta_2] \subset [\chi(\alpha/2)\beta_1, \chi'\beta_2]$. Thus we have verified the conditions in the hypothesis for the choices as above. Since η is of length $p+1$, by the induction hypothesis the assertion of the Proposition holds for η. For any admissible sequence ζ let $X'(\zeta)$ be the set corresponding to $X(\zeta)$ as in the proposition with respect to the choices as above. Let X' be the union of $X'(\zeta)$ over all admissible sequences extending η. Then we have

$$\ell(X') \geqslant (\tau^{-1} - \sigma^{-1})^{r-p-1}(\sigma - 1)(s_0 - x) \tag{3.8}$$

It is straightforward to verify by substitution that for any admissible sequence ζ extending η and $t \in X'(\zeta)$, $x + t \in X(\zeta) \cap [s_0, s_1]$; recall for this purpose that $[s_0 - x, \sigma(s_0 - x)] = [s_0 - x, s_1 - x] \subset [s - x, \sigma s - x]$. Hence by (3.8) we get that

$$\begin{aligned} \ell(X \cap [s_0, s_1]) &\geqslant (\tau^{-1} - \sigma^{-1})^{r-p-1}(\sigma - 1)(s_0 - x) \\ &= (1 - \sigma^{-1})(\tau^{-1} - \sigma^{-1})^{r-p-1}(s_1 - x). \end{aligned}$$

Now choose $x' = s_1$. Then, since $x \leqslant s_0$, the above relation shows that (3.6) is satisfied.

Also by condition b) above there exists a $y \in [x, s_0] \subset [x, x']$ such that (3.7) holds. Thus we have produced a x' for which (3.6) and (3.7) hold.

To complete the proof we construct a finite sequence $x_0, x_1, \ldots, x_n$ in $[s, \sigma s]$ as follows. We choose $x_0 = s$. Let $k \geqslant 0$ and suppose that $x_0, \ldots, x_k$ have been chosen. If $x_k \leqslant \tau^{-1}\sigma s$ then we choose $x_{k+1} \in [x_k, \tau x_k]$ as follows: If there exists $x' \in (x_k, \sigma s)$ such that $[x_k, x') \subset X$ then we choose x_{k+1} to be such that $[x_k, x_{k+1}) \subset X$ but $[x_k, x'')$ is not contained in X for any $x'' > x_{k+1}$. If there does not exist any $x' > x_k$ with $[x_k, x') \subset X$ then, as $x_k \in [s, \tau^{-1}\sigma s]$, by what we proved above (see (3.6)) there exists a $x_{k+1} \in (x_k, \tau x_k]$ such that

$$\ell(X \cap [x_k, x_{k+1}]) \geqslant (1 - \sigma^{-1})(\tau^{-1} - \sigma^{-1})^{r-p-1}(x_{k+1} - x_k) \tag{3.9}$$

and there exist $(j, \mu) \in \mathscr{C}(\xi)$ and $y \in [x_k, x_{k+1}]$ such that

$$d_j^2(u_y g\mu) \geqslant 2d_j^2(u_{x_k} g\mu). \tag{3.10}$$

Observe that since $x_k \leqslant \tau^{-1}\sigma s$, $x_{k+1} \leqslant \sigma s$. Lastly, if $x_k > \tau^{-1}\sigma s$ we terminate the sequence, setting $n = k$.

We show that the sequence as defined above does terminate in finitely many steps. For this purpose observe that if for some $k \geqslant 0$, $[x_k, x_{k+1}) \subset X$ then $[x_{k+1}, x')$ is not contained in X for any $x' > x_{k+1}$. In view of this, to show that the sequence terminates it is enough to show that there exists a $c > 0$ such that $x_{k+1} - x_k \geqslant c$ for any $k \geqslant 0$ such that $[x_k, x_{k+1})$ is not contained in X. In view of Lemma 2.6 there exists a $c > 0$ such that if for some $i \in \{1, \ldots, r\}, h \in G$ and $t \geqslant 0$, $d_i(u_t h)/d_i(h) \geqslant \sqrt{2}$ then $t \geqslant c$. Recall that when $[x_k, x_{k+1})$ is not contained in X there exist $(j, \mu) \in \mathscr{C}(\xi)$ and $y \in [x_k, x_{k+1}]$ such that (3.7) holds and in that case, by the above observation, $y - x_k \geqslant c$ and in turn $x_{k+1} - x_k \geqslant c$, as desired. Hence the sequence indeed terminates (in at most $2(\tau^{-1}\sigma - 1)s/c$ steps!) at a $x_n > \tau^{-1}\sigma s$.

Now we have

$$\ell(X) \geqslant \sum_{k=0}^{n-1} \ell(X \cap [x_k, x_{k+1}]) \geqslant (1 - \sigma^{-1})(\tau^{-1} - \sigma^{-1})^{r-p-1}(x_n - x_0),$$

by (3.9). Since $(x_n - x_0) > (\tau^{-1}\sigma s - s) = \sigma(\tau^{-1} - \sigma^{-1})s$, this yields that

$$\ell(X) \geqslant (\sigma - 1)(\tau^{-1} - \sigma^{-1})^{r-p} s$$

thus proving the Proposition.

Proof of Theorem 1. Let $F \subset \mathbf{G}_{\mathbf{Q}}$ be a finite subset such that $\Lambda(\phi) = \Gamma F$ (cf. (1.2)). Now let $\varepsilon > 0$ and $\theta > 0$ be as in the hypothesis of the Theorem and $\sigma > 1$ such that $(1 - \sigma^{-1})^r > (1 - \varepsilon)$. Let $\tau > 1$, $\rho > 1$, $\alpha \in (0, 1)$ and $0 < \beta_1 < 1 < \beta_2$ be the constants chosen as in the beginning of the section starting with σ. For any admissible sequence ζ of length q let

$$W(\zeta) = \{g \in G \mid d_i^2(g\lambda) \geqslant \theta(\alpha/2)^{q+1}\, \forall (i, \lambda) \in \mathscr{C}(\zeta) \text{ and}$$
$$(\alpha/2)^q \theta\beta_1 \leqslant d_i^2(g\lambda) \leqslant \theta\beta_2\, \forall (i, \lambda) \in \operatorname{supp}\zeta\}$$

and let

$$C = \cup_\zeta \overline{W(\zeta)\Gamma}/\Gamma$$

where the union is taken over all admissible sequences ζ. By Proposition 1.8 there are only finitely many distinct subsets involved in the union and by Proposition 1.3 each of them is compact. Hence C is a compact subset of G/Γ. We shall show that the contention of the Theorem holds for the compact set C and σ as above.

Let a unipotent one-parameter subgroup $\{u_t\}$ *in* $G, g\in G$ and $T\geqslant 0$ be given. For any admissible sequence ζ of length q let

$$X(\zeta)=\{t\in[T,\sigma T]\,|\,d_i^2(u_tg\lambda)\geqslant\theta(\alpha/2)^{q+1}\,\forall(i,\lambda)\in\mathscr{C}(\zeta)\text{ and }(\alpha/2)^q\theta\beta_1\leqslant d_i^2(u_tg\lambda)\geqslant\theta\beta_2\,\forall(i,\lambda)\in\operatorname{supp}\zeta\}$$

and let

$$X=\cup_\zeta X(\zeta)$$

the union being taken over all admissible sequences ζ. Applying Proposition 3.5 to the empty sequence ϕ, with $s=T$ and $\chi=\chi'=\theta^2$ we see that either there exists a $(i,\lambda)\in\mathscr{C}(\phi)$ such that $d_i(u_tg\lambda)<\theta$ for all $t\in[0,T]$ or.

$$\ell(X)\geqslant(\tau^{-1}-\sigma^{-1})^r(\sigma-1)\,T.$$

Observe that if $t\in X$ then $u_tg\Gamma\in C$. Recall also that by choice $(\tau^{-1}-\sigma^{-1})^r\geqslant(1-\varepsilon)$ and that for $i\in\{1,\ldots,r\}$, $(i,\lambda)\in\mathscr{C}(\phi)$ if and only if $\lambda\in\Lambda(\phi)=\Gamma F$. Hence the above conclusion implies the assertion in the theorem, that either

$$\ell(\{t\in[T,\sigma T]\,u_tg\Gamma\in C\})\geqslant(1-\varepsilon)(\sigma-1)\,T$$

or these exist $\lambda\in\Gamma F$ and $i\in\{1,\ldots,r\}$ such that $d_i(u_tg\lambda)<\theta$ for all $t\in[0,T]$.

4. Proofs of the other theorems

We shall now deduce the other theorems stated in the introduction. We follow the same notation as before.

Proof of Theorem 2. Let $\varepsilon>0$ and $\theta>0$ be given and let C be a compact subset of G/Γ for which the contention of Theorem 1 holds for $\varepsilon/2$ and θ in the place of ε and θ respectively. Let $\{u_t\}$ be a unipotent one-parameter subgroup of G and let $g\in G$. Let $\sigma>1$ be such that $(1-\sigma^{-1})^r>(1-\varepsilon/2)$. Then by Theorem 1 for any $T\geqslant 0$ either there exist $j\in\{1,\ldots,r\}$ and $\mu\in\Gamma F$ such that $d_j(u_tg\mu)<\theta$ for all $t\in[0,\sigma^{-1}T]$ or

$$\ell(\{t\in[\sigma^{-1}T,T]\,|\,u_tg\Gamma\in C\})\geqslant(1-\varepsilon/2)(\sigma-1)\sigma^{-1}T>(1-\varepsilon)\,T.$$

Hence if the first condition in the conclusion of Theorem 2 does not hold then for each $T\geqslant 0$ there exist $j\in\{1,\ldots,r\}$ and $\mu\in\Gamma F$ such that $d_j(u_tg\mu)<\theta$ for all $t\in[0,\sigma^{-1}T]$. By Lemma 2.4 the set

$$\{(j,\mu)\,|\,1\leqslant j\leqslant r,\mu\in\Gamma F,d_j(g\mu)<\theta\}$$

is finite. Therefore the above condition implies that there exist $i\in\{1,\ldots,r\}$ and $\lambda\in\Gamma F$ such that $d_i(u_tg\lambda)<\theta$ for all $t\geqslant 0$. By Proposition 2.3, $d_i^2(u_tg\lambda)$ is a polynomial in t and hence the preceding condition implies that $d_i(u_tg\lambda)=d_i(g\lambda)$ for all $t\in\mathbf{R}$. This implies, by the second part of Proposition 2.3 that $\lambda^{-1}g^{-1}u_t g\lambda \in Q_i$, or equivalently, $g^{-1}u_t g \in \lambda Q_i \lambda^{-1}$ for all $t \in \mathbf{R}$. This proves the theorem.

Proof of Theorem 3. Let F be a finite subset of $\mathbf{G}_\mathbf{Q}$ and C be a compact subset of G/Γ such that the contention of Theorem 2 holds, for some choice of $\varepsilon > 0$ and $\theta > 0$. Let V and $\{x_k\}$, satisfying the conditions as in the statement of the Theorem, and $g \in G$ be given. If $\{u_t\}$ be any one-parameter subgroup of V and $k \geqslant 1$ then by Theorem 2 either there exists a $t \geqslant 0$ such that $u_t x_k g\Gamma \in C$ or there exist $i \in \{1, \ldots, r\}$ and $\lambda \in \Gamma F$ such that $g^{-1}x_k^{-1}u_t x_k g \in \lambda Q_i \lambda^{-1}$ for all $t \in \mathbf{R}$ and $d_i(x_k g\lambda) < \theta$. Let $k \geqslant 1$ be such that $C \cap Vx_k g\Gamma/\Gamma$ is empty. Then by the last observation every one-parameter subgroup of V is contained in one of the subgroups $x_k g\mu Q_j \mu^{-1}g^{-1}x_k^{-1}$ for some $1 \leqslant j \leqslant r$ and $\mu \in \Gamma F$ such that $d_j(x_k g\mu) < \theta$. Since the latter is a countable family of subgroups and V is an analytic subgroup, this implies that there exist $i \in \{1, \ldots, r\}$ and $\lambda \in \Gamma F$ such that $V \subset x_k g\lambda Q_i \lambda^{-1}g^{-1}x_k^{-1}$ and $d_i(x_k g\lambda) < \theta$. Since $Q_i \subset P_i$ and V is in general position we also get that $x_k g\lambda \in P_i$. Thus for any $k \geqslant 1$ such that $C \cap Vx_k g\Gamma/\Gamma = \phi$ there exist $i \in \{1, \ldots, r\}$ and a $\lambda \in \Gamma F$ such that $x_k g\lambda \in P_i$ and $d_i(x_k g\lambda) < \theta$.

Now suppose that the assertion in the Theorem does not hold for the compact set C as above. Then by the above observation there exist a subsequence of $\{x_k\}$, say $\{y_k\}$, $i \in \{1, \ldots, r\}$ and a sequence $\{\lambda_k\}$ in ΓF such that $y_k g\lambda_k \in P_i$ and $d_i(y_k g\lambda_k) < \theta$ for all k. Since $y_k \in P_0 \subset P_i$ and $y_k g\lambda_k \in P_i$ we get that $d_i(y_k g\lambda_k) = d_i(y_k)d_i(g\lambda_k)$ for all k. Now while $d_i(y_k g\lambda_k) < \theta$ for all k, since $\{y_k\}$ is a subsequence of $\{x_k\}$, by hypothesis $d_i(y_k) \to \infty$. Therefore we get that $d_i(g\lambda_k) \to 0$ as $k \to \infty$. But by Lemma 2.4 this is impossible since $\{\lambda_k\}$ is contained in ΓF which is finite union of cosets of the form $\Gamma f, f \in \mathbf{G}_\mathbf{Q}$.

Proof of Theorem 4. Let Γ be a lattice in $SL(3, \mathbf{R})$. If $SL(3, \mathbf{R})/\Gamma$ is compact then the assertion is obvious. We shall therefore assume that G/Γ is noncompact. Then by the arithmeticity theorem (cf. [11]) there exists an algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$ such that $SL(3, \mathbf{R})$ is Lie isomorphic to $\mathbf{G}_\mathbf{R}$ and under the isomorphism Γ corresponds to an arithmetic lattice in $\mathbf{G}_\mathbf{R}$ with respect to the $\mathbf{Q}$-structure on $\mathbf{G}$. We now follow the notation as before with respect to this $\mathbf{G}$ and identify $G = \mathbf{G}_\mathbf{R}$ with $SL(3, \mathbf{R})$ via an isomorphism. We note that since G/Γ is noncompact the $\mathbf{Q}$-rank r of $\mathbf{G}$ is at least 1. On the other hand clearly $r \leqslant 2$, which is the $\mathbf{R}$-rank of $SL(3, \mathbf{R})$. Now let F be a finite subset of $\mathbf{G}_\mathbf{Q}$ and C be a compact subset of G/Γ such that the contentions of Theorems 2 and 3 hold (the former for some choices of $\varepsilon > 0$ and $\theta > 0$). Let $g \in G$ be given. Suppose that one of the sets $\{t \geqslant 0 | v_1(t)g\Gamma \in C\}$ and $\{t \leqslant 0 | v_1(t)g\Gamma \in C\}$ is bounded. Then by Theorem 2, applied to either $\{v_1(t)\}$ or $\{v_1(-t)\}$ in the place of $\{u_t\}$, we get that there exist an $i \in \{1, r\}$ and a $\lambda \in \Gamma F$ such that $g^{-1}v_1(t)g \in \lambda Q_i \lambda^{-1}$ for all $t \in \mathbf{R}$. Put $P = \lambda P_i \lambda^{-1}$. Let L be the closed subgroup generated by all unipotent elements in P. Then we have $g^{-1}v_1(t)g \in L$ for all $t \in \mathbf{R}$. Also L is the group of $\mathbf{R}$-elements of an algebraic subgroup $\mathbf{L}$ which is defined over $\mathbf{Q}$ and has no character defined over $\mathbf{Q}$. This implies that $L\Gamma$ is closed and $L \cap \Gamma$ is a lattice in L (cf. [4] § 2). This shows that condition a) as in the definition of Condition $(*)$ holds for the set for the set C (as above).

Let P_0 be the minimal $\mathbf{Q}$-parabolic subgroup of G as before. It is easy to see that $N(V_1)$ is contained in a Borel subgroup, specifically the group of upper triangular matrices. Hence there exists a $h \in G$ such that $hN(V_1)h^{-1} \subset P_0$. We shall show that condition b) holds for the compact set $h^{-1}C$. This would imply that Condition $(*)$

holds for the compact set $C \cup h^{-1}C$ (in the place of C in the definition). Let $\{f(t)\}_{t\geqslant 0}$ be a curve in $N(V_1)$ such that $|\det f(t)|\, W| \to \infty$ as $t \to \infty$ for every proper nonzero $N(V_1)$-invariant subspace. Put $V = hV_1h^{-1}$ and $\varphi(t) = hf(t)h^{-1}$ for all $t \geqslant 0$. Then $\{\varphi(t)\}_{t\geqslant 0}$ is a curve in $N(V) \subset P_0$ and $|\det \varphi(t)|\, W| \to \infty$ for every proper nonzero $N(V)$-invariant subspace. We shall deduce from this that $d_i(\varphi(t)) \to \infty$ as $t \to \infty$ for any $i \in \{1, r\}$. We first assume this and complete the proof. By Theorem 3 it yields that $C \cap V\varphi(t)hg\Gamma/\Gamma$ is nonempty for all large t. Substituting for V and $\varphi(t)$ we get that $C \cap hV_1f(t)g\Gamma/\Gamma$ is nonempty for all large t, or equivalently, $h^{-1}C \cap V_1f(t)g\Gamma/\Gamma$ is nonempty for all large t. This shows that condition b) holds for the compact set $h^{-1}C$, as desired.

It remains to prove that $d_i(\varphi(t)) \to \infty$ as $t \to \infty$ for any $i \in \{1, r\}$. Let $i \in \{1, r\}$ be given. First suppose that P_i is a maximal **R**-parabolic subgroup. Then there exists a subspace W of $\mathbf{R}^3$ such that

$$P_i = \{g \in G \mid g(W_i) = W_i\}.$$

Further it is easy to see that in this case $d_i(x) = |\det x|\, W_i|^2$ for all $x \in P_i$. Since $|\det \varphi(t)|\, W| \to \infty$ for every proper nonzero $N(V)$-invariant subspace and $N(V) \subset P_0 \subset P_i$, this yields that $d_i(\varphi(t)) \to \infty$ as $t \to \infty$. Now suppose that P_i is not a maximal **R**-parabolic subgroup. Since **G** has **R**-rank 2, this implies that $\mathbf{P}_i$ is a minimal **R**-parabolic subgroup. In turn we get $r = 1$, $i = 1$ and $P_0 = P_1$ and they are conjugate to the subgroup B consisting of upper triangular matrices; in fact $P_1 = hBh^{-1}$, since $h^{-1}P_1h$ has to be the Borel subgroup containing V_1. Using this we see that for all $t \geqslant 0$, $d_1(\varphi(t)) = (a_1(t)/a_3(t))^2 = a_1^4(t)a_2^2(t)$, where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are the diagonal entries of $f(t)$. Since $|\det f(t)|\, W| \to \infty$ for any $N(V_1)$-invariant proper non-zero subgroup, and $N(V_1) \subset B_1$, we get that $a_1^2(t) \to \infty$ and $a_1^2(t)a_2^2(t) \to \infty$ as $t \to \infty$. Hence $d_1(\varphi(t)) \to \infty$ as sought to be proved. This proves the Theorem.

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